A precision calculation of neutrino decoupling

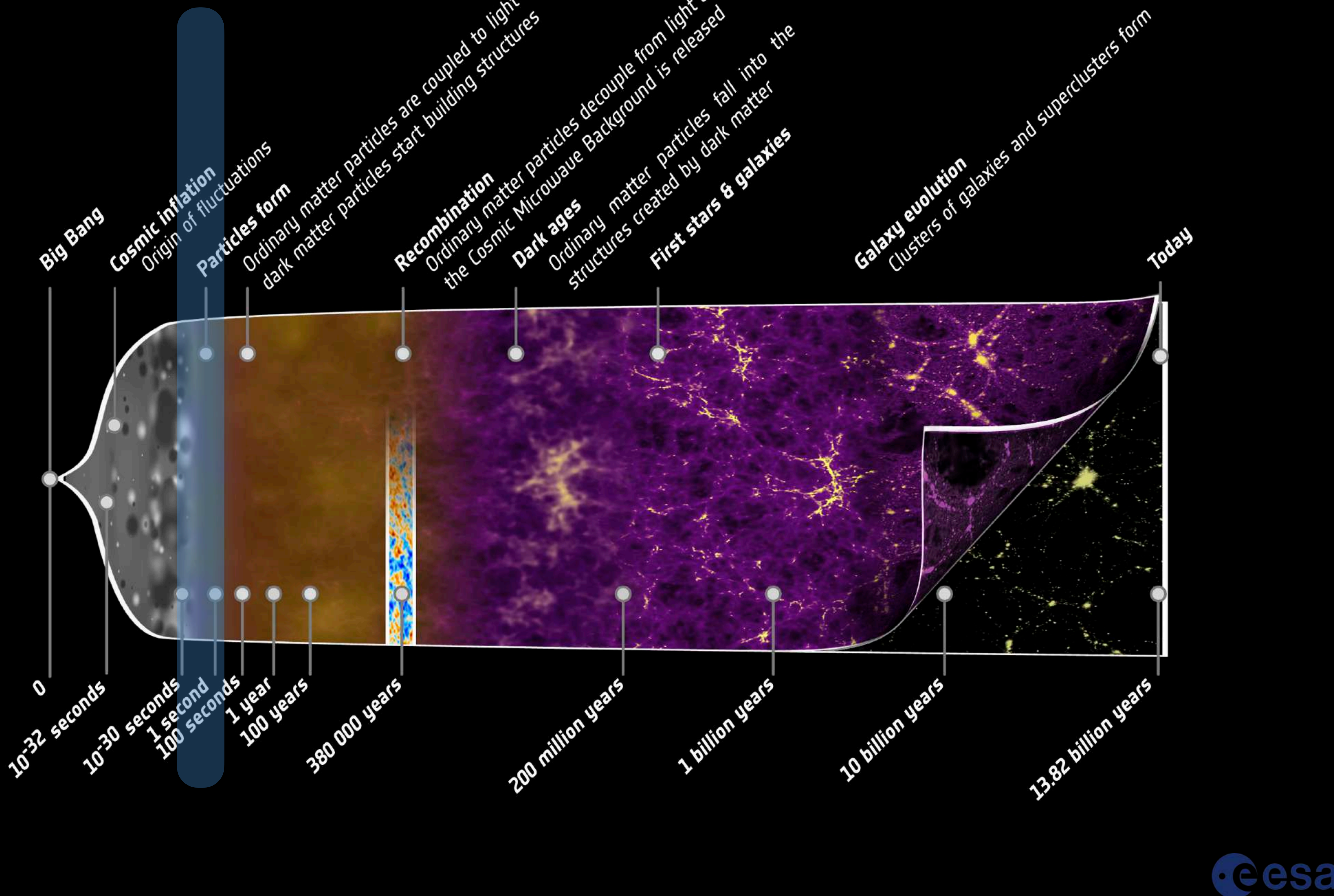
Julien Froustey

with Cyril Pitrou (IAP), Maria Cristina Volpe (APC)

LUTH seminar ○ 05/11/2020

[1912.09378] **JF**, C. Pitrou, *Phys. Rev. D* 101, 043524 (2020)
[2008.01074] **JF**, C. Pitrou, M.C. Volpe, *to appear in JCAP*

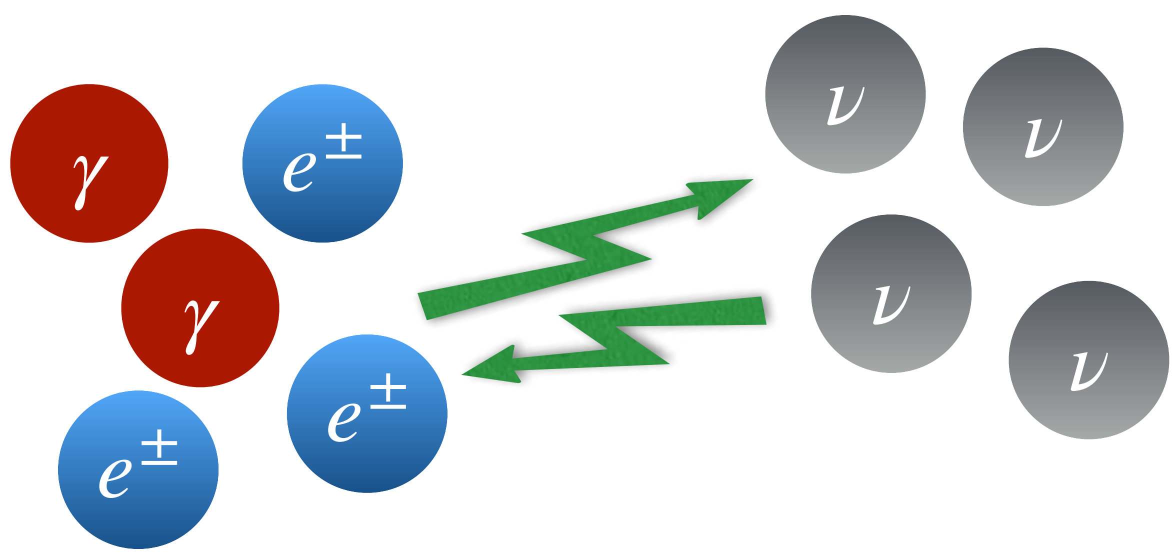
“MeV age”



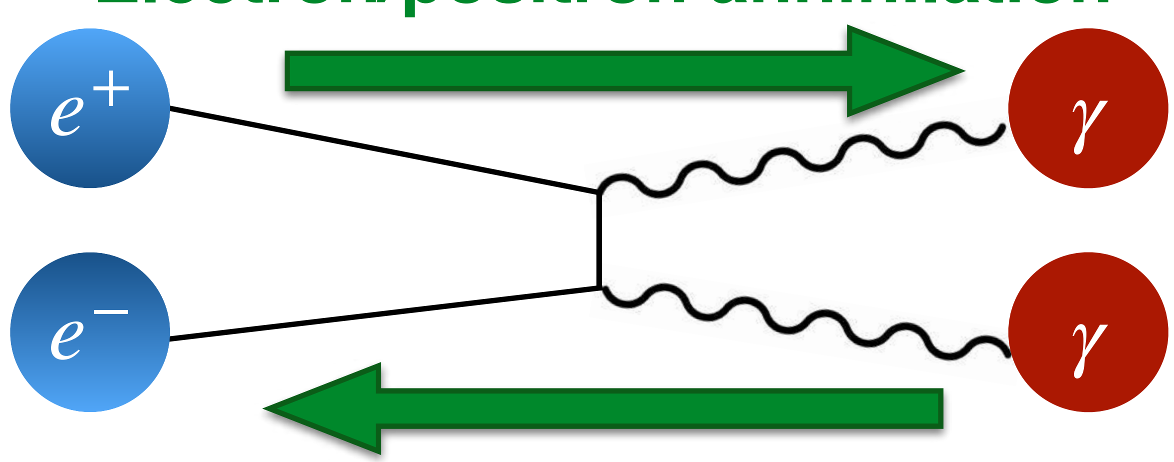
Big Bang
Cosmic inflation
Origin of fluctuations
Particles form
Ordinary matter particles are coupled to light and dark matter particles start building structures
Recombination
Ordinary matter particles decouple from light and the Cosmic Microwave Background is released
Dark ages
Ordinary matter particles fall into the structures created by dark matter
First stars & galaxies
Galaxy evolution
Clusters of galaxies and superclusters form
Today
0
10^{-32} seconds
10^{-30} seconds
1 second
100 seconds
1 year
100 years
380 000 years
200 million years
1 billion years
10 billion years
13.82 billion years
esa

The MeV age

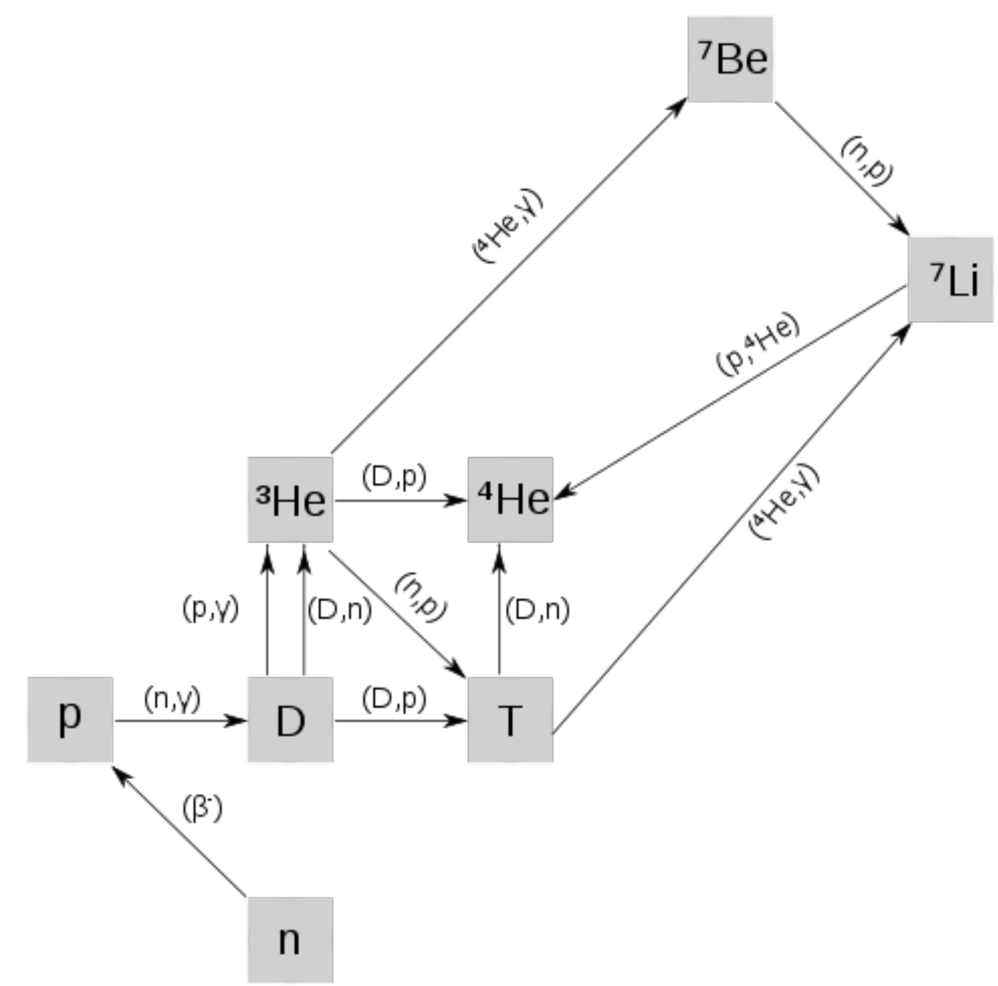
Neutrino decoupling



Electron/positron annihilation

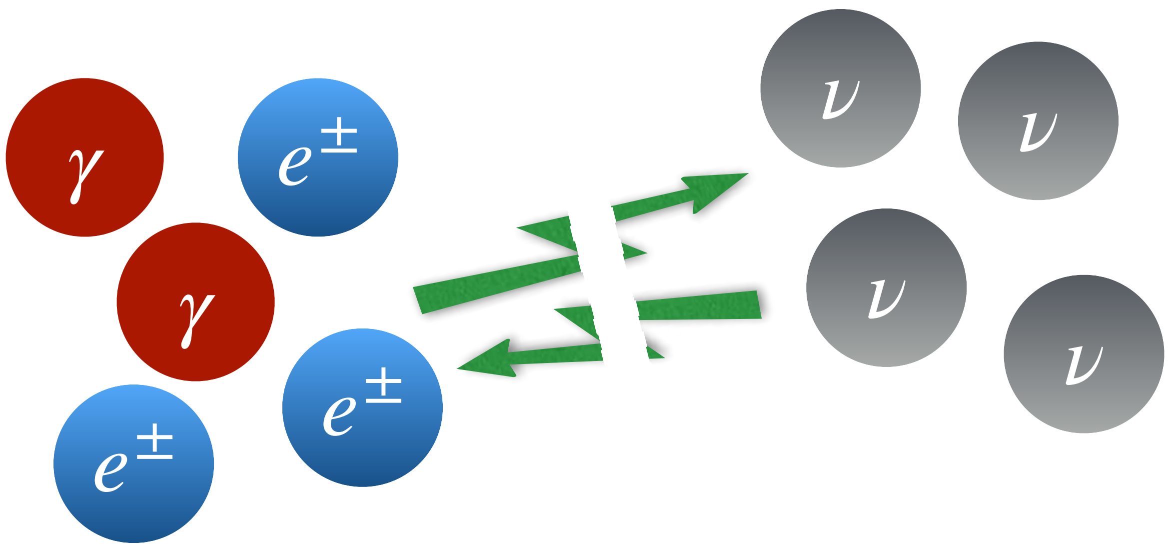


Big Bang Nucleosynthesis



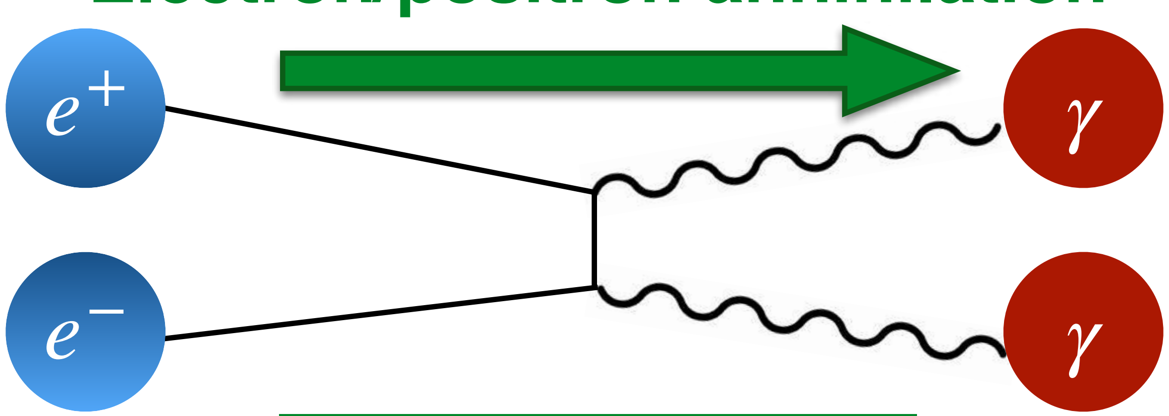
The MeV age

Neutrino decoupling



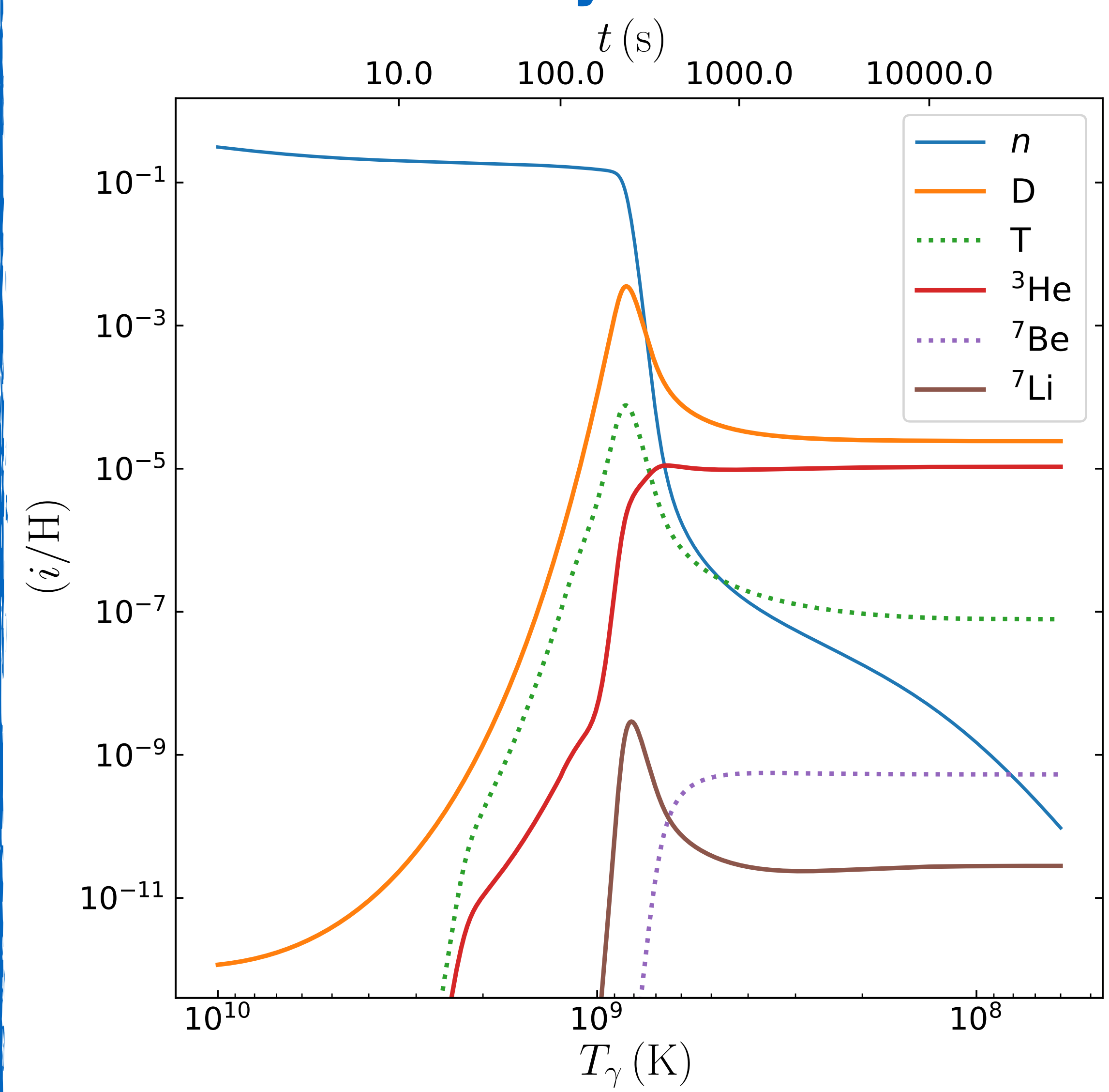
$T \sim 1\,\mathrm{MeV}$

Electron/positron annihilation



$T \sim 0.511\,\mathrm{MeV}$

Big Bang Nucleosynthesis

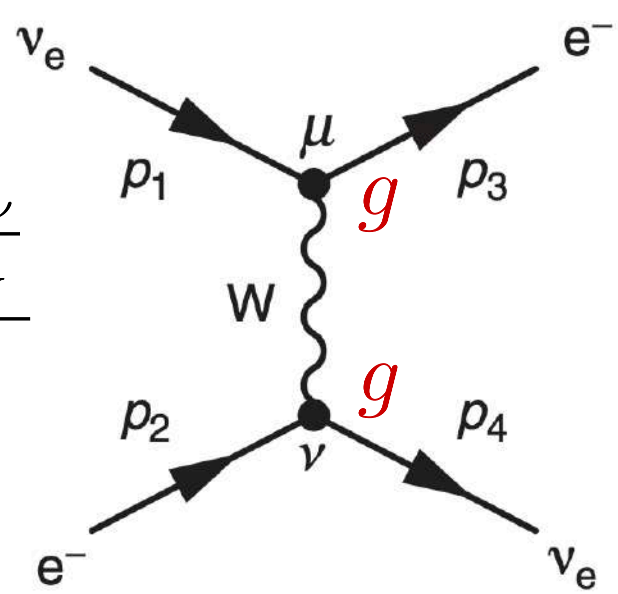


$T \sim 0.1\,\mathrm{MeV}$

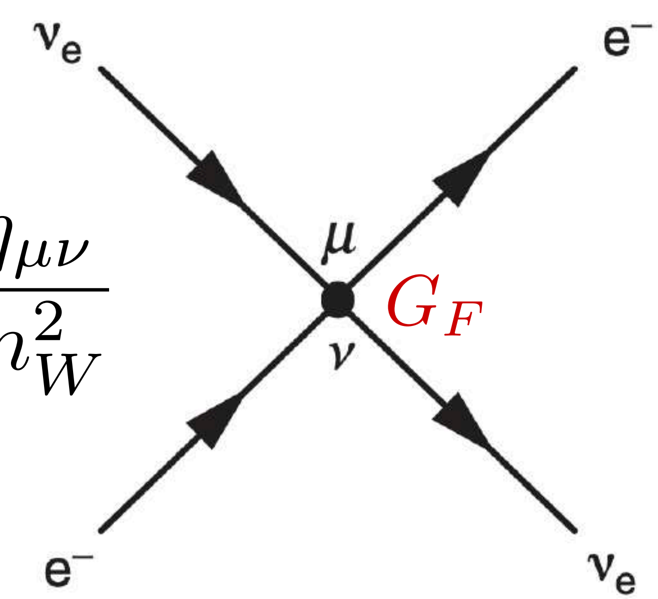
Instantaneous neutrino decoupling

- Weak interactions : low energy 4-Fermi theory

$$-i\frac{\eta_{\mu\nu}-\frac{q_\mu q_\nu}{m_W^2}}{q^2-m_W^2}$$



$$\longrightarrow \quad i\frac{\eta_{\mu\nu}}{m_W^2}$$

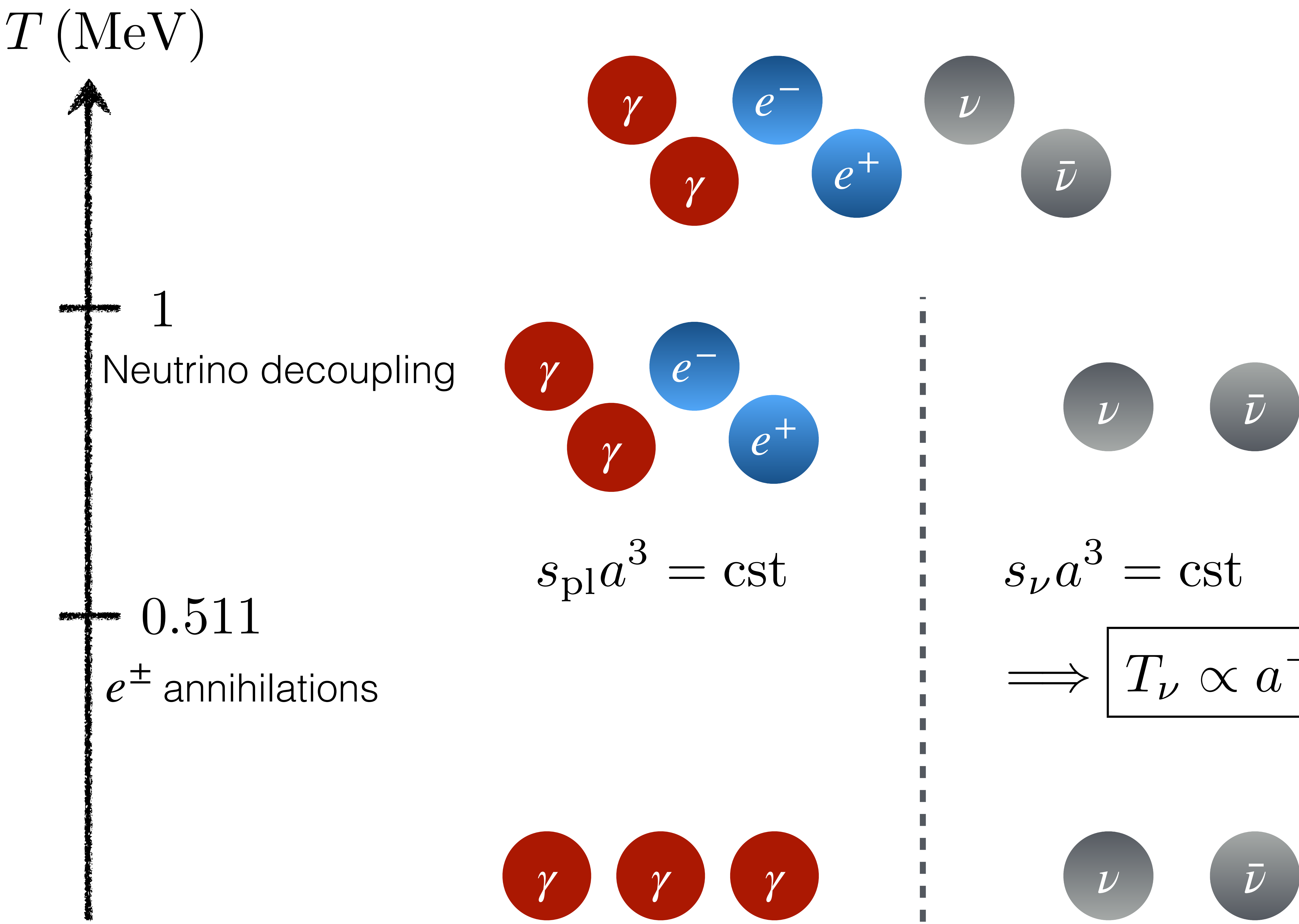


$$\boxed{G_F \simeq 1.17 \times 10^{-5}\,\mathrm{GeV}^{-2}}$$

- Decoupling temperature

$$\frac{\Gamma}{H} = \frac{G_F^2 T^5}{T^2/m_{\mathrm{Pl}}} \simeq \left(\frac{T}{1\ \mathrm{MeV}}\right)^3$$

Instantaneous neutrino decoupling - Entropy conservation



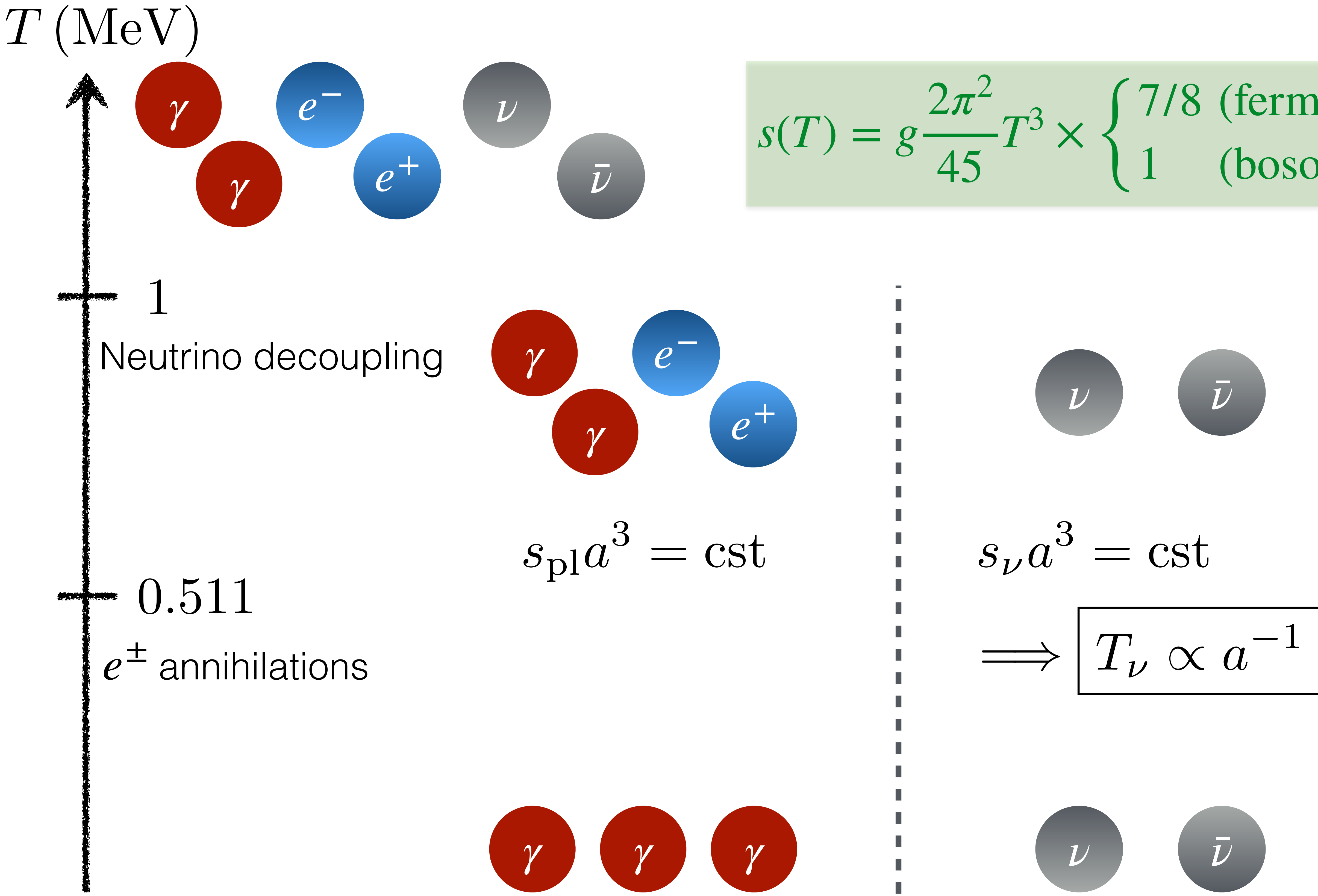
$s_{\mathrm{pl}} a^3 = \mathrm{cst}$

$s_\nu a^3 = \mathrm{cst}$

$\Longrightarrow \boxed{T_\nu \propto a^{-1}}$

Instantaneous neutrino decoupling - Entropy conservation

$$s(T) = g\frac{2\pi^2}{45}T^3 \times \begin{cases} 7/8 & \text{(fermions)} \\ 1 & \text{(bosons)} \end{cases}$$



Instantaneous neutrino decoupling - Entropy conservation

$T\,(\mathrm{MeV})$

γ e^- ν

γ e^+ $\bar{\nu}$

$$s(T) = g\frac{2\pi^2}{45}T^3 \times \begin{cases} 7/8 & \text{(fermions)} \\ 1 & \text{(bosons)} \end{cases}$$

1

Neutrino decoupling

γ e^- ν $\bar{\nu}$

γ e^+

$$s_{\mathrm{pl}}a^3 = \mathrm{cst} \qquad s_\nu a^3 = \mathrm{cst}$$

0.511

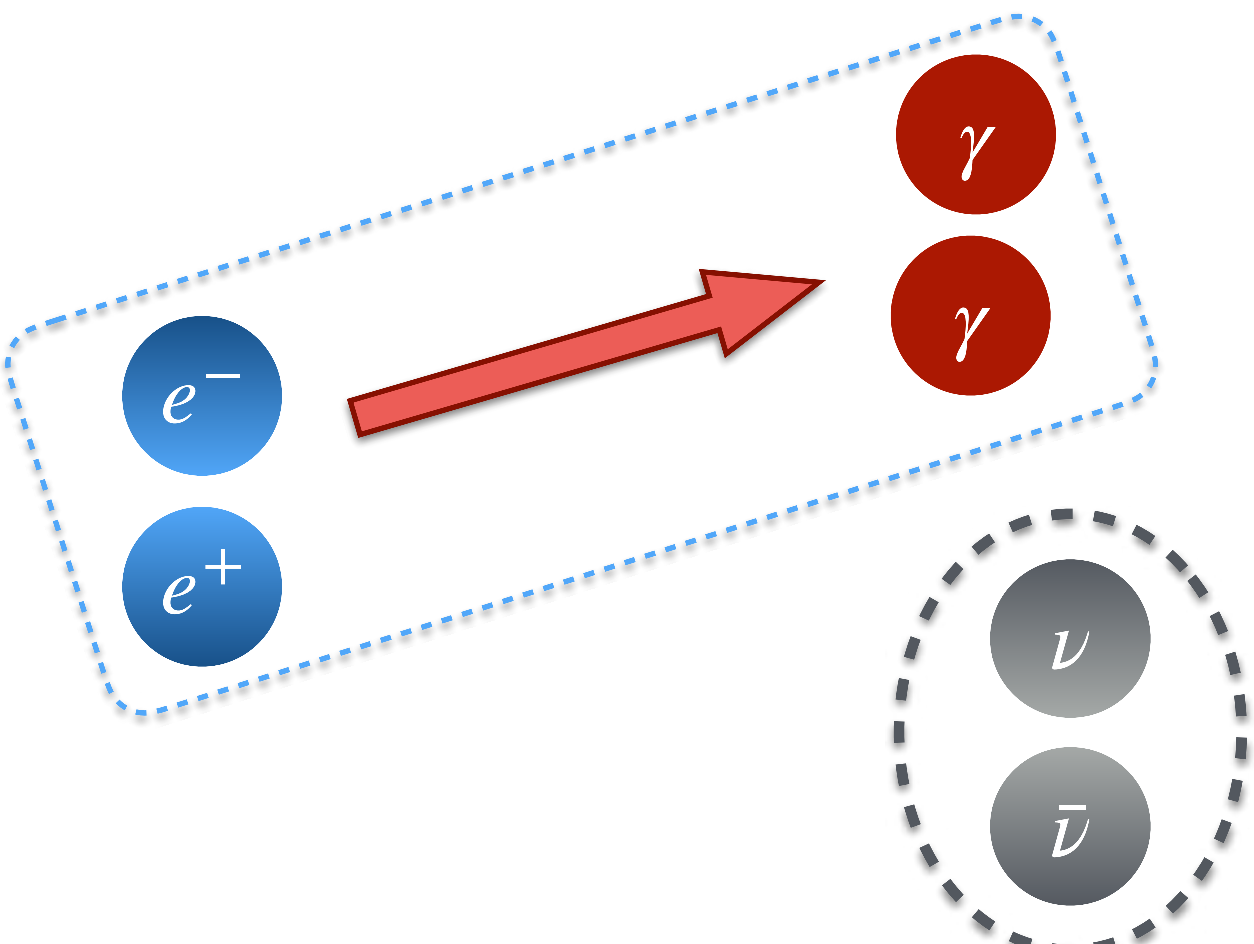
$e^\pm$ annihilations

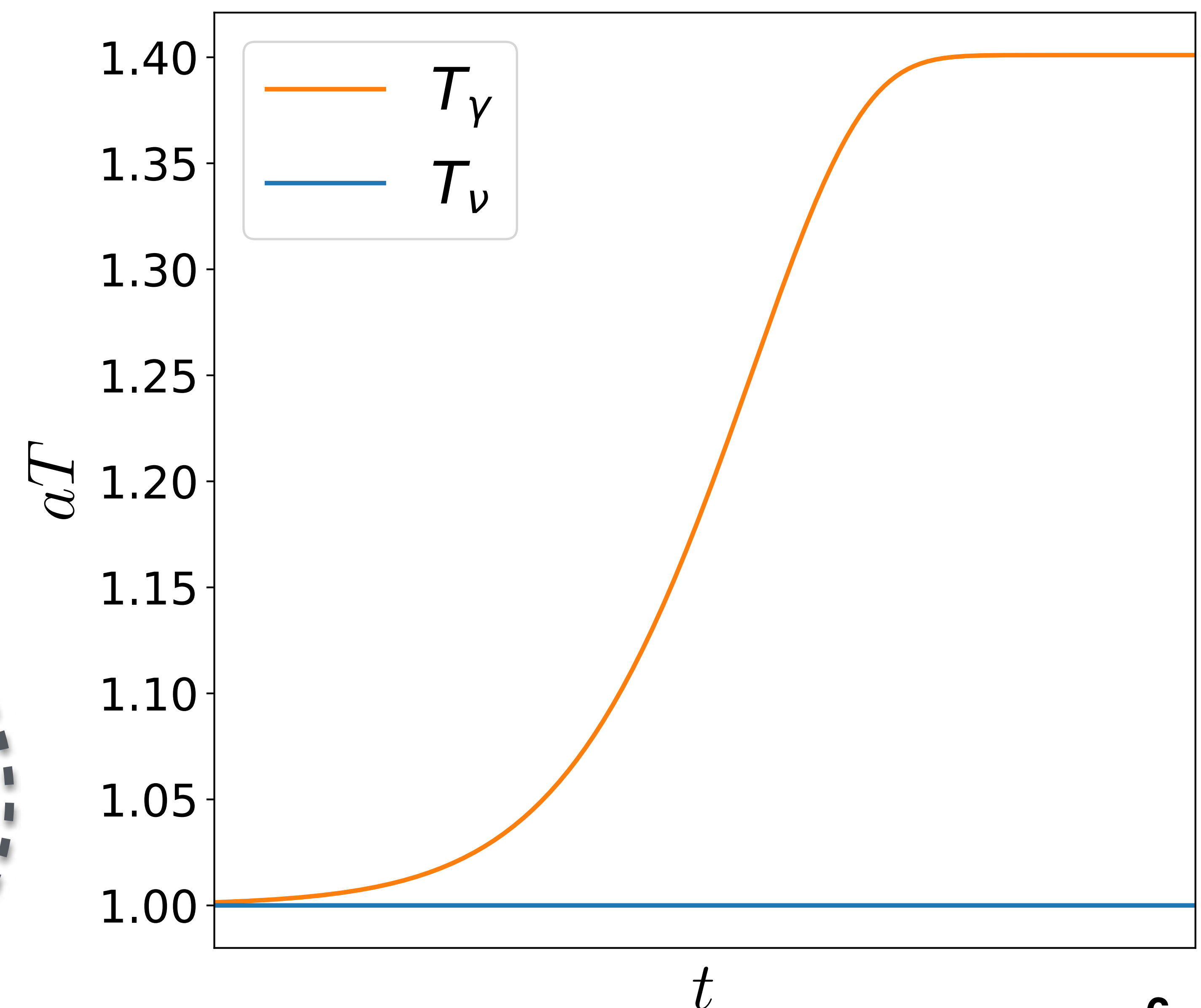
$$\left(\frac{T_\gamma}{T_\nu}\right)_{\mathrm{today}} = \left(\frac{11}{4}\right)^{1/3} \simeq 1.40102$$

γ γ γ ν $\bar{\nu}$

Beyond the instantaneous decoupling approximation

- Overlap between decoupling and $e^{\pm}$ annihilations

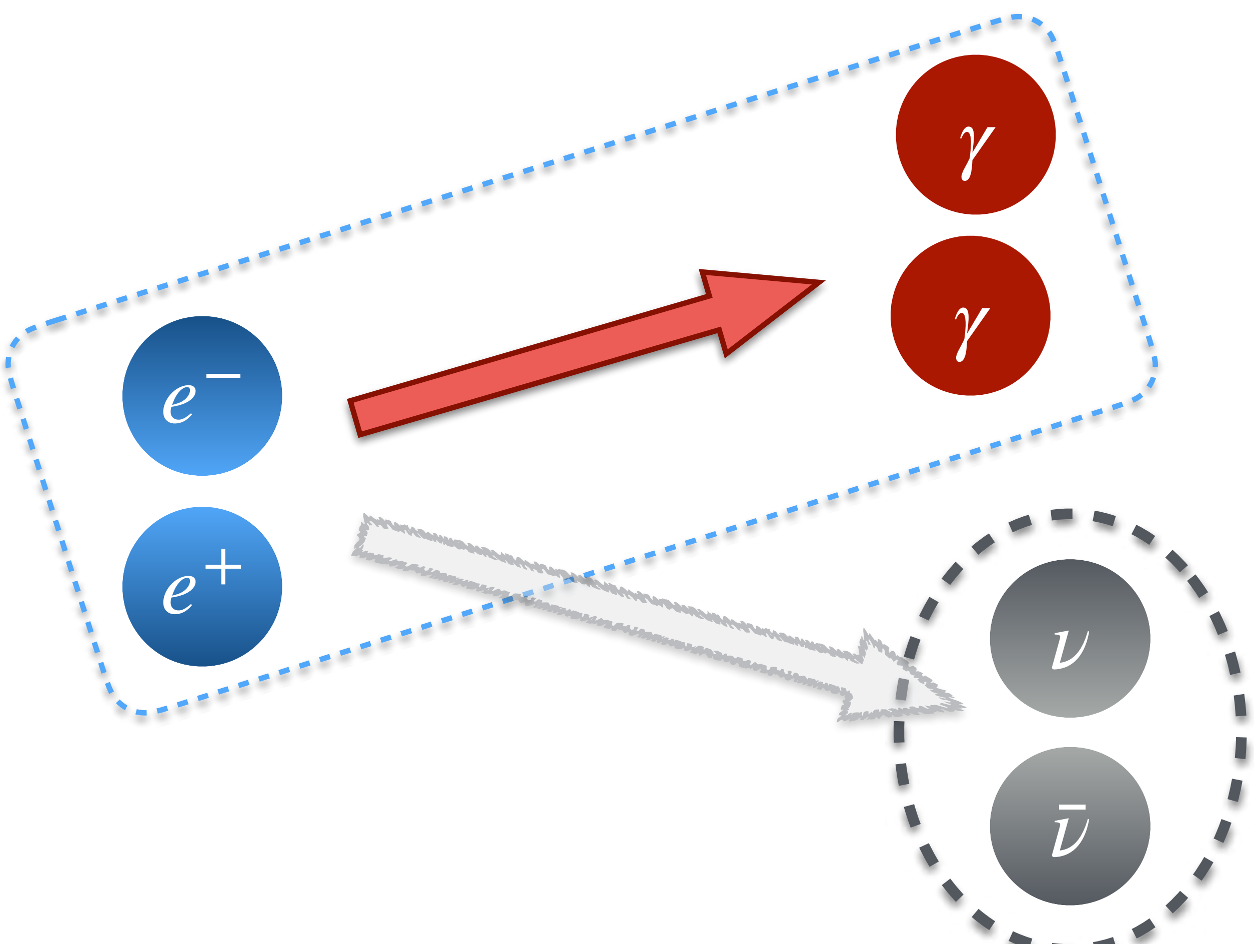


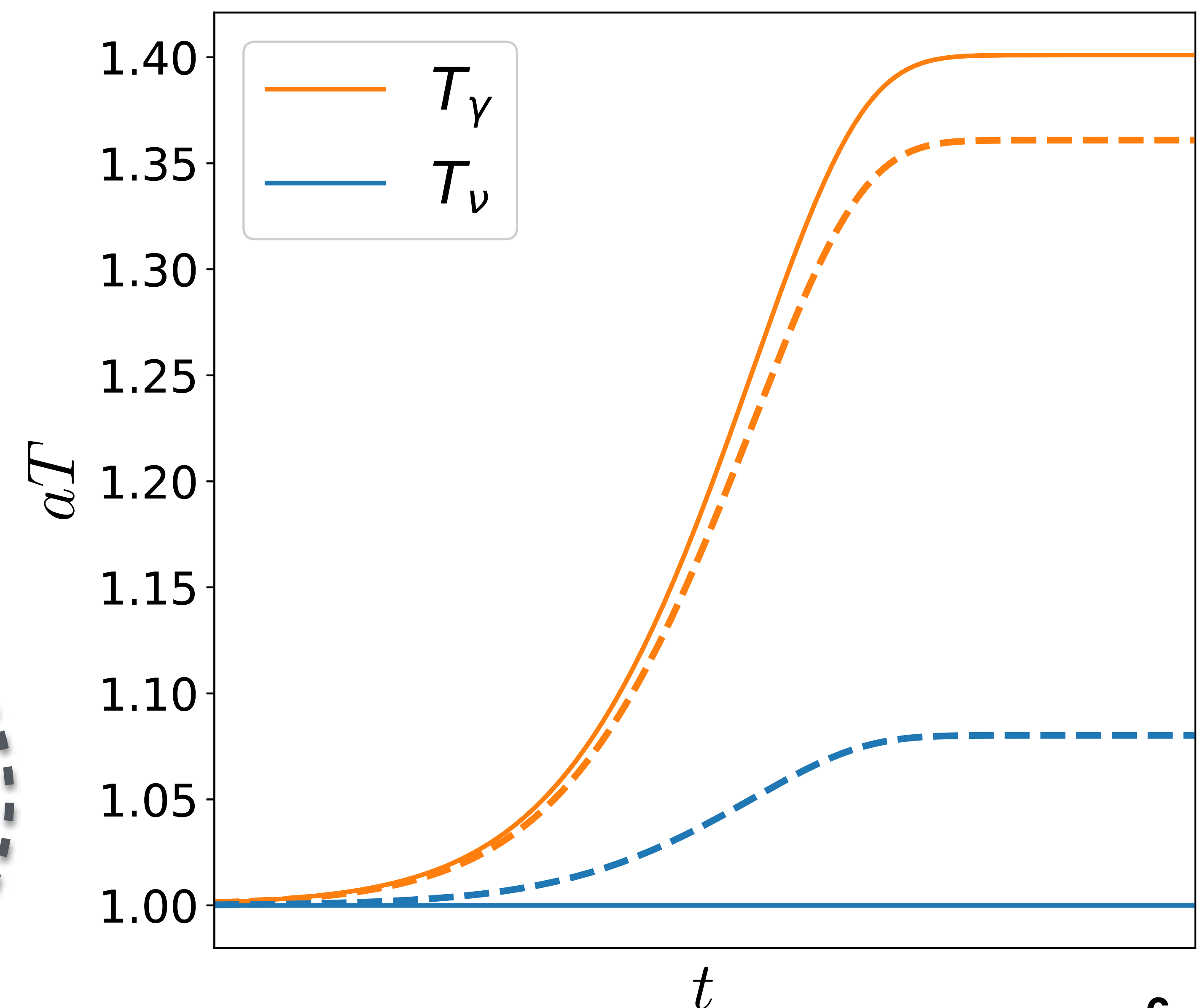


Beyond the instantaneous decoupling approximation

- Overlap between decoupling and $e^{\pm}$ annihilations

$\Longrightarrow$ smaller T_γ and increased T_ν



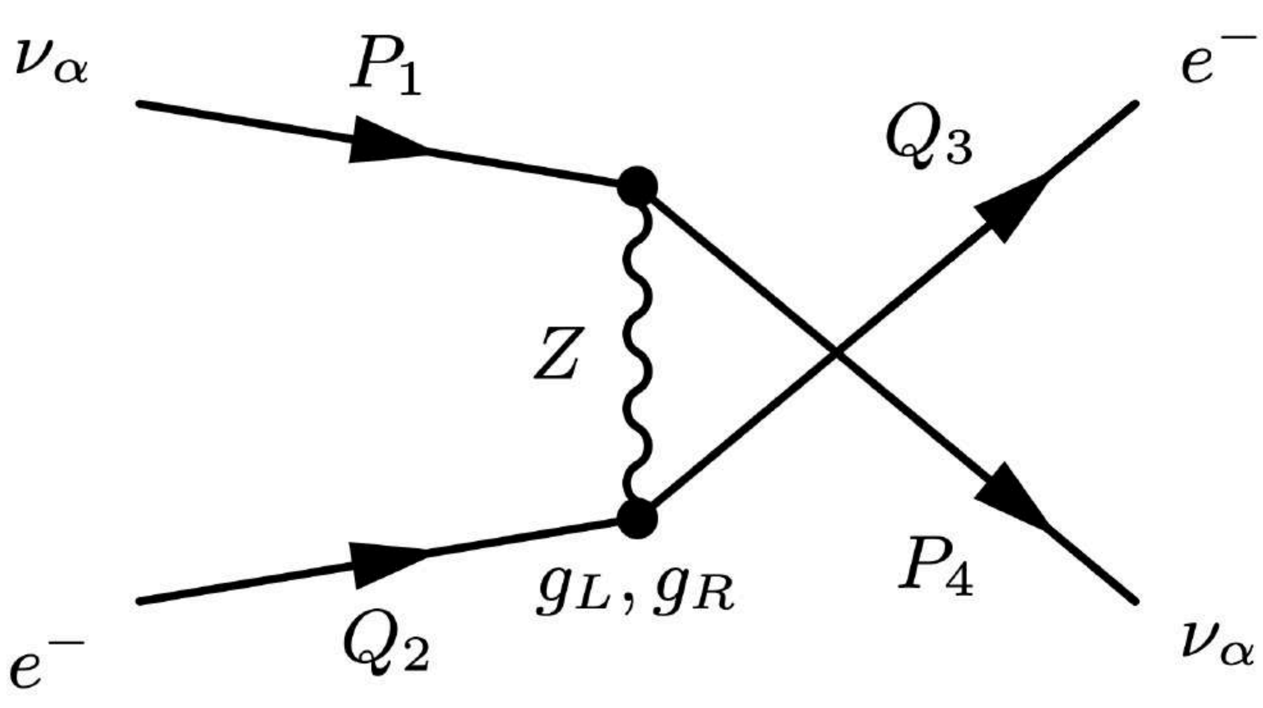


Beyond the instantaneous decoupling approximation

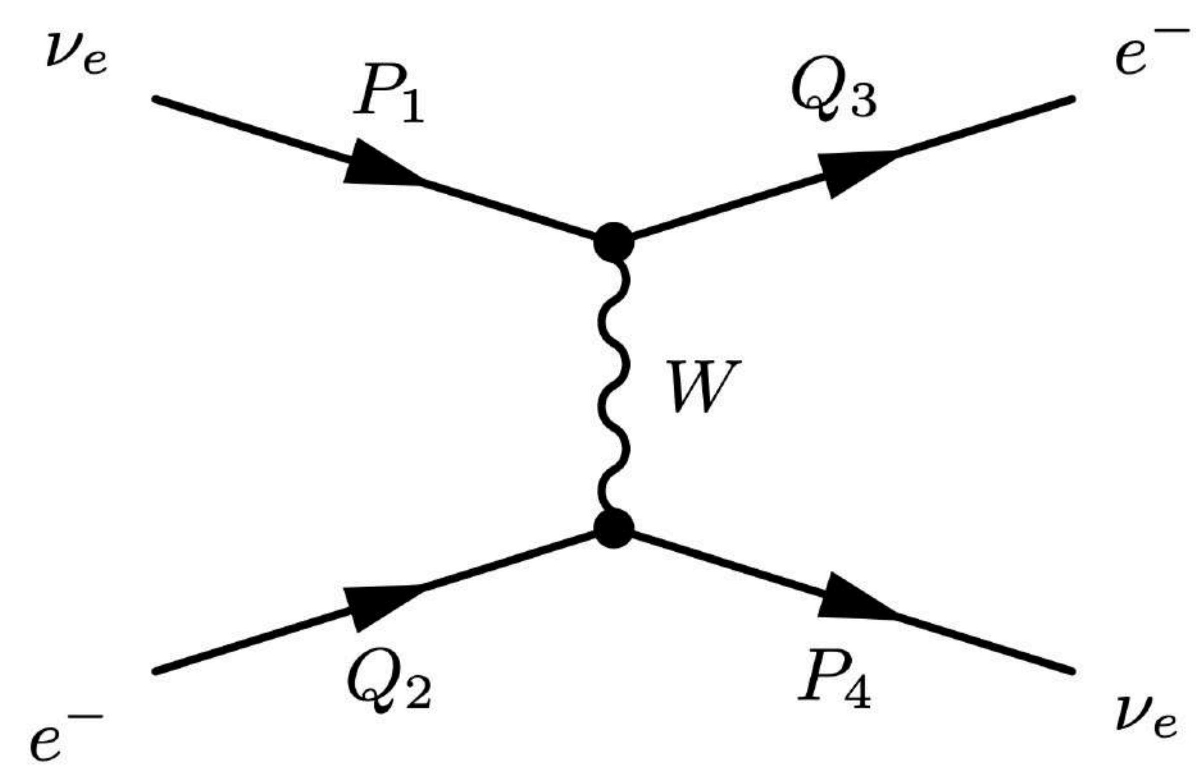
- Overlap between decoupling and $e^{\pm}$ annihilations

  $\Longrightarrow$ smaller T_γ and increased T_ν

- Different interactions of ν_e and $\nu_{\mu,\tau}$



Neutral current



Charged current

Beyond the instantaneous decoupling approximation

- Overlap between decoupling and $e^{\pm}$ annihilations

  $\Longrightarrow$ smaller T_γ and increased T_ν

- Different interactions of ν_e and $\nu_{\mu,\tau}$

  $\Longrightarrow$ later decoupling for ν_e + higher energy transfer

Beyond the instantaneous decoupling approximation

- Overlap between decoupling and $e^{\pm}$ annihilations

  $\Longrightarrow$ smaller T_γ and increased T_ν

- Different interactions of ν_e and $\nu_{\mu,\tau}$

  $\Longrightarrow$ later decoupling for ν_e + higher energy transfer

- More energetic neutrinos remain in thermal contact longer

  $\Longrightarrow$ spectral distortions

Beyond the instantaneous decoupling approximation

- Overlap between decoupling and $e^{\pm}$ annihilations

  $\Longrightarrow$ smaller T_γ and increased T_ν

- Different interactions of ν_e and $\nu_{\mu,\tau}$

  $\Longrightarrow$ later decoupling for ν_e + higher energy transfer

- More energetic neutrinos remain in thermal contact longer

  $\Longrightarrow$ spectral distortions

We need to numerically evolve the distribution functions

$$\boxed{f_{\nu_e}(p,t) \neq f_{\nu_{\mu,\tau}}(p,t) \neq f_{\mathrm{Fermi-Dirac}}}$$

Neutrino decoupling - standard calculations

- Homogeneous and isotropic cosmology

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right] \qquad\qquad H \equiv \frac{\dot{a}}{a}$$

$\Longrightarrow$ Distribution function $f(\vec{r}, \vec{p}, t) = f(p, t)$

- Boltzmann equation + energy conservation equation

$$\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right] f = \mathcal{C}[f] \qquad\qquad \dot{\rho} + 3H(\rho + P) = 0$$

$$\nu_\alpha + e^- \to \nu_\alpha + e^- \qquad \nu_\alpha + \bar{\nu}_\alpha \to e^- + e^+$$

$$\nu_\alpha + \nu_\beta \to \nu_\alpha + \nu_\beta \qquad \nu_\alpha + \bar{\nu}_\alpha \to \nu_\beta + \bar{\nu}_\beta$$

Neutrino decoupling - standard calculations

- Use $T_{\rm cm} = T_\nu^{(0)} \propto a^{-1}$ as the integration variable.

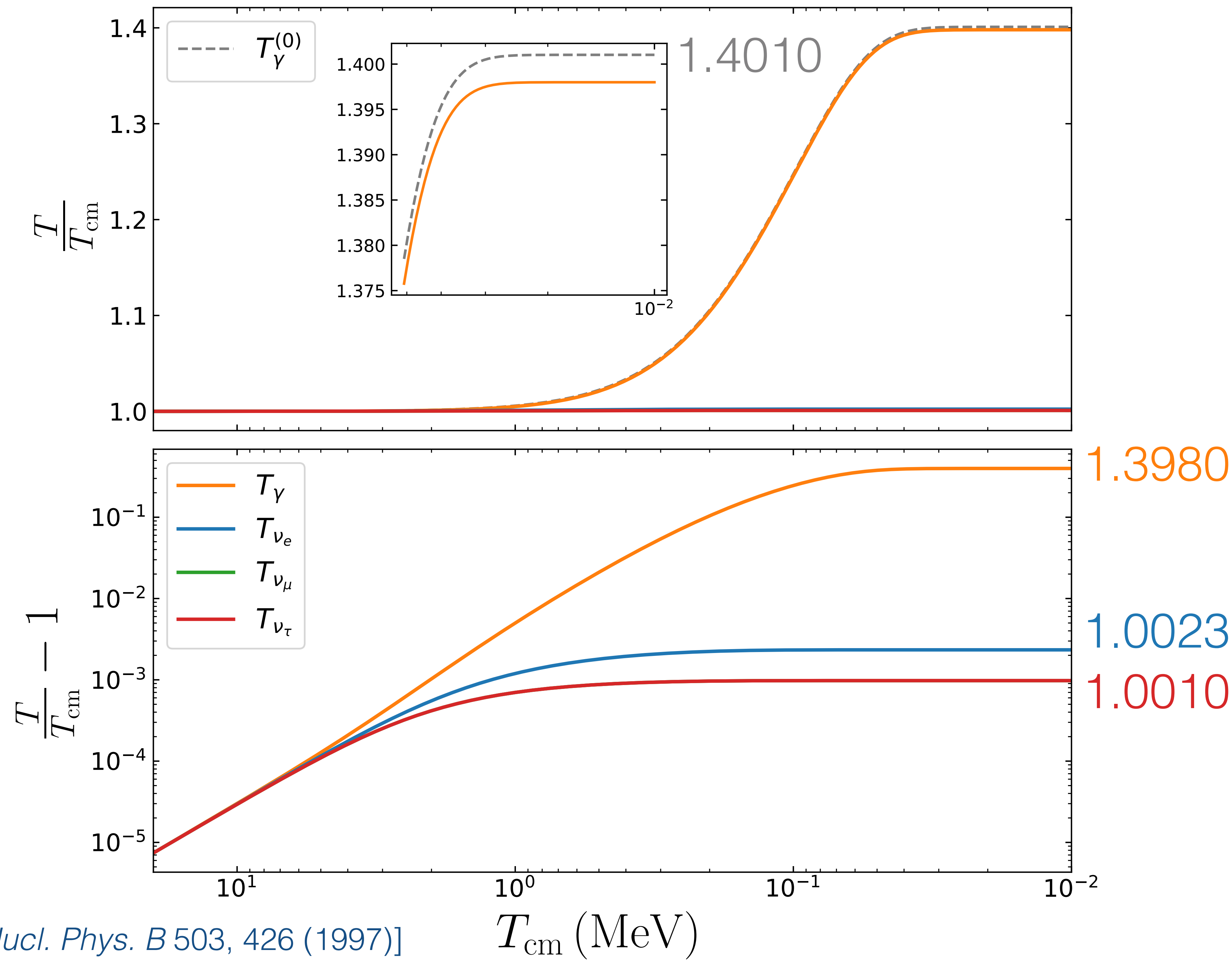
- Parametrization $\quad f_{\nu_\alpha}(p,t) \equiv \dfrac{1}{e^{p/T_{\nu_\alpha}}+1}\left[1+\delta g_{\nu_\alpha}(p,t)\right]$

$$\rho_{\nu_\alpha} \equiv \frac{7}{8}\frac{\pi^2}{30}T_{\nu_\alpha}^4$$

- Initially ($T_{\rm cm}^{\rm (in)} = 20\ {\rm MeV}$), all species are coupled

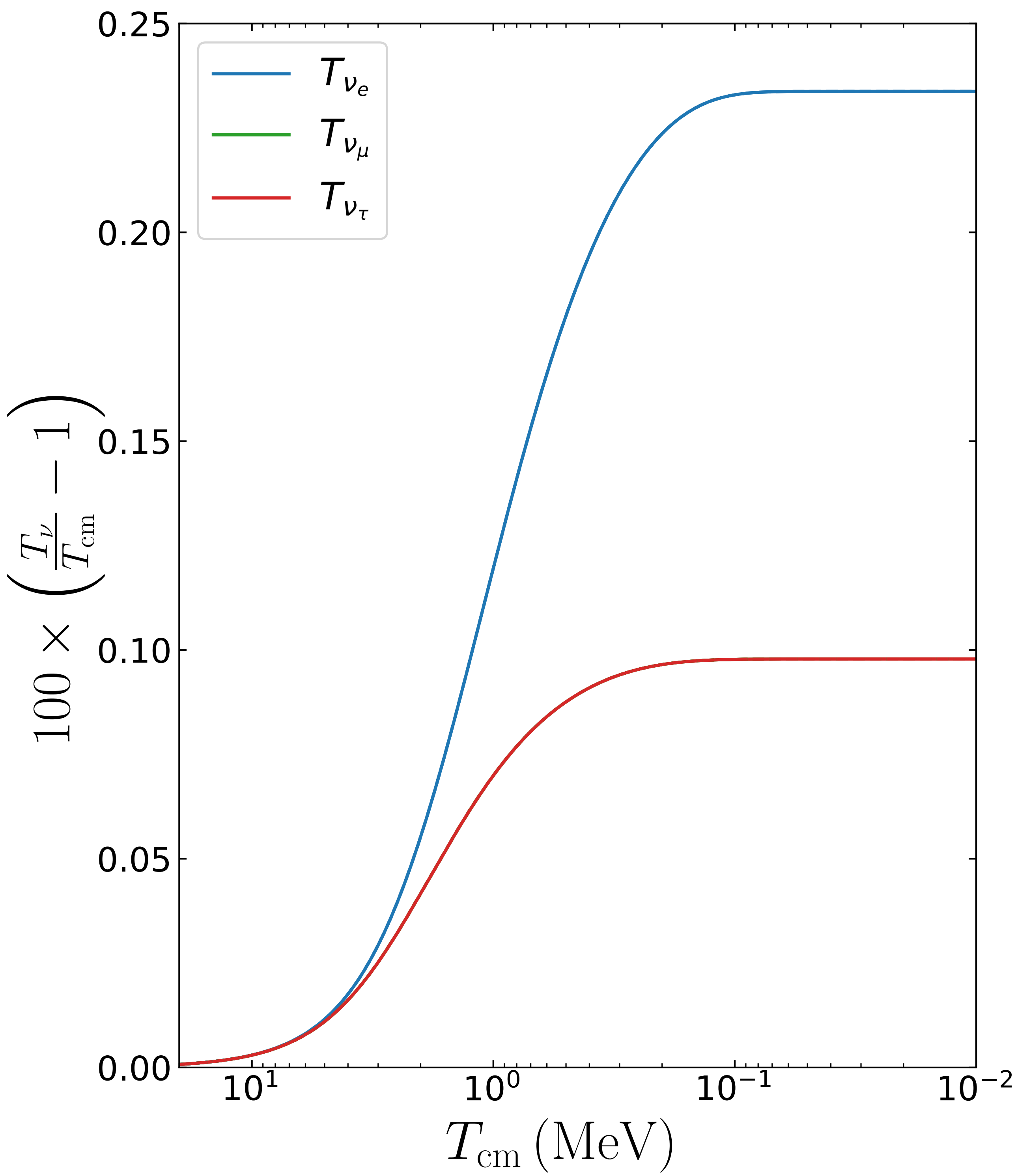
$$f_{\nu_\alpha}^{\rm (in)}(p,t) = \frac{1}{e^{p/T_\gamma^{\rm (in)}}+1}$$

Neutrino decoupling - standard calculations

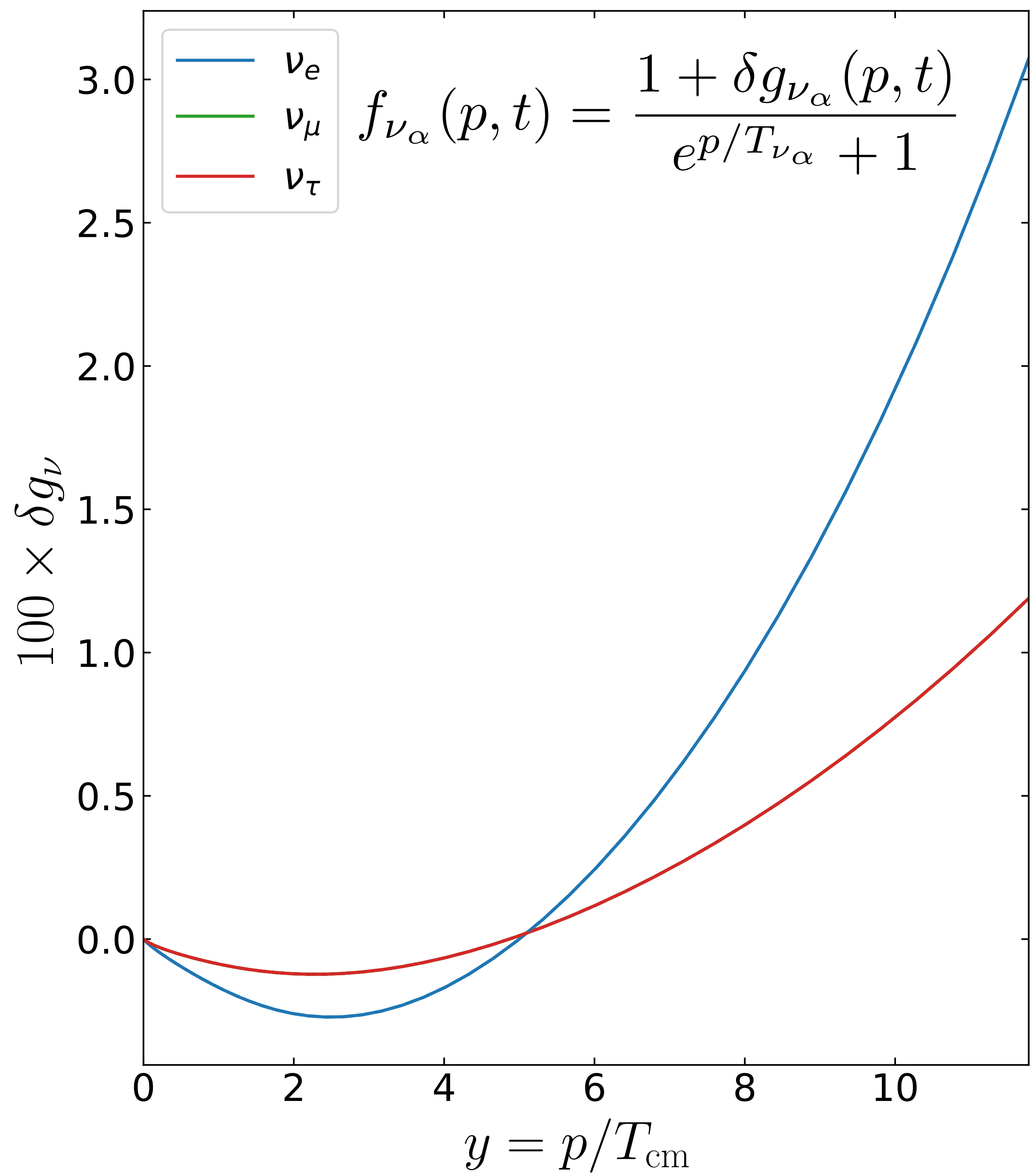


[A. Dolgov et al., *Nucl. Phys. B* 503, 426 (1997)]
[S. Esposito et al., *Nucl. Phys. B* 590, 539 (2000)]
[G. Mangano et al., *Phys. Lett. B* 534, 8 (2002)]
[E. Grohs et al., *Phys. Rev. D* 93, 083522 (2016)]
[**JF**, C. Pitrou, *Phys. Rev. D* 101, 043524 (2020)]

Neutrino decoupling - standard calculations



Effective temperatures



Effective distortions

Effective number of neutrinos N_{eff}

Increased energy density of neutrinos $\Longleftrightarrow$ $N_{\mathrm{eff}} > 3$ species of neutrinos that instantaneously decouple

Effective number of neutrinos $N_{\rm eff}$

Increased energy density of neutrinos $\Longleftrightarrow$ $N_{\rm eff} > 3$ species of neutrinos that instantaneously decouple

$$\rho_\nu^{(0)} = 2 \times \frac{7}{8} \times \frac{\pi^2}{30} \times 3 \times \left(\frac{4}{11}\right)^{4/3} T_\gamma^4$$

Instantaneous decoupling

$$\rho_\gamma = 2 \times \frac{\pi^2}{30} \times T_\gamma^4 \qquad \Longrightarrow \quad \rho_\nu^{(0)} = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times 3 \times \rho_\gamma$$

Effective number of neutrinos $N_{\rm eff}$

Increased energy density of neutrinos $\Longleftrightarrow$ $N_{\rm eff} > 3$ species of neutrinos that instantaneously decouple

$$\rho_\nu^{(0)} = \underbrace{2}_{\text{neutrinos + antineutrinos}} \times \underbrace{\frac{7}{8}}_{\text{fermions}} \times \frac{\pi^2}{30} \times \underbrace{3}_{e,\mu,\tau} \times \underbrace{\left(\frac{4}{11}\right)^{4/3} T_\gamma^4}_{T_{\rm cm}^4}$$

Instantaneous decoupling

$$\rho_\gamma = \underbrace{2}_{\text{d.o.f.s}} \times \frac{\pi^2}{30} \times T_\gamma^4 \qquad \Longrightarrow \qquad \rho_\nu^{(0)} = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times 3 \times \rho_\gamma$$

Effective number of neutrinos $N_{\rm eff}$

Increased energy density of neutrinos $\Longleftrightarrow$ $N_{\rm eff} > 3$ species of neutrinos that instantaneously decouple

$$\rho_\nu^{(0)} = 2 \times \frac{7}{8} \times \frac{\pi^2}{30} \times 3 \times \underbrace{\left(\frac{4}{11}\right)^{4/3} T_\gamma^4}_{T_{\rm cm}^4}$$

Instantaneous decoupling

neutrinos + antineutrinos

fermions

e, μ, τ

d.o.f.s

$$\rho_\gamma = 2 \times \frac{\pi^2}{30} \times T_\gamma^4 \qquad \Longrightarrow \quad \rho_\nu^{(0)} = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times 3 \times \rho_\gamma$$

$$\rho_\nu = 2 \times \frac{7}{8} \times \frac{\pi^2}{30} \left(T_{\nu_e}^4 + T_{\nu_\mu}^4 + T_{\nu_\tau}^4\right)$$

Incomplete decoupling

$$\Longrightarrow \quad \rho_\nu = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times N_{\rm eff} \times \rho_\gamma$$

Effective number of neutrinos $N_{\rm eff}$

Increased energy density of neutrinos $\Longleftrightarrow$ $N_{\rm eff} > 3$ species of neutrinos that instantaneously decouple

$$\rho_\nu^{(0)} = 2 \times \frac{7}{8} \times \frac{\pi^2}{30} \times 3 \times \underbrace{\left(\frac{4}{11}\right)^{4/3} T_\gamma^4}_{T_{\rm cm}^4}$$

neutrinos + antineutrinos

fermions

e, μ, τ

Instantaneous decoupling

d.o.f.s

$$\rho_\gamma = 2 \times \frac{\pi^2}{30} \times T_\gamma^4 \qquad \Longrightarrow \qquad \rho_\nu^{(0)} = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times 3 \times \rho_\gamma$$

$$\rho_\nu = 2 \times \frac{7}{8} \times \frac{\pi^2}{30}\left(T_{\nu_e}^4 + T_{\nu_\mu}^4 + T_{\nu_\tau}^4\right)$$

Incomplete decoupling

$$= \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times N_{\rm eff} \times \rho_\gamma$$

$N_{\rm eff} \simeq 3.0434$

Planck

$N_{\rm eff} = 2.99 \pm 0.17 \ (68\%)$

Towards a precision calculation

Physical phenomena to take into account:

- Boltzmann equation with collisions ✓
- Proper distributions (Fermi-Dirac) ✓
- Neutrino masses and mixings
- …

Towards a precision calculation

Physical phenomena to take into account:

- Boltzmann equation with collisions ✓ ✗
- Proper distributions (Fermi-Dirac) ✓
- Neutrino masses and mixings ✗
- …

Previous works:

[G. Mangano et al., *Nucl. Phys. B* 729, 221 (2005)]
[P.F. de Salas, S. Pastor, *JCAP* 07, 051 (2016)]
[K. Akita, M. Yamaguchi, *JCAP* 08, 012 (2020)]

Outline

1. Neutrino evolution with mixing:

   Quantum Kinetic Equations

2. An approximation:

   Adiabatic Transfer of Averaged Oscillations

3. Results for neutrino decoupling

Outline

1. Neutrino evolution with mixing:

   Quantum Kinetic Equations

2. An approximation:

   Adiabatic Transfer of Averaged Oscillations

3. Results for neutrino decoupling

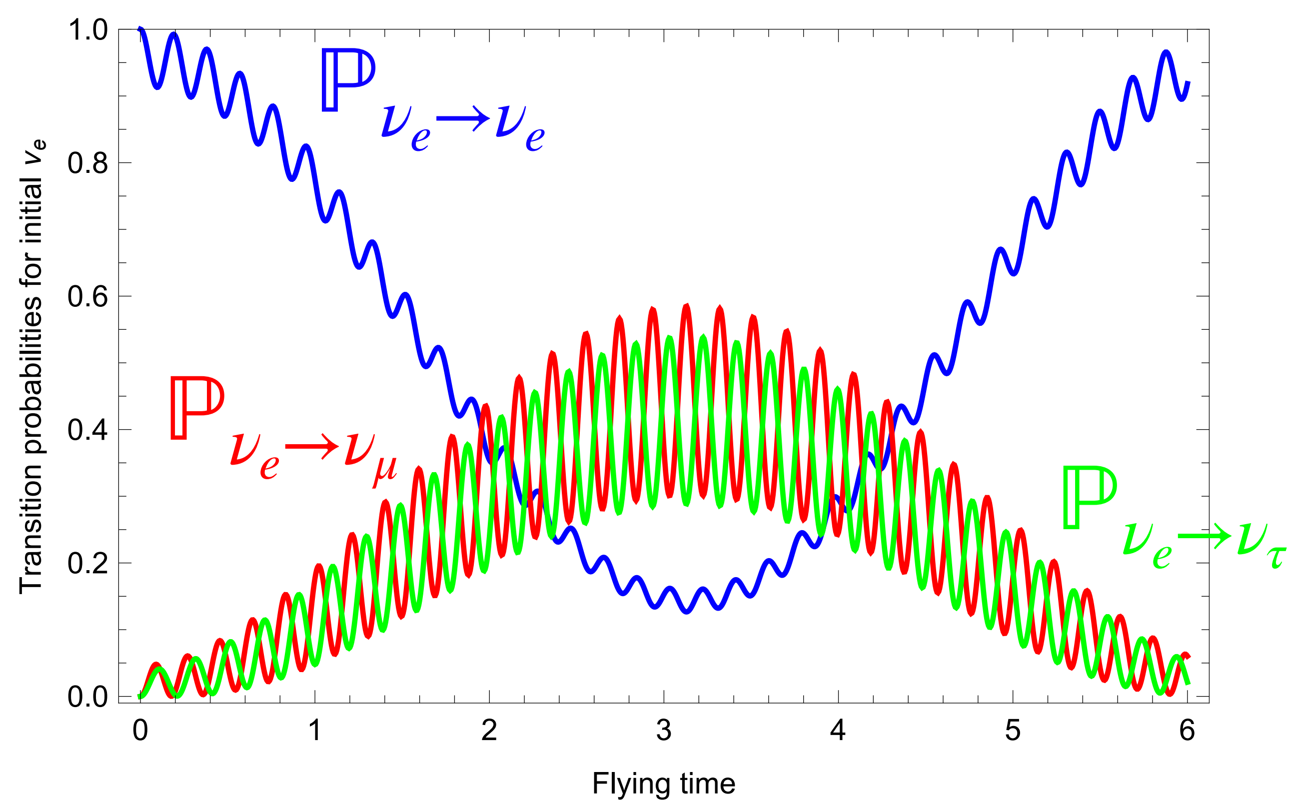
Massive neutrinos (1)

- Standard model: 3 species of massless neutrinos ν_L
- Homestake experiment, Solar Neutrino Problem…
  $\rightarrow$ massive neutrinos

$$\text{Flavor states} \quad \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad \text{Mass states}$$

PMNS mixing matrix

$\Longrightarrow$ neutrino oscillations



Massive neutrinos (2)

- Parametrization of the PMNS matrix (no CP violating phase)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} \\ 0 & 1 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}$$

- Mixing angles

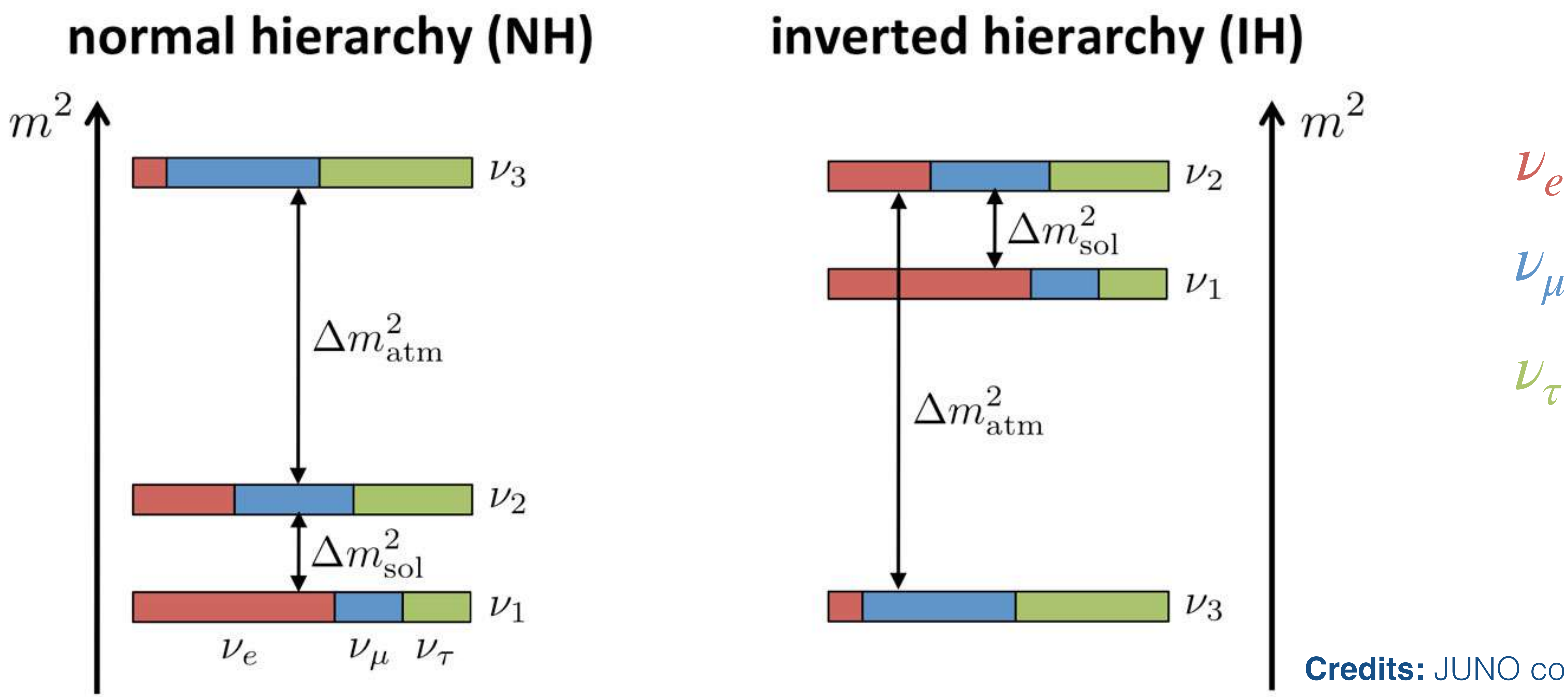
$$\sin^2\theta_{12} \simeq 0.307 \quad , \quad \sin^2\theta_{23} \simeq 0.545 \quad , \quad \sin^2\theta_{13} \simeq 0.0218$$

[Particle Data Group (2020)]

Massive neutrinos (3)

- Neutrino mass hierarchy

PLANCK $\boxed{\sum m_\nu < 0.12 \text{ eV}}$



Credits: JUNO collaboration

$$\Delta m^2_{21} = m^2_{\nu_2} - m^2_{\nu_1} = 7.53 \times 10^{-5} \text{ eV}^2$$

$$\Delta m^2_{32} = m^2_{\nu_3} - m^2_{\nu_2} = \pm\, 2.45 \times 10^{-3} \text{ eV}^2$$

Massive neutrinos (4)

- Flavor mixing → the distribution functions f_{ν_α} are not sufficient to describe the neutrino ensemble

$$\begin{pmatrix} f_{\nu_e} & & \\ & f_{\nu_\mu} & \\ & & f_{\nu_\tau} \end{pmatrix} \longrightarrow \begin{pmatrix} \langle \hat{a}^\dagger_{\nu_e} \hat{a}_{\nu_e} \rangle & \langle \hat{a}^\dagger_{\nu_\mu} \hat{a}_{\nu_e} \rangle & \langle \hat{a}^\dagger_{\nu_\tau} \hat{a}_{\nu_e} \rangle \\ \langle \hat{a}^\dagger_{\nu_e} \hat{a}_{\nu_\mu} \rangle & \langle \hat{a}^\dagger_{\nu_\mu} \hat{a}_{\nu_\mu} \rangle & \langle \hat{a}^\dagger_{\nu_\tau} \hat{a}_{\nu_\mu} \rangle \\ \langle \hat{a}^\dagger_{\nu_\tau} \hat{a}_{\nu_e} \rangle & \langle \hat{a}^\dagger_{\nu_\mu} \hat{a}_{\nu_\tau} \rangle & \langle \hat{a}^\dagger_{\nu_\tau} \hat{a}_{\nu_\tau} \rangle \end{pmatrix}$$

$\Longrightarrow$ Density matrix description

Which evolution equation? → generalization of Boltzmann equation

Extended BBGKY formalism

- Central object: *s*-body reduced density matrix

$$\varrho^{i_1 \cdots i_s}_{j_1 \cdots j_s} \equiv \langle \hat{a}^\dagger_{j_s} \cdots \hat{a}^\dagger_{j_1} \hat{a}_{i_1} \cdots \hat{a}_{i_s} \rangle$$

In particular, one-body density matrix $\varrho^i_j \equiv \langle \hat{a}^\dagger_j \hat{a}_i \rangle$

$$\left(\varrho^{\phi_i(\vec{p}_i, h_i)}_{\phi_j(\vec{p}_j, h_j)} = \langle \hat{a}^\dagger_{\phi_j}(\vec{p}_j, h_j)\, \hat{a}_{\phi_i}(\vec{p}_i, h_i) \rangle \right)$$

species, momentum, helicity

- Hamiltonian (second quantization)

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int} = \sum_{i,j} t^i_j \, \hat{a}^\dagger_i \hat{a}_j + \frac{1}{4} \sum_{i,j,k,l} \tilde{v}^{ik}_{jl} \, \hat{a}^\dagger_i \hat{a}^\dagger_k \hat{a}_l \hat{a}_j$$

Kinetic term

Two-body interactions

[C. Volpe et al., *Phys. Rev. D* 87, 113010 (2013)]

Extended BBGKY formalism

Ehrenfest theorem

$$\boxed{i\frac{\mathrm{d}\langle \hat{a}_j^\dagger \hat{a}_i\rangle}{\mathrm{d}t} = \langle [\hat{a}_j^\dagger \hat{a}_i, \hat{H}]\rangle}$$

- BBGKY hierarchy

$$\begin{cases} i\dfrac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left(t^i_k \varrho^k_j - \varrho^i_k t^k_j\right) + \dfrac{1}{2}\left(\tilde{v}^{ik}_{ml}\varrho^{ml}_{jk} - \varrho^{ik}_{ml}\tilde{v}^{ml}_{jk}\right) \\ i\dfrac{\mathrm{d}\varrho^{ik}_{jl}}{\mathrm{d}t} = \left(t^i_r \varrho^{rk}_{jl} + t^k_p \varrho^{ip}_{jl} + \dfrac{1}{2}\tilde{v}^{ik}_{rp}\varrho^{rp}_{jl} - \varrho^{ik}_{rl} t^r_j - \varrho^{ik}_{jp} t^p_l - \dfrac{1}{2}\varrho^{ik}_{rp}\tilde{v}^{rp}_{jl}\right) \\ \qquad\qquad + \dfrac{1}{2}\left(\tilde{v}^{im}_{rn}\varrho^{rkn}_{jlm} + \tilde{v}^{km}_{pn}\varrho^{ipn}_{jlm} - \varrho^{ikm}_{rln}\tilde{v}^{rn}_{jm} - \varrho^{ikm}_{jpn}\tilde{v}^{pn}_{lm}\right) \end{cases}$$

1-body density matrix   2-body density matrix   3-body density matrix

Need to truncate this hierarchy $\Longrightarrow$ Hartree-Fock (mean-field),…

Extended BBGKY formalism

- Correlated and uncorrelated contributions

$$\varrho^{ik}_{jl} \equiv 2\varrho^{i}_{[j}\varrho^{k}_{l]} + C^{ik}_{jl} \equiv \varrho^{i}_{j}\varrho^{k}_{l} - \varrho^{i}_{l}\varrho^{k}_{j} + C^{ik}_{jl}$$

$$i\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left(t^{i}_{k}\varrho^{k}_{j} - \varrho^{i}_{k}t^{k}_{j}\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml}\varrho^{ml}_{jk} - \varrho^{ik}_{ml}\tilde{v}^{ml}_{jk}\right)$$

$$\Longrightarrow\ i\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left(\left[t^{i}_{k} + \Gamma^{i}_{k}\right]\varrho^{k}_{j} - \varrho^{i}_{k}\left[t^{k}_{j} + \Gamma^{k}_{j}\right]\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml}C^{ml}_{jk} - C^{ik}_{ml}\tilde{v}^{ml}_{jk}\right)$$

Extended BBGKY formalism

- Correlated and uncorrelated contributions

$$\varrho^{ik}_{jl} \equiv 2\varrho^{i}_{[j}\varrho^{k}_{l]} + C^{ik}_{jl} \equiv \varrho^{i}_{j}\varrho^{k}_{l} - \varrho^{i}_{l}\varrho^{k}_{j} + C^{ik}_{jl}$$

$$i\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left(t^{i}_{k}\varrho^{k}_{j} - \varrho^{i}_{k}t^{k}_{j}\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml}\varrho^{ml}_{jk} - \varrho^{ik}_{ml}\tilde{v}^{ml}_{jk}\right)$$

$$\Longrightarrow \; i\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left(\left[t^{i}_{k} + \Gamma^{i}_{k}\right]\varrho^{k}_{j} - \varrho^{i}_{k}\left[t^{k}_{j} + \Gamma^{k}_{j}\right]\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml}C^{ml}_{jk} - C^{ik}_{ml}\tilde{v}^{ml}_{jk}\right)$$

Mean-field potential $\boxed{\Gamma^{i}_{j} = \sum_{k,l}\tilde{v}^{ik}_{jl}\,\varrho^{l}_{k}}$

Extended BBGKY formalism

- Correlated and uncorrelated contributions

$$\varrho^{ik}_{jl} \equiv 2\varrho^{i}_{[j}\varrho^{k}_{l]} + C^{ik}_{jl} \equiv \varrho^{i}_{j}\varrho^{k}_{l} - \varrho^{i}_{l}\varrho^{k}_{j} + \cancel{C^{ik}_{jl}}$$

$$i\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left(t^{i}_{k}\varrho^{k}_{j} - \varrho^{i}_{k}t^{k}_{j}\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml}\varrho^{ml}_{jk} - \varrho^{ik}_{ml}\tilde{v}^{ml}_{jk}\right)$$

$$\Longrightarrow \quad i\frac{\mathrm{d}\varrho^{i}_{j}}{\mathrm{d}t} = \left(\left[t^{i}_{k} + {\color{red}\Gamma^{i}_{k}}\right]\varrho^{k}_{j} - \varrho^{i}_{k}\left[t^{k}_{j} + {\color{red}\Gamma^{k}_{j}}\right]\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml}\cancel{C^{ml}_{jk}} - \cancel{C^{ik}_{ml}}\tilde{v}^{ml}_{jk}\right)$$

Mean-field potential $\boxed{\Gamma^{i}_{j} = \sum_{k,l}\tilde{v}^{ik}_{jl}\,\varrho^{l}_{k}}$

- Simplest closure: Hartree-Fock (or *mean-field*) approximation

but need to account for correlations due to two-body collisions…

Extended BBGKY formalism

- *Molecular chaos* assumption = correlations are built from collisions between uncorrelated particles

$$
\begin{aligned}
i\frac{\mathrm{d}C^{ik}_{jl}}{\mathrm{d}t} = {} & \left[t^i_r C^{rk}_{jl} + t^k_p C^{ip}_{jl} - C^{ik}_{rl} t^r_j - C^{ik}_{jp} t^p_l\right] \\
& + (\hat{1}-\varrho)^i_r (\hat{1}-\varrho)^k_p \, \tilde{v}^{rp}_{sq} \, \varrho^s_j \varrho^q_l - \varrho^i_r \varrho^k_p \, \tilde{v}^{rp}_{sq} \, (\hat{1}-\varrho)^s_j (\hat{1}-\varrho)^q_l
\end{aligned}
$$

Pauli-blocking factors

Extended BBGKY formalism

- *Molecular chaos* assumption = correlations are built from collisions between uncorrelated particles

$$i\frac{\mathrm{d}C^{ik}_{jl}}{\mathrm{d}t} = \left[t^i_r C^{rk}_{jl} + t^k_p C^{ip}_{jl} - C^{ik}_{rl} t^r_j - C^{ik}_{jp} t^p_l\right] + (\hat{1}-\varrho)^i_r (\hat{1}-\varrho)^k_p \, \tilde{v}^{rp}_{sq} \, \varrho^s_j \varrho^q_l - \varrho^i_r \varrho^k_p \, \tilde{v}^{rp}_{sq} \, (\hat{1}-\varrho)^s_j (\hat{1}-\varrho)^q_l$$

Pauli-blocking factors

$$C^{ik}_{jl}(t) = \int_0^t (\cdots) \to \frac{1}{2}\int_{-\infty}^{+\infty} (\cdots)$$

Duration of one collision $\ll$ Time scale of evolution of ϱ

Extended BBGKY formalism

- *Molecular chaos* assumption = correlations are built from collisions between uncorrelated particles

$$i\frac{\mathrm{d}C^{ik}_{jl}}{\mathrm{d}t} = \left[t^i_r C^{rk}_{jl} + t^k_p C^{ip}_{jl} - C^{ik}_{rl} t^r_j - C^{ik}_{jp} t^p_l\right] + (\hat{1}-\varrho)^i_r (\hat{1}-\varrho)^k_p \, \tilde{v}^{rp}_{sq} \, \varrho^s_j \varrho^q_l - \varrho^i_r \varrho^k_p \, \tilde{v}^{rp}_{sq} \, (\hat{1}-\varrho)^s_j (\hat{1}-\varrho)^q_l$$

Pauli-blocking factors

$$C^{ik}_{jl}(t) = \int_0^t (\cdots) \to \frac{1}{2}\int_{-\infty}^{+\infty} (\cdots)$$

Duration of one collision $\ll$ Time scale of evolution of ϱ

- Evolution equation

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left(\left[t^i_k + \Gamma^i_k\right] \varrho^k_j - \varrho^i_k \left[t^k_j + \Gamma^k_j\right]\right) + \frac{1}{2}\left(\tilde{v}^{ik}_{ml} C^{ml}_{jk} - C^{ik}_{ml} \tilde{v}^{ml}_{jk}\right)$$

$$= \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^i_j + i\,\hat{C}^i_j$$

Quantum Kinetic Equation for neutrinos

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^i_j + i\,\hat{\mathcal{C}}^i_j$$

$$\mathcal{C}^{i_1}_{i'_1} = \frac{1}{4}\Big(\tilde{v}^{i_1 i_2}_{i_3 i_4}\,\varrho^{i_3}_{j_3}\,\varrho^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,(\hat{1}-\varrho)^{j_1}_{i'_1}\,(\hat{1}-\varrho)^{j_2}_{i_2} - \tilde{v}^{i_1 i_2}_{i_3 i_4}\,(\hat{1}-\varrho)^{i_3}_{j_3}\,(\hat{1}-\varrho)^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\varrho^{j_1}_{i'_1}\,\varrho^{j_2}_{i_2}$$
$$+(\hat{1}-\varrho)^{i_1}_{j_1}\,(\hat{1}-\varrho)^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,\varrho^{j_3}_{i_3}\,\varrho^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2} - \varrho^{i_1}_{j_1}\,\varrho^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,(\hat{1}-\varrho)^{j_3}_{i_3}\,(\hat{1}-\varrho)^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2}\Big)$$

Quantum Kinetic Equation for neutrinos

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^i_j + i\,\hat{\mathcal{C}}^i_j$$

$$\mathcal{C}^{i_1}_{i_1'} = \frac{1}{4}\Big(\tilde{v}^{i_1 i_2}_{i_3 i_4}\,\varrho^{i_3}_{j_3}\,\varrho^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,(\hat{1}-\varrho)^{j_1}_{i_1'}\,(\hat{1}-\varrho)^{j_2}_{i_2} - \tilde{v}^{i_1 i_2}_{i_3 i_4}\,(\hat{1}-\varrho)^{i_3}_{j_3}\,(\hat{1}-\varrho)^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\varrho^{j_1}_{i_1'}\,\varrho^{j_2}_{i_2}$$
$$+(\hat{1}-\varrho)^{i_1}_{j_1}\,(\hat{1}-\varrho)^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,\varrho^{j_3}_{i_3}\,\varrho^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i_1' i_2} - \varrho^{i_1}_{j_1}\,\varrho^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,(\hat{1}-\varrho)^{j_3}_{i_3}\,(\hat{1}-\varrho)^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i_1' i_2}\Big)$$

Gain

Loss

Quantum Kinetic Equation for neutrinos

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^i_j + i\,\hat{\mathcal{C}}^i_j$$

$$\mathcal{C}^{i_1}_{i'_1} = \frac{1}{4}\Big(\tilde{v}^{i_1 i_2}_{i_3 i_4}\,\varrho^{i_3}_{j_3}\,\varrho^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}(\hat{1}-\varrho)^{j_1}_{i'_1}(\hat{1}-\varrho)^{j_2}_{i_2} - \tilde{v}^{i_1 i_2}_{i_3 i_4}(\hat{1}-\varrho)^{i_3}_{j_3}(\hat{1}-\varrho)^{i_4}_{j_4}\,\tilde{v}^{j_3 j_4}_{j_1 j_2}\,\varrho^{j_1}_{i'_1}\,\varrho^{j_2}_{i_2}$$
$$+(\hat{1}-\varrho)^{i_1}_{j_1}(\hat{1}-\varrho)^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}\,\varrho^{j_3}_{i_3}\,\varrho^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2} - \varrho^{i_1}_{j_1}\,\varrho^{i_2}_{j_2}\,\tilde{v}^{j_1 j_2}_{j_3 j_4}(\hat{1}-\varrho)^{j_3}_{i_3}(\hat{1}-\varrho)^{j_4}_{i_4}\,\tilde{v}^{i_3 i_4}_{i'_1 i_2}\Big)$$

Gain **Loss**

- Neutrinos in the early universe (homogeneous, isotropic)

$$\langle \hat{a}^\dagger_{\nu_\beta}(\vec{p}\,', h')\hat{a}_{\nu_\alpha}(\vec{p}, h)\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}\,')\delta_{hh'}\, \varrho^\alpha_\beta(p,t)\, \delta_{h-}$$
$$\langle \hat{b}^\dagger_{\nu_\alpha}(\vec{p}, h)\hat{b}_{\nu_\beta}(\vec{p}\,', h')\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}\,')\delta_{hh'}\, \bar{\varrho}^\alpha_\beta(p,t)\, \delta_{h+}$$

Quantum Kinetic Equation for neutrinos

$$i\frac{\mathrm{d}\varrho^i_j}{\mathrm{d}t} = \left[\hat{t} + \hat{\Gamma}, \hat{\varrho}\right]^i_j + i\,\hat{\mathcal{C}}^i_j$$

$$\mathcal{C}^{i_1}_{i'_1} = \frac{1}{4}\Big(\underbrace{\tilde{v}^{i_1 i_2}_{i_3 i_4}\varrho^{i_3}_{j_3}\varrho^{i_4}_{j_4}\tilde{v}^{j_3 j_4}_{j_1 j_2}(\hat{1}-\varrho)^{j_1}_{i'_1}(\hat{1}-\varrho)^{j_2}_{i_2} + (\hat{1}-\varrho)^{i_1}_{j_1}(\hat{1}-\varrho)^{i_2}_{j_2}\tilde{v}^{j_1 j_2}_{j_3 j_4}\varrho^{j_3}_{i_3}\varrho^{j_4}_{i_4}\tilde{v}^{i_3 i_4}_{i'_1 i_2}}_{\text{Gain}}\; \underbrace{-\, \tilde{v}^{i_1 i_2}_{i_3 i_4}(\hat{1}-\varrho)^{i_3}_{j_3}(\hat{1}-\varrho)^{i_4}_{j_4}\tilde{v}^{j_3 j_4}_{j_1 j_2}\varrho^{j_1}_{i'_1}\varrho^{j_2}_{i_2} - \varrho^{i_1}_{j_1}\varrho^{i_2}_{j_2}\tilde{v}^{j_1 j_2}_{j_3 j_4}(\hat{1}-\varrho)^{j_3}_{i_3}(\hat{1}-\varrho)^{j_4}_{i_4}\tilde{v}^{i_3 i_4}_{i'_1 i_2}}_{\text{Loss}}\Big)$$

- Neutrinos in the early universe (homogeneous, isotropic)

$$\langle \hat{a}^\dagger_{\nu_\beta}(\vec{p}\,', h')\hat{a}_{\nu_\alpha}(\vec{p}, h)\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}\,')\delta_{hh'}\,\boxed{\varrho^\alpha_\beta(p,t)}\,\delta_{h-}$$

$$\langle \hat{b}^\dagger_{\nu_\alpha}(\vec{p}, h)\hat{b}_{\nu_\beta}(\vec{p}\,', h')\rangle = (2\pi)^3\, 2E_p\, \delta^{(3)}(\vec{p}-\vec{p}\,')\delta_{hh'}\,\boxed{\bar{\varrho}^\alpha_\beta(p,t)}\,\delta_{h+}$$

$$\begin{pmatrix} \varrho^e_e & \varrho^e_\mu & \varrho^e_\tau \\ \varrho^\mu_e & \varrho^\mu_\mu & \varrho^\mu_\tau \\ \varrho^\tau_e & \varrho^\tau_\mu & \varrho^\tau_\tau \end{pmatrix} = \begin{pmatrix} f_{\nu_e} & \varrho^e_\mu & \varrho^e_\tau \\ \varrho^\mu_e & f_{\nu_\mu} & \varrho^\mu_\tau \\ \varrho^\tau_e & \varrho^\tau_\mu & f_{\nu_\tau} \end{pmatrix}$$

Quantum Kinetic Equation for neutrinos

- Example of interaction matrix element ($\nu - e^-$ scattering)

$$\tilde{v}^{\nu_\alpha(1)e(2)}_{\nu_\beta(3)e(4)} = 2\sqrt{2}G_F\,(2\pi)^3\delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)$$
$$\times\,[\bar{u}^{h_1}_{\nu_\alpha}(\vec{p}_1)\gamma^\mu P_L u^{h_3}_{\nu_\beta}(\vec{p}_3)]\;[\bar{u}^{h_2}_e(\vec{p}_2)\gamma_\mu(G^{\alpha\beta}_L P_L + G^{\alpha\beta}_R P_R)u^{h_4}_e(\vec{p}_4)]$$

$$G^L = \begin{pmatrix} g_L + 1 & 0 & 0 \\ 0 & g_L & 0 \\ 0 & 0 & g_L \end{pmatrix} \quad G^R = \begin{pmatrix} g_R & 0 & 0 \\ 0 & g_R & 0 \\ 0 & 0 & g_R \end{pmatrix} \qquad \begin{aligned} g_L &= -\frac{1}{2} + \sin^2\theta_W \\ g_R &= \sin^2\theta_W \end{aligned}$$

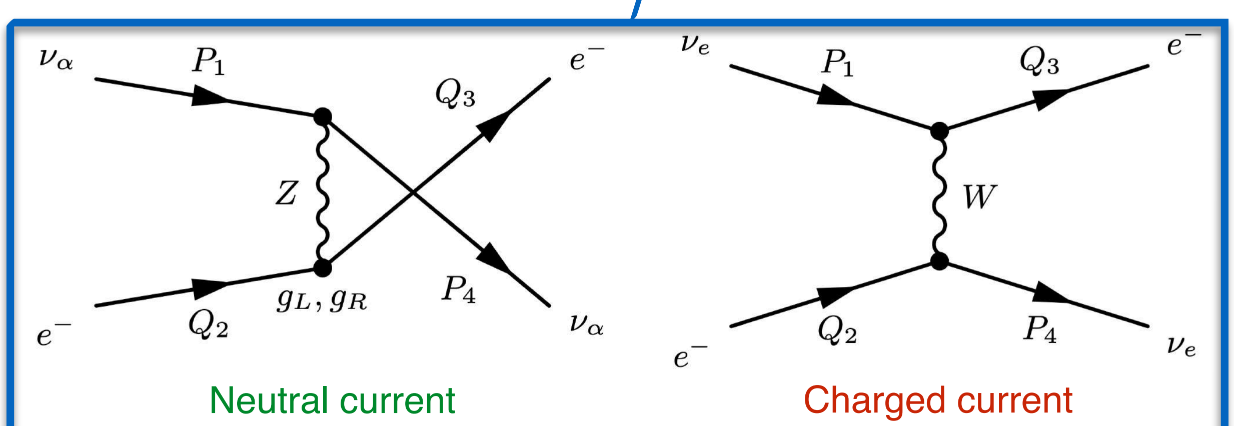
Quantum Kinetic Equation for neutrinos

- Example of interaction matrix element ($\nu - e^-$ scattering)

$$\tilde{v}^{\nu_\alpha(1)e(2)}_{\nu_\beta(3)e(4)} = 2\sqrt{2}G_F\,(2\pi)^3\delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$
$$\times\,[\bar{u}^{h_1}_{\nu_\alpha}(\vec{p}_1)\gamma^\mu P_L u^{h_3}_{\nu_\beta}(\vec{p}_3)]\;[\bar{u}^{h_2}_e(\vec{p}_2)\gamma_\mu(G^{\alpha\beta}_L P_L + G^{\alpha\beta}_R P_R)u^{h_4}_e(\vec{p}_4)]$$

$$G^L = \begin{pmatrix} g_L+1 & 0 & 0 \\ 0 & g_L & 0 \\ 0 & 0 & g_L \end{pmatrix} \quad G^R = \begin{pmatrix} g_R & 0 & 0 \\ 0 & g_R & 0 \\ 0 & 0 & g_R \end{pmatrix}$$

$$g_L = -\frac{1}{2} + \sin^2\theta_W$$
$$g_R = \sin^2\theta_W$$



Quantum Kinetic Equations

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_e + \mathbb{P}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

Vacuum Mean-field Collisions

$$\sqrt{p^2 + m_{\nu_i}^2} \simeq p + \frac{m_{\nu_i}^2}{2p}$$

$$\mathbb{E}_e + \mathbb{P}_e = \begin{pmatrix} \rho_e + P_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Reminder:

$$\varrho = \begin{pmatrix} \varrho_e^e & \varrho_\mu^e & \varrho_\tau^e \\ \varrho_e^\mu & \varrho_\mu^\mu & \varrho_\tau^\mu \\ \varrho_e^\tau & \varrho_\mu^\tau & \varrho_\tau^\tau \end{pmatrix} = \begin{pmatrix} f_{\nu_e} & \varrho_\mu^e & \varrho_\tau^e \\ \varrho_e^\mu & f_{\nu_\mu} & \varrho_\tau^\mu \\ \varrho_e^\tau & \varrho_\mu^\tau & f_{\nu_\tau} \end{pmatrix}$$

[G. Sigl, G. Raffelt, *Nucl. Phys. B* 406, 423 (1993)]
[C. Volpe et al., *Phys. Rev. D* 87, 113010 (2013)]
[D. Blaschke, V. Cirigliano, *Phys. Rev. D* 94, 033009 (2016)]
[**JF**, C. Pitrou, M.C. Volpe, 2008.01074]

Quantum Kinetic Equations

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_e + \mathbb{P}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

Vacuum Mean-field Collisions

$$\boxed{\mathcal{C} = \mathcal{C}^{[\nu e^- \to \nu e^-]} + \mathcal{C}^{[\nu e^+ \to \nu e^+]} + \mathcal{C}^{[\nu\bar{\nu} \to e^- e^+]} + \mathcal{C}^{[\nu\nu]}}$$

$$\begin{aligned}
\mathcal{C}^{[\nu e^- \to \nu e^-]} =& \frac{1}{2}\frac{2^5 G_F^2}{2E_1}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4} \\
&\times (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \\
&\times \Big[4(p_1\cdot p_2)(p_3\cdot p_4)F_{\mathrm{sc}}^{LL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\
&\quad + 4(p_1\cdot p_4)(p_2\cdot p_3)F_{\mathrm{sc}}^{RR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\
&\quad - 2(p_1\cdot p_3)m_e^2\Big(F_{\mathrm{sc}}^{LR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) + F_{\mathrm{sc}}^{RL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)})\Big)\Big]
\end{aligned}$$

Quantum Kinetic Equations

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_e + \mathbb{P}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

Vacuum Mean-field Collisions

$$\boxed{\mathcal{C} = \mathcal{C}^{[\nu e^- \to \nu e^-]} + \mathcal{C}^{[\nu e^+ \to \nu e^+]} + \mathcal{C}^{[\nu\bar{\nu} \to e^- e^+]} + \mathcal{C}^{[\nu\nu]}}$$

$$\begin{aligned}\mathcal{C}^{[\nu e^- \to \nu e^-]} =& \frac{1}{2}\frac{2^5 G_F^2}{2E_1}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4} \\ &\times (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \\ &\times \Big[4(p_1\cdot p_2)(p_3\cdot p_4)F_{\mathrm{sc}}^{LL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\ &\quad + 4(p_1\cdot p_4)(p_2\cdot p_3)F_{\mathrm{sc}}^{RR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\ &\quad - 2(p_1\cdot p_3)m_e^2\Big(F_{\mathrm{sc}}^{LR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) + F_{\mathrm{sc}}^{RL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)})\Big)\Big]\end{aligned}$$

<u>Statistical factor</u>

$$F_{\mathrm{sc}}^{AB}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) = \underbrace{f_4(1-f_2)\left[G^A\varrho_3 G^B(1-\varrho_1)\right]}_{\text{“gain”}} - \underbrace{(1-f_4)f_2\left[G^A(1-\varrho_3)G^B\varrho_1\right]}_{\text{“loss”}} + \text{h.c.}$$

Quantum Kinetic Equations

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_e + \mathbb{P}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

Vacuum Mean-field Collisions

$$\boxed{\mathcal{C} = \mathcal{C}^{[\nu e^- \to \nu e^-]} + \mathcal{C}^{[\nu e^+ \to \nu e^+]} + \mathcal{C}^{[\nu\bar{\nu} \to e^- e^+]} + \mathcal{C}^{[\nu\nu]}}$$

$$\begin{aligned}\mathcal{C}^{[\nu e^- \to \nu e^-]} =& \frac{1}{2}\frac{2^5 G_F^2}{2E_1}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4} \\ &\times (2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4) \\ &\times \Big[4(p_1\cdot p_2)(p_3\cdot p_4)F_{\mathrm{sc}}^{LL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\ &\quad + 4(p_1\cdot p_4)(p_2\cdot p_3)F_{\mathrm{sc}}^{RR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) \\ &\quad - 2(p_1\cdot p_3)m_e^2\Big(F_{\mathrm{sc}}^{LR}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) + F_{\mathrm{sc}}^{RL}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)})\Big)\Big]\end{aligned}$$

Statistical factor

Pauli-blocking

$$F_{\mathrm{sc}}^{AB}(\nu^{(1)} + e^{(2)} \to \nu^{(3)} + e^{(4)}) = f_4(1 - f_2)\left[G^A\varrho_3 G^B(1 - \varrho_1)\right] - (1 - f_4)f_2\left[G^A(1 - \varrho_3)G^B\varrho_1\right] + \text{h.c.}$$

"gain" "loss"

Quantum Kinetic Equations

(Anti)neutrino self-interactions

$$\mathcal{C}^{[\nu\nu]} = \frac{1}{2}\frac{2^5 G_F^2}{2E_1}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4}$$

$$\times (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\times\Big[(p_1\cdot p_2)(p_3\cdot p_4)F_{\mathrm{sc}}(\nu^{(1)}+\nu^{(2)}\to\nu^{(3)}+\nu^{(4)})$$

$$+(p_1\cdot p_4)(p_2\cdot p_3)\Big(F_{\mathrm{sc}}(\nu^{(1)}+\bar{\nu}^{(2)}\to\nu^{(3)}+\bar{\nu}^{(4)})+F_{\mathrm{ann}}(\nu^{(1)}+\bar{\nu}^{(2)}\to\nu^{(3)}+\bar{\nu}^{(4)})\Big)\Big]$$

$$F_{\mathrm{sc}}(\nu^{(1)}+\nu^{(2)}\to\nu^{(3)}+\nu^{(4)}) = [\varrho_4(1-\varrho_2)+\mathrm{Tr}(\cdots)]\,\varrho_3(1-\varrho_1) - [(1-\varrho_4)\varrho_2+\mathrm{Tr}(\cdots)]\,(1-\varrho_3)\varrho_1 + \mathrm{h.c.}$$

$$F_{\mathrm{sc}}(\nu^{(1)}+\bar{\nu}^{(2)}\to\nu^{(3)}+\bar{\nu}^{(4)}) = [(1-\bar{\varrho}_2)\bar{\varrho}_4+\mathrm{Tr}(\cdots)]\,\varrho_3(1-\varrho_1) - [\bar{\varrho}_2(1-\bar{\varrho}_4)+\mathrm{Tr}(\cdots)]\,(1-\varrho_3)\varrho_1 + \mathrm{h.c.}$$

$$F_{\mathrm{ann}}(\nu^{(1)}+\bar{\nu}^{(2)}\to\nu^{(3)}+\bar{\nu}^{(4)}) = [\varrho_3\bar{\varrho}_4+\mathrm{Tr}(\cdots)]\,(1-\bar{\varrho}_2)(1-\varrho_1) - [(1-\varrho_3)(1-\bar{\varrho}_4)+\mathrm{Tr}(\cdots)]\,\bar{\varrho}_2\varrho_1 + \mathrm{h.c.}$$

Quantum Kinetic Equations

(Anti)neutrino self-interactions

$$\mathcal{C}^{[\nu\nu]} = \frac{1}{2}\frac{2^5 G_F^2}{2E_1}\int \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 2E_4}$$

$$\times (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4) \times \Big[(p_1\cdot p_2)(p_3\cdot p_4) F_{\mathrm{sc}}(\nu^{(1)}+\nu^{(2)}\to\nu^{(3)}+\nu^{(4)})$$

$$+ (p_1\cdot p_4)(p_2\cdot p_3)\Big(F_{\mathrm{sc}}(\nu^{(1)}+\bar\nu^{(2)}\to\nu^{(3)}+\bar\nu^{(4)}) + F_{\mathrm{ann}}(\nu^{(1)}+\bar\nu^{(2)}\to\nu^{(3)}+\bar\nu^{(4)})\Big)\Big]$$

$$F_{\mathrm{sc}}(\nu^{(1)}+\nu^{(2)}\to\nu^{(3)}+\nu^{(4)}) = [\varrho_4(1-\varrho_2)+\mathrm{Tr}(\cdots)]\,\varrho_3(1-\varrho_1) - [(1-\varrho_4)\varrho_2+\mathrm{Tr}(\cdots)]\,(1-\varrho_3)\varrho_1 + \mathrm{h.c.}$$

$$F_{\mathrm{sc}}(\nu^{(1)}+\bar\nu^{(2)}\to\nu^{(3)}+\bar\nu^{(4)}) = [(1-\bar\varrho_2)\bar\varrho_4+\mathrm{Tr}(\cdots)]\,\varrho_3(1-\varrho_1) - [\bar\varrho_2(1-\bar\varrho_4)+\mathrm{Tr}(\cdots)]\,(1-\varrho_3)\varrho_1 + \mathrm{h.c.}$$

$$F_{\mathrm{ann}}(\nu^{(1)}+\bar\nu^{(2)}\to\nu^{(3)}+\bar\nu^{(4)}) = [\varrho_3\bar\varrho_4+\mathrm{Tr}(\cdots)]\,(1-\bar\varrho_2)(1-\varrho_1) - [(1-\varrho_3)(1-\bar\varrho_4)+\mathrm{Tr}(\cdots)]\,\bar\varrho_2\varrho_1 + \mathrm{h.c.}$$

9 dimensions $\longrightarrow$ 5 dimensions $\longrightarrow$ 2 dimensions

Outline

1. Neutrino evolution with mixing:

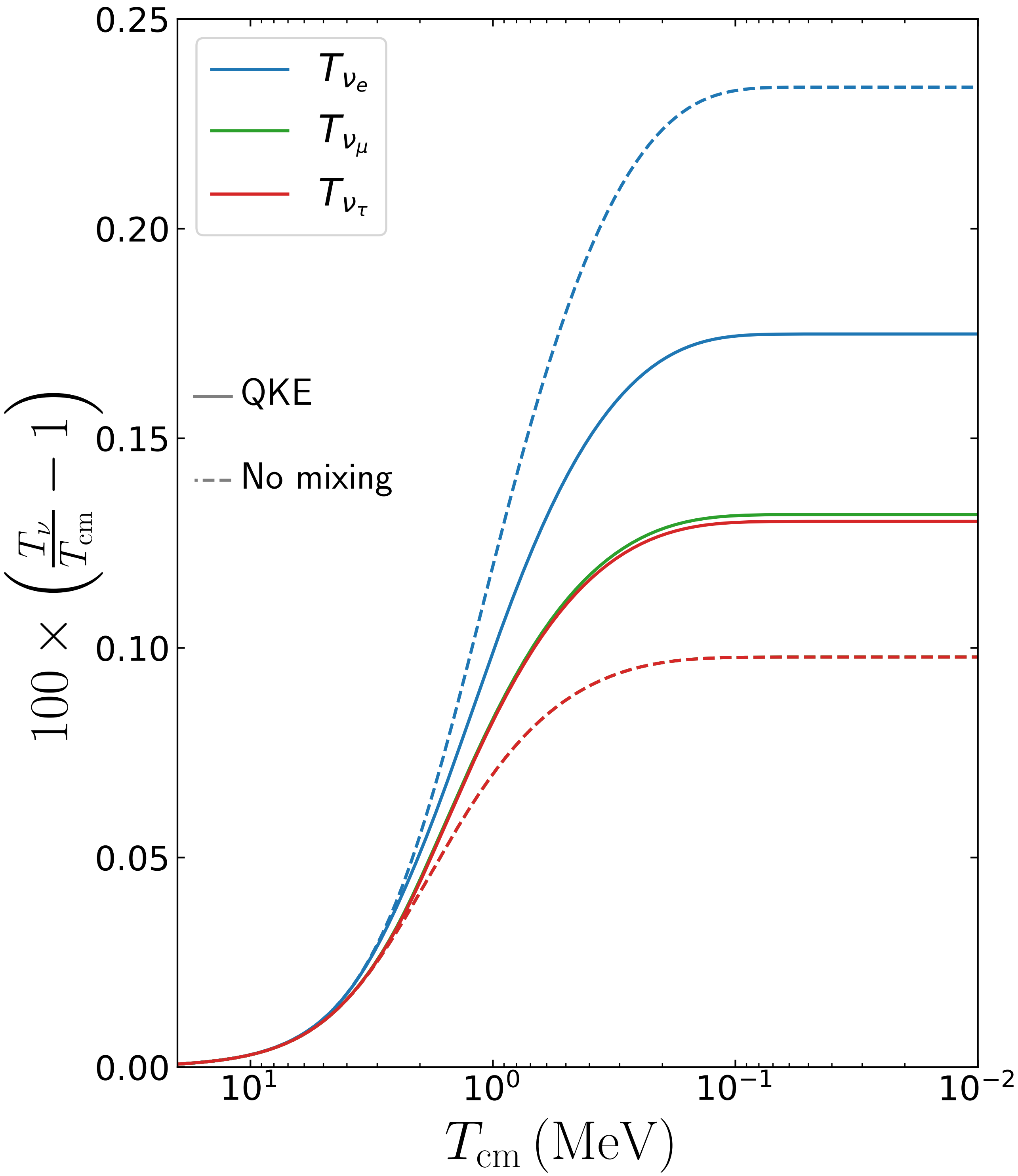
   Quantum Kinetic Equations

2. An approximation:

   Adiabatic Transfer of Averaged Oscillations

3. Results for neutrino decoupling

Approximation scheme for neutrino oscillations



"No visible oscillations"

$\Longrightarrow$ averaged oscillations?

$\Longrightarrow$ approximate scheme?

Approximation scheme for neutrino oscillations

- For simplicity, discard (for now) the mean-field term + two-neutrino mixing

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -i\left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] + \mathcal{C} \quad \Longleftrightarrow \quad \frac{\mathrm{d}\varrho_m}{\mathrm{d}t} = -i\left[\frac{\mathbb{M}^2}{2p}, \varrho_m\right] + U^\dagger \mathcal{C} U$$

$$\varrho_m \equiv U^\dagger \varrho U$$

Approximation scheme for neutrino oscillations

- For simplicity, discard (for now) the mean-field term + two-neutrino mixing

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -i\left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] + \mathcal{C} \quad \Longleftrightarrow \quad \boxed{\frac{\mathrm{d}\varrho_m}{\mathrm{d}t} = -i\left[\frac{\mathbb{M}^2}{2p}, \varrho_m\right]} + U^\dagger \mathcal{C} U$$

$$\varrho_m \equiv U^\dagger \varrho U$$

$$\varrho_m = \begin{pmatrix} f_1 & a\, e^{i\frac{\Delta m^2}{2p}t} \\ a\, e^{-i\frac{\Delta m^2}{2p}t} & f_2 \end{pmatrix}$$

Approximation scheme for neutrino oscillations

- For simplicity, discard (for now) the mean-field term + two-neutrino mixing

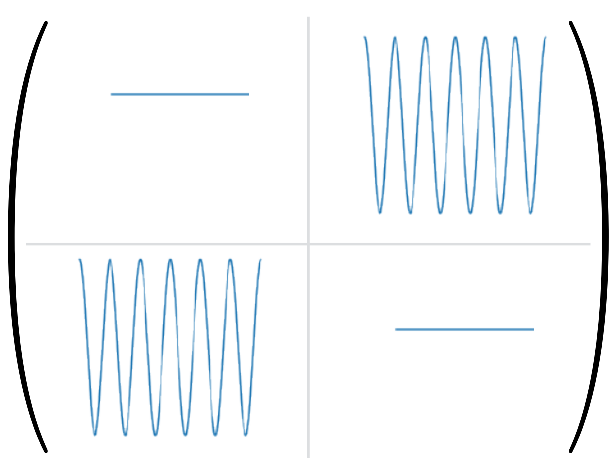
$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -i\left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] + \mathcal{C} \quad \Longleftrightarrow \quad \frac{\mathrm{d}\varrho_m}{\mathrm{d}t} = -i\left[\frac{\mathbb{M}^2}{2p}, \varrho_m\right] + U^\dagger \mathcal{C} U$$

$$\varrho_m \equiv U^\dagger \varrho U$$

$$\varrho_m = \begin{pmatrix} f_1 & a\, e^{i\frac{\Delta m^2}{2p}t} \\ a\, e^{-i\frac{\Delta m^2}{2p}t} & f_2 \end{pmatrix}$$

Schematically,

$\varrho_m =$



Localized neutrino injection
$(U^\dagger \mathcal{C} U \sim K \times \delta(0))$

Approximation scheme for neutrino oscillations

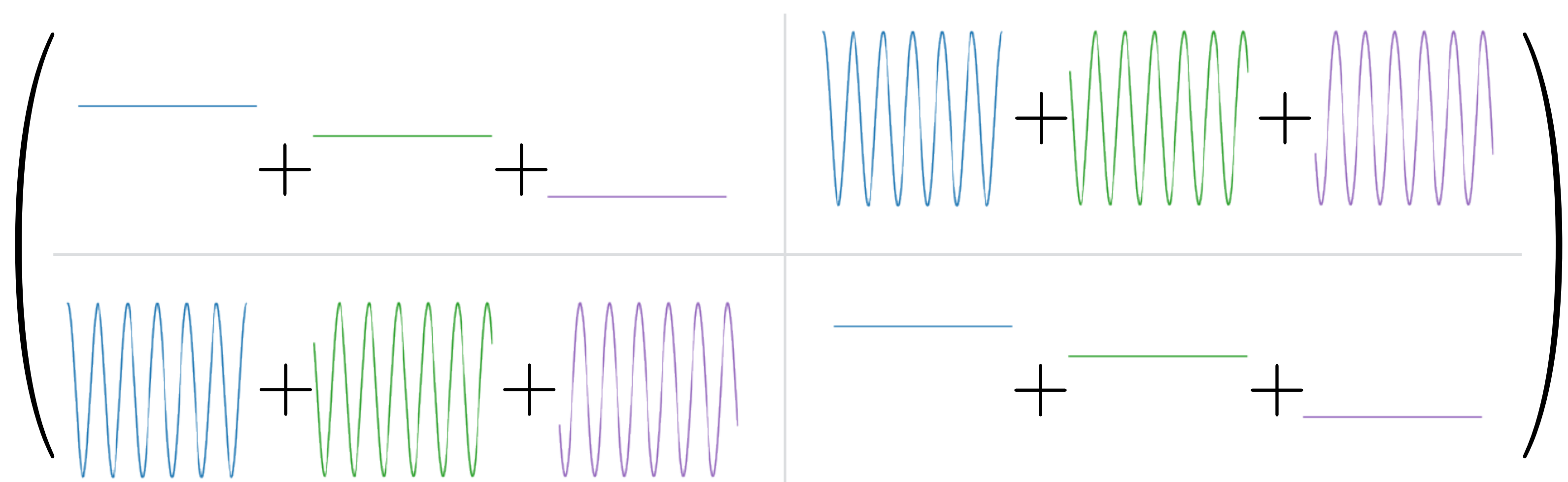
- For simplicity, discard (for now) the mean-field term + two-neutrino mixing

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -i\left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] + \mathcal{C} \quad \Longleftrightarrow \quad \boxed{\frac{\mathrm{d}\varrho_m}{\mathrm{d}t} = -i\left[\frac{\mathbb{M}^2}{2p}, \varrho_m\right]} + U^\dagger \mathcal{C} U$$

$$\varrho_m \equiv U^\dagger \varrho U$$

$$\varrho_m = \begin{pmatrix} f_1 & a\, e^{i\frac{\Delta m^2}{2p}t} \\ a\, e^{-i\frac{\Delta m^2}{2p}t} & f_2 \end{pmatrix}$$

Schematically,

$\varrho_m =$ 

Random neutrino injection

Approximation scheme for neutrino oscillations

- For simplicity, discard (for now) the mean-field term + two-neutrino mixing

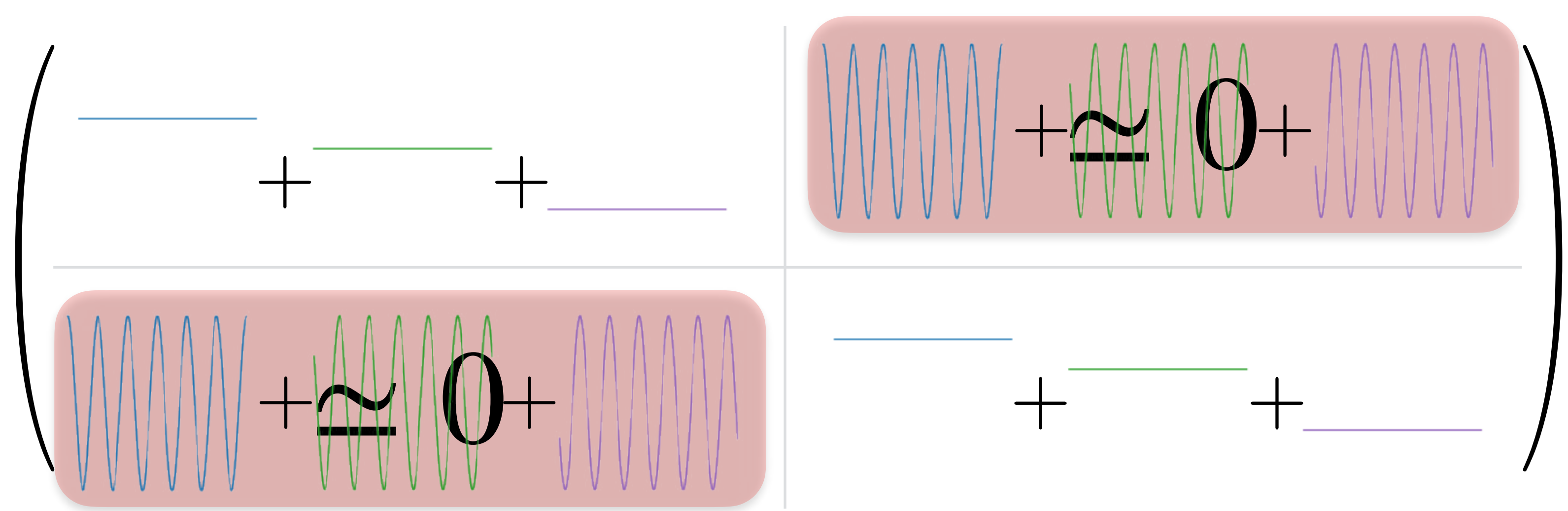
$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = -i\left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] + \mathcal{C} \quad \Longleftrightarrow \quad \boxed{\frac{\mathrm{d}\varrho_m}{\mathrm{d}t} = -i\left[\frac{\mathbb{M}^2}{2p}, \varrho_m\right]} + U^\dagger \mathcal{C} U$$

$$\varrho_m \equiv U^\dagger \varrho U$$

$$\varrho_m = \begin{pmatrix} f_1 & a\, e^{i\frac{\Delta m^2}{2p}t} \\ a\, e^{-i\frac{\Delta m^2}{2p}t} & f_2 \end{pmatrix}$$

Schematically,

$\varrho_m =$



Random neutrino injection

Approximation scheme for neutrino oscillations

- Generalization of the previous argument
  - ✦ Expansion
  - ✦ 3-neutrino mixing
  - ✦ Mean-field term

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_e + \mathbb{P}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

Approximation scheme for neutrino oscillations

- Generalization of the previous argument
  - ✦ Expansion — New variables $x = (m_e/T_{\rm cm}) \propto a$, $y = p/T_{\rm cm}$
  - ✦ 3-neutrino mixing — 3 oscillation frequencies
  - ✦ Mean-field term — Mass basis → *matter* basis

$$i\left[\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right]\varrho(p,t) = \left[U\frac{\mathbb{M}^2}{2p}U^\dagger, \varrho\right] - 2\sqrt{2}G_F p\left[\frac{\mathbb{E}_e + \mathbb{P}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

Approximation scheme for neutrino oscillations

- Generalization of the previous argument
  - ✦ Expansion — New variables $x = (m_e/T_{\rm cm}) \propto a$, $y = p/T_{\rm cm}$
  - ✦ 3-neutrino mixing — 3 oscillation frequencies
  - ✦ Mean-field term — Mass basis → *matter* basis

$$i\,xH\frac{\partial\varrho(x,y)}{\partial x} = \frac{x}{m_e}\left[U\frac{\mathbb{M}^2}{2y}U^\dagger,\varrho\right] - 2\sqrt{2}G_F y\left(\frac{m_e}{x}\right)^5\left[\frac{\bar{\mathbb{E}}_e+\bar{\mathbb{P}}_e}{m_W^2},\varrho\right] + i\,\mathcal{C}[\varrho,\bar\varrho]$$

$$\boxed{\frac{\partial\varrho}{\partial x} = -i[\mathcal{H},\varrho]+\mathcal{K}}$$

Approximation scheme for neutrino oscillations

- Generalization of the previous argument
    - ✦ Expansion — New variables $x = (m_e/T_{\rm cm}) \propto a$, $y = p/T_{\rm cm}$
    - ✦ 3-neutrino mixing — 3 oscillation frequencies
    - ✦ Mean-field term — Mass basis → *matter* basis

$$i\,xH\frac{\partial \varrho(x,y)}{\partial x} = \frac{x}{m_e}\left[U\frac{\mathbb{M}^2}{2y}U^\dagger, \varrho\right] - 2\sqrt{2}G_F y\left(\frac{m_e}{x}\right)^5\left[\frac{\bar{\mathbb{E}}_e + \bar{\mathbb{P}}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

$$\boxed{\frac{\partial \varrho}{\partial x} = -i[\mathcal{H}, \varrho] + \mathcal{K}}$$

$$\boxed{\frac{\partial \varrho_m}{\partial x} = -i[\mathcal{H}_m, \varrho_m] - \left[U_m^\dagger\frac{\partial U_m}{\partial x}, \varrho_m\right] + U_m^\dagger \mathcal{K} U_m}$$

$$\varrho_m \equiv U_m^\dagger \varrho U_m$$

Approximation scheme for neutrino oscillations

- Generalization of the previous argument
  - ✦ Expansion — New variables $x = (m_e/T_{\rm cm}) \propto a$, $y = p/T_{\rm cm}$
  - ✦ 3-neutrino mixing — 3 oscillation frequencies
  - ✦ Mean-field term — Mass basis → *matter* basis

$$i\, x H \frac{\partial \varrho(x,y)}{\partial x} = \frac{x}{m_e}\left[U \frac{\mathbb{M}^2}{2y} U^\dagger, \varrho\right] - 2\sqrt{2} G_F y \left(\frac{m_e}{x}\right)^5 \left[\frac{\bar{\mathbb{E}}_e + \bar{\mathbb{P}}_e}{m_W^2}, \varrho\right] + i\,\mathcal{C}[\varrho, \bar{\varrho}]$$

$$\boxed{\frac{\partial \varrho}{\partial x} = -i[\mathcal{H}, \varrho] + \mathcal{K}}$$

$$\boxed{\frac{\partial \varrho_m}{\partial x} = -i[\mathcal{H}_m, \varrho_m] - \cancel{\left[U_m^\dagger \frac{\partial U_m}{\partial x}, \varrho_m\right]} + U_m^\dagger \mathcal{K} U_m}$$

$$\varrho_m \equiv U_m^\dagger \varrho U_m$$

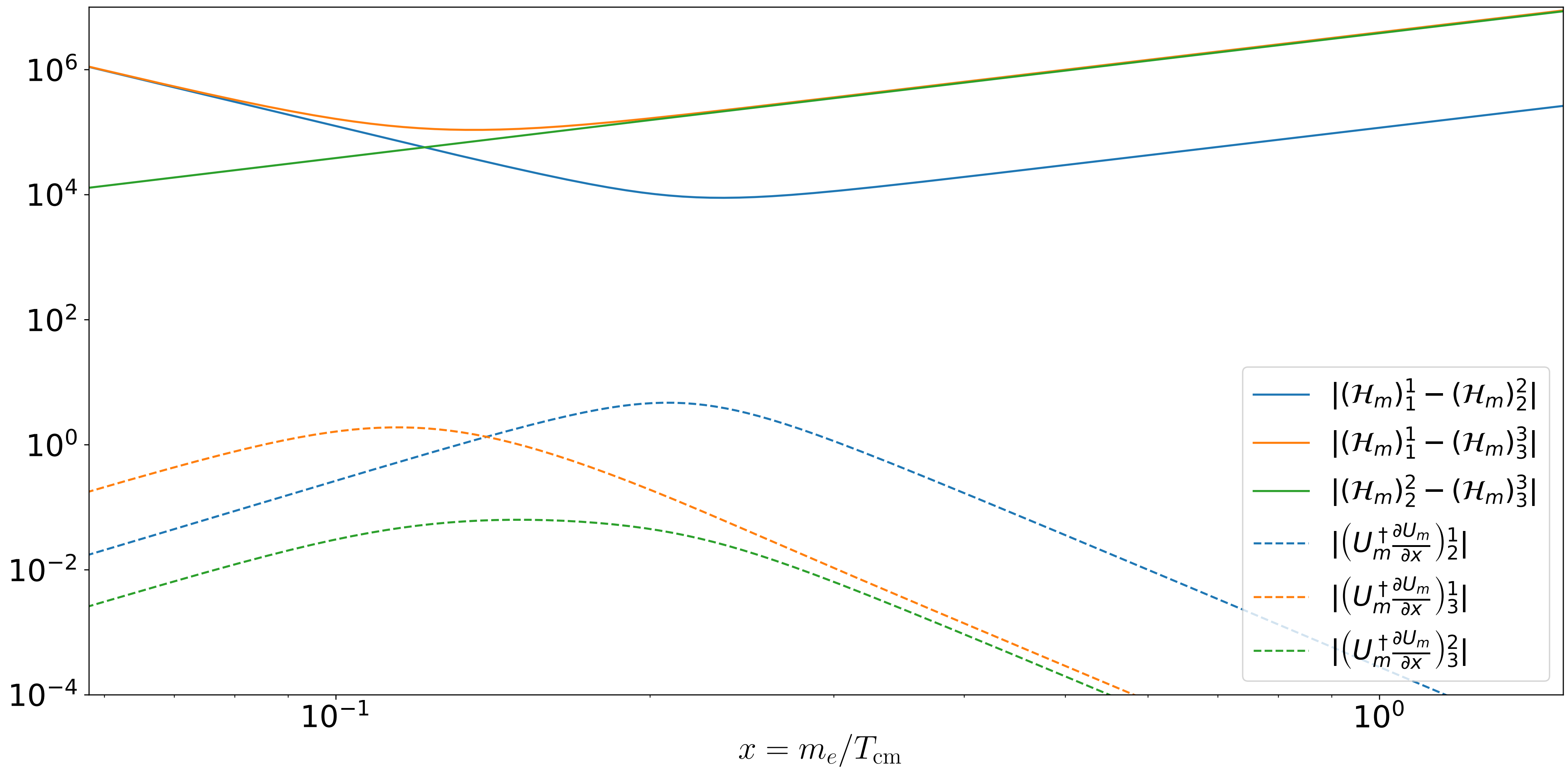
adiabatic approximation

diagonal

Checking the adiabatic approximation

$$\frac{\partial \varrho_m}{\partial x} = -i[\mathcal{H}_m, \varrho_m] - [U_m^\dagger \frac{\partial U_m}{\partial x}, \varrho_m] + \mathcal{K}_m$$

Effective oscillation frequencies



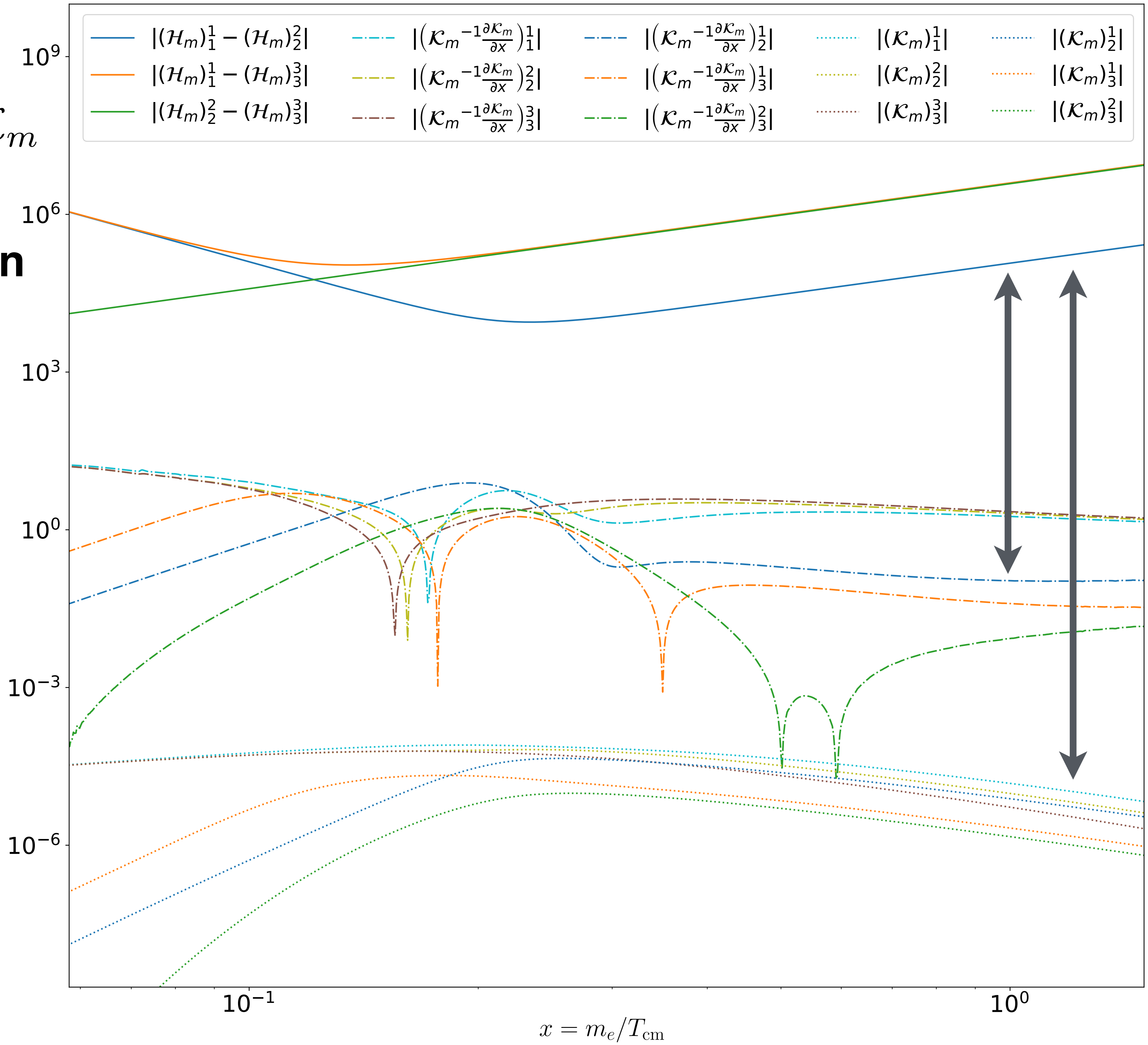
Checking that oscillations are averaged

$$\frac{\partial \varrho_m}{\partial x} = -i[\mathcal{H}_m, \varrho_m] + \mathcal{K}_m$$

Effective oscillation frequencies

Relative variation of collision term

Collision rate



Adiabatic Transfer of Averaged Oscillations

- Non-diagonal components of the density matrix in matter basis are *averaged out*

$$\varrho_m = \begin{pmatrix} * & \sim & \sim \\ \sim & * & \sim \\ \sim & \sim & * \end{pmatrix} \longrightarrow \tilde{\varrho}_m = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

- Effective "ATAO" equation

$$\boxed{\frac{\partial \tilde{\varrho}_m}{\partial x} = \widetilde{U_m^\dagger \mathcal{K} U_m}}$$

keep only the diagonal

Adiabatic Transfer of Averaged Oscillations

Instead of solving the full QKE, we can

1. Go to matter basis, where the effective Hamiltonian (vacuum + mean-field) is diagonal.

   This matter basis evolves *adiabatically*.

2. Evolve the diagonal components of ϱ_m (off-diagonal components are *averaged* out).

3. Read the results in flavor basis.

Outline

1. Neutrino evolution with mixing:

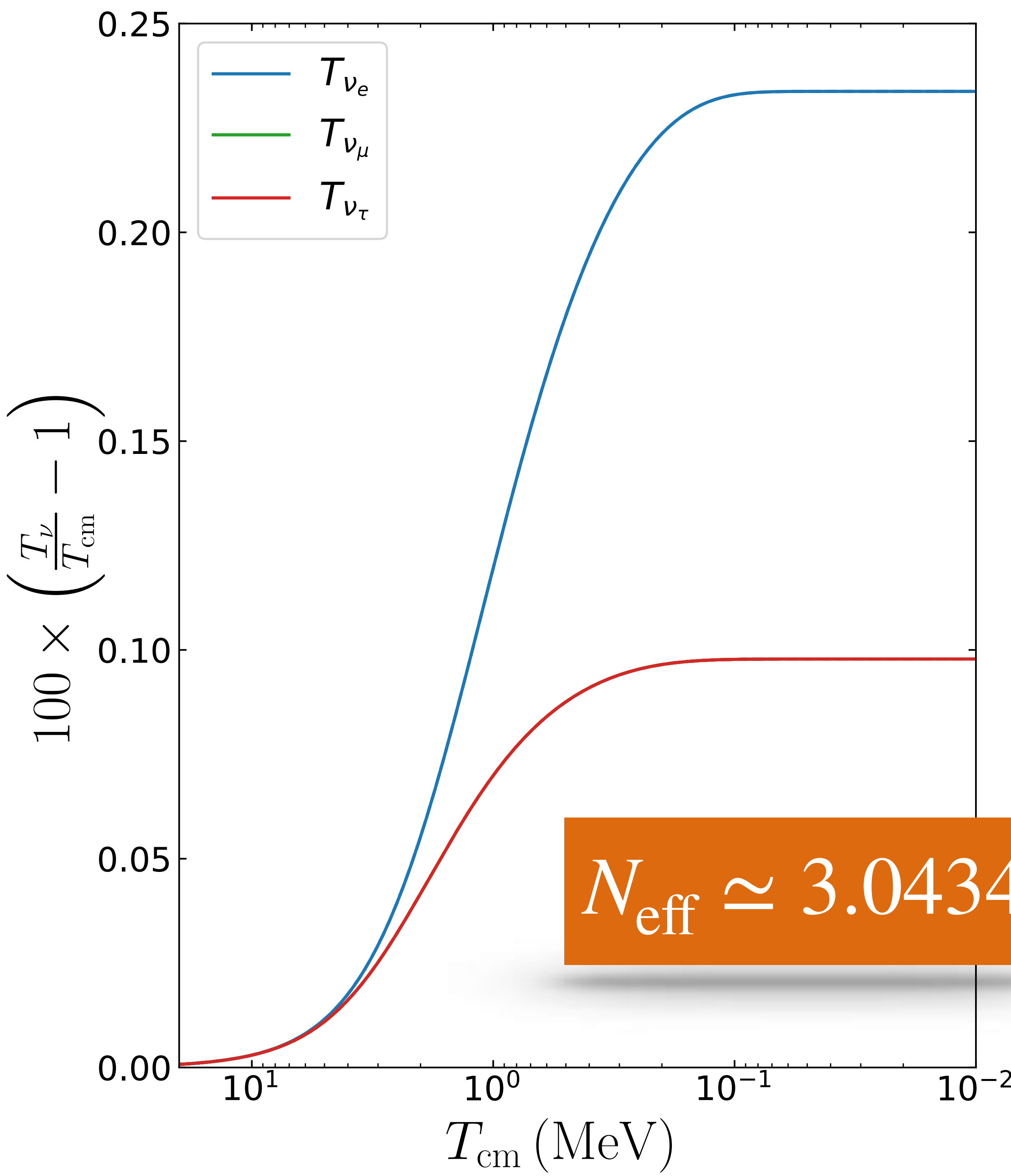
   Quantum Kinetic Equations

2. An approximation:

   Adiabatic Transfer of Averaged Oscillations

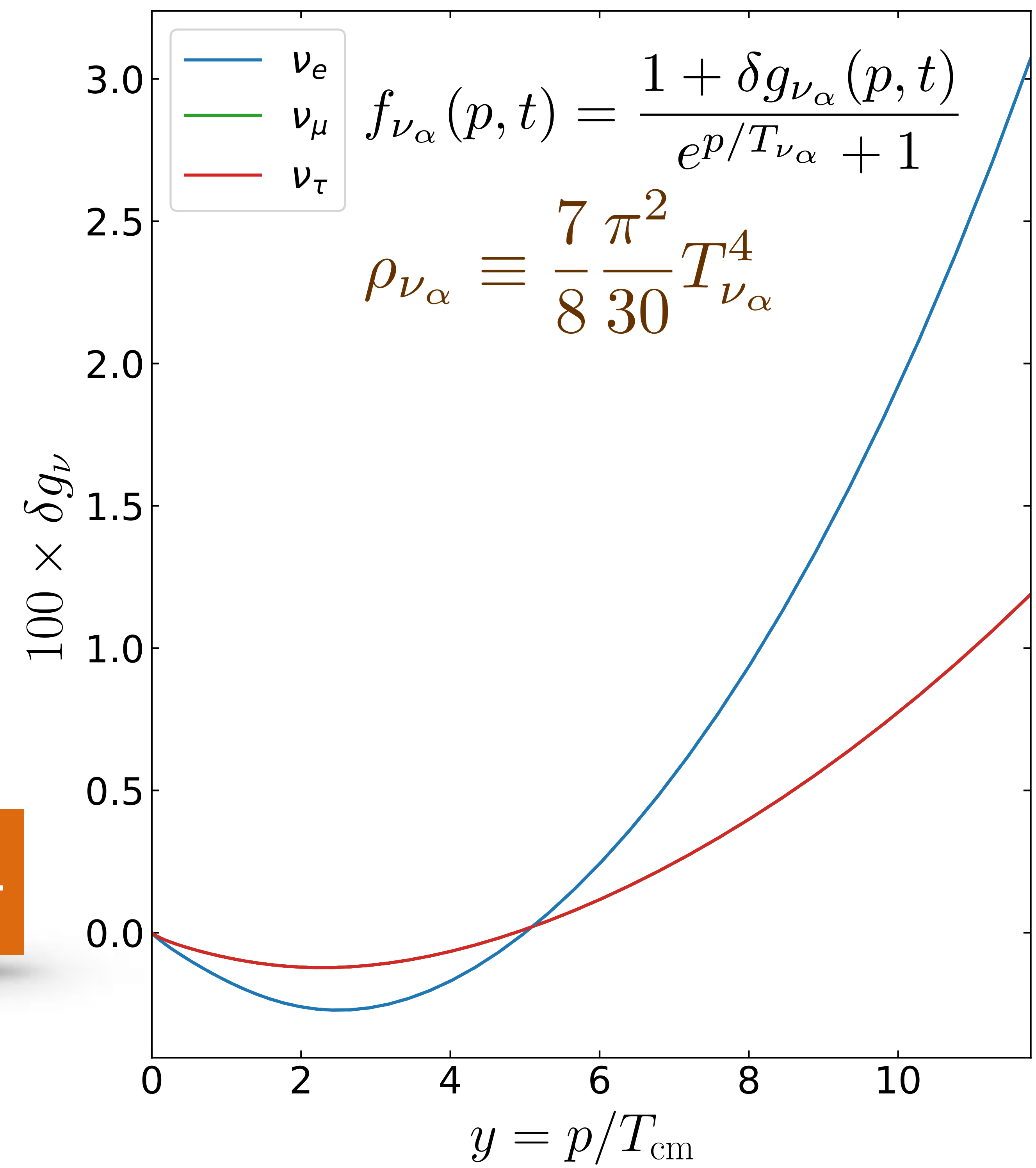
3. Results for neutrino decoupling

Neutrino decoupling without flavor oscillations



$N_{\rm eff} \simeq 3.0434$

Effective temperatures

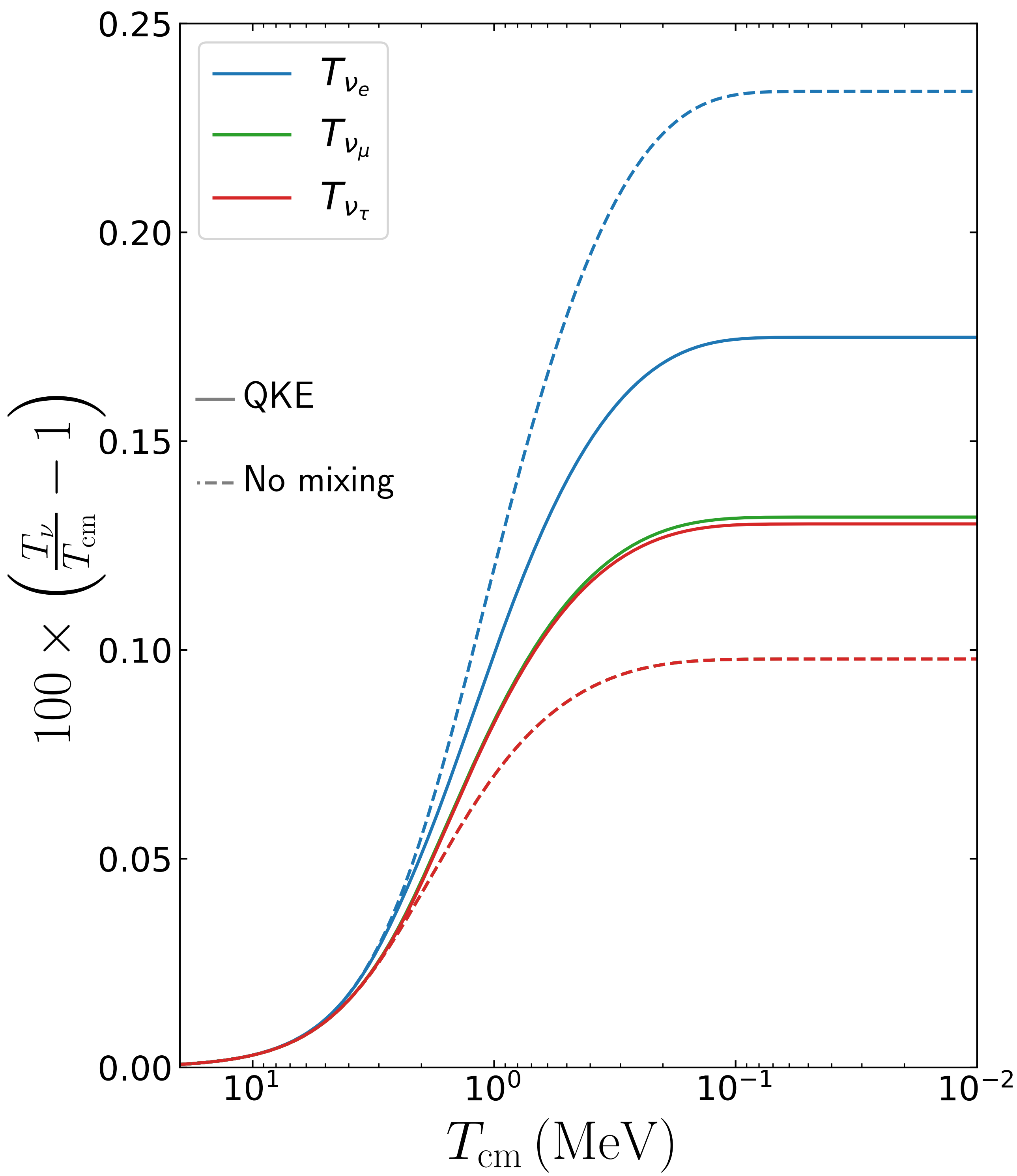


$$f_{\nu_\alpha}(p,t) = \frac{1 + \delta g_{\nu_\alpha}(p,t)}{e^{p/T_{\nu_\alpha}} + 1}$$

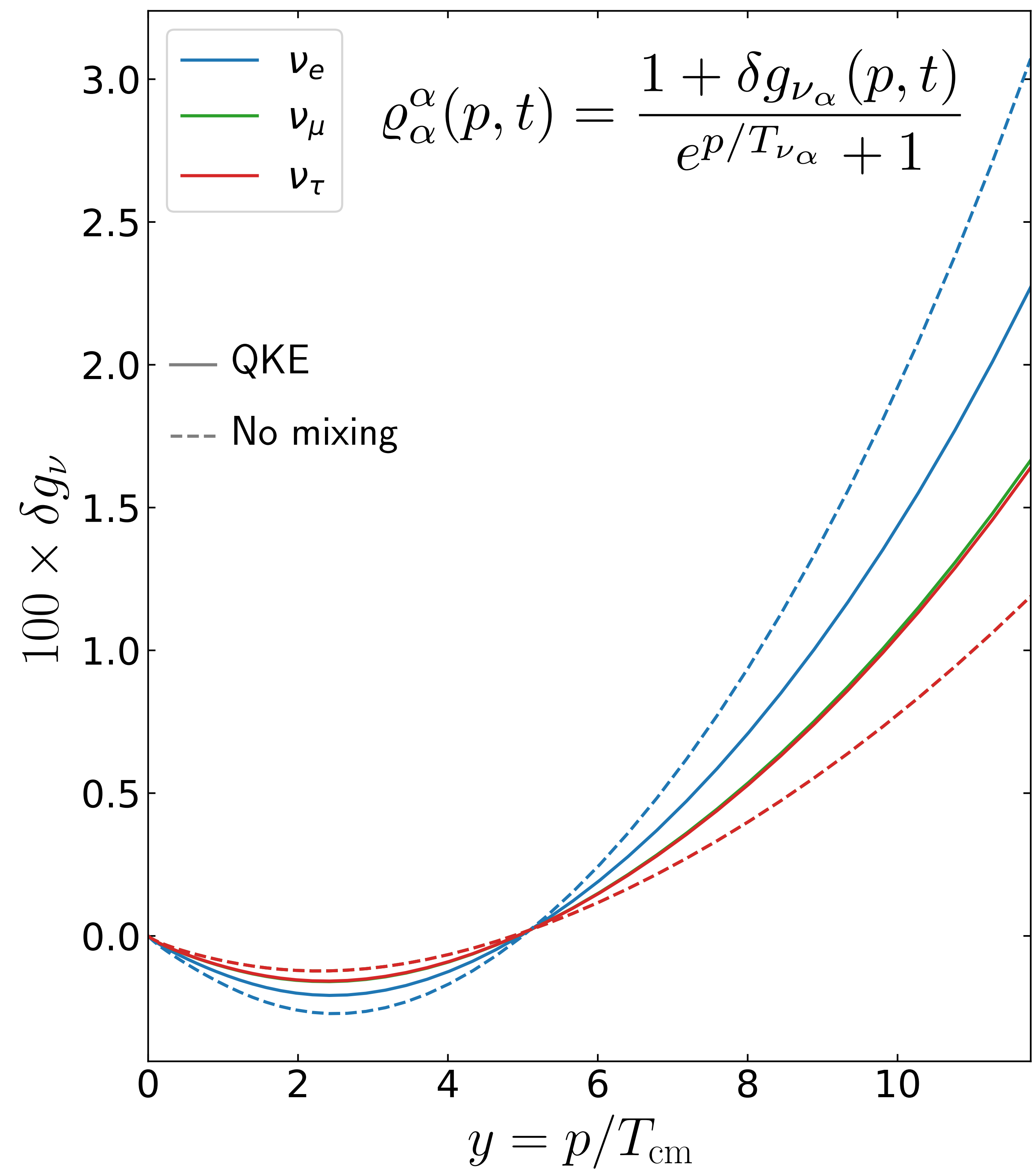
$$\rho_{\nu_\alpha} \equiv \frac{7}{8}\frac{\pi^2}{30}T_{\nu_\alpha}^4$$

Effective distortions

Neutrino decoupling with flavor oscillations

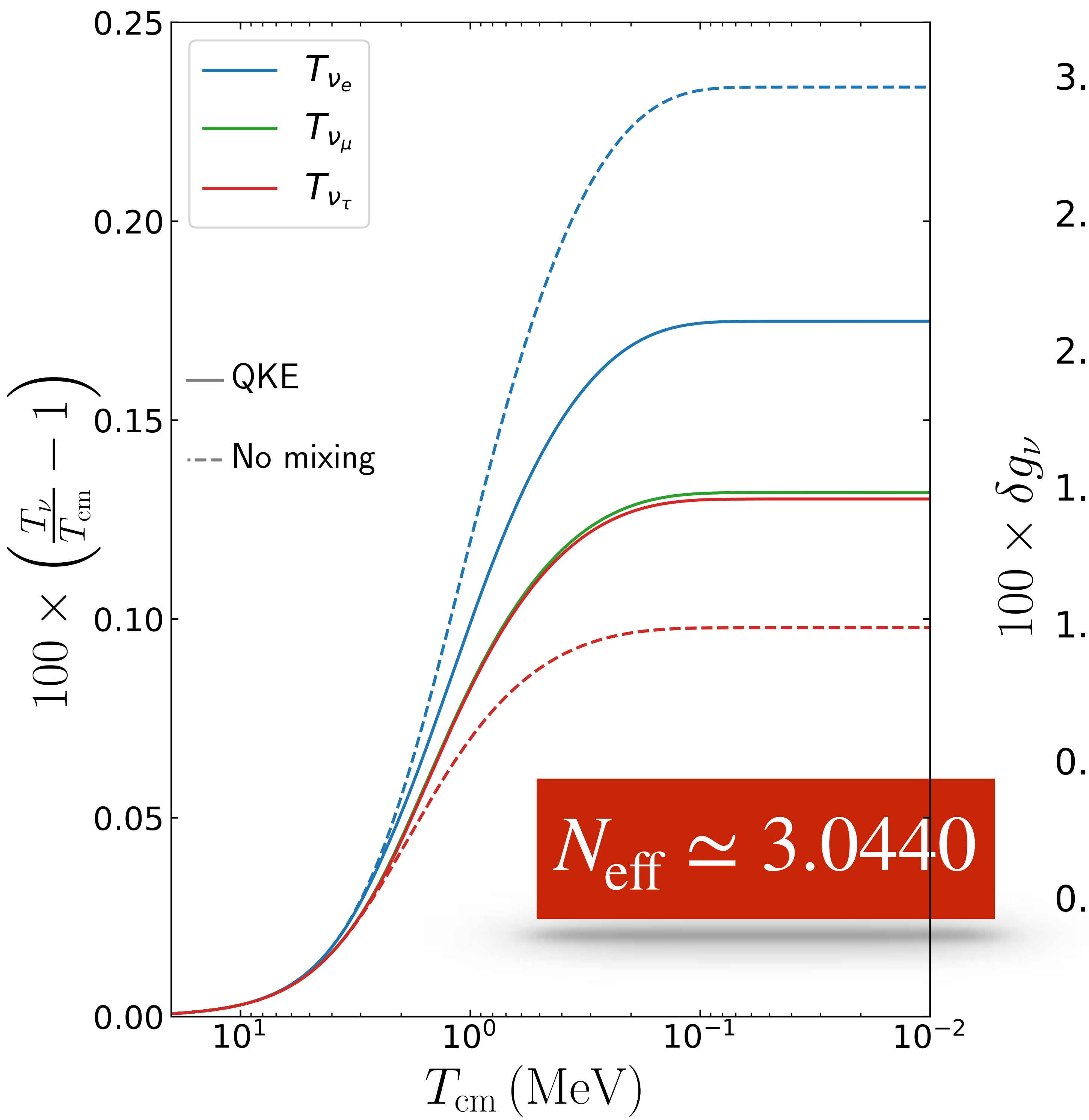


Effective temperatures

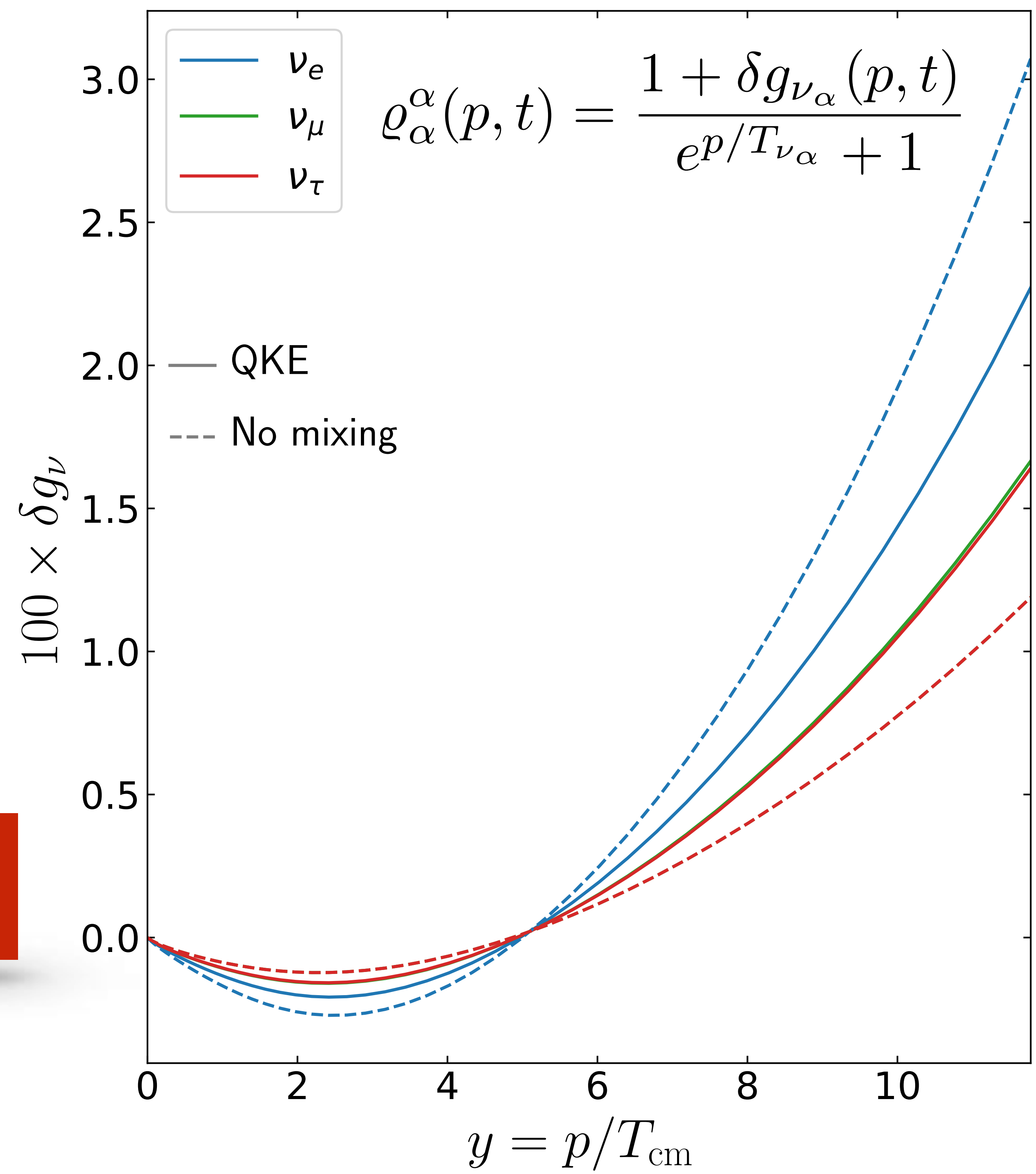


Effective distortions

Neutrino decoupling with flavor oscillations



Effective temperatures



Effective distortions

Decoupling with flavor oscillations - Comments

- Excellent accuracy of ATAO approximation ($<10^{-6}$).
- Slight increase of $N_{\rm eff}$ (3.0434 → 3.0440)

  flavor conversion of ν_e $\Longrightarrow$ more phase space for $e^{\pm}$ annihilations

- Higher precision?
  - Full QED corrections
  - Inhomogeneous cosmology

$\Delta N_{\rm eff} > 10^{-5}$

Conclusion

Neutrino decoupling

- Neutrinos capture part of the entropy released by $e^{\pm}$ annihilations
- Increased effective temperatures + spectral distortions
- Exact or approximate treatment of neutrino mixing
- $N_{\text{eff}} \simeq 3.044$

Consequences on BBN, CMB…

[**JF**, C. Pitrou, *Phys. Rev. D* 101, 043524 (2020)]