Putting Infinity on the Grid

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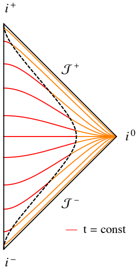
- Class.Quant.Grav. 35 (2018) 5, with E. Harms, M. Bugner. H. Rüter. B. Brügmann.
- Class.Quant.Grav. 36 (2019) 19, with E. Gasperin.
- Class.Quant.Grav. 37 (2020) 3 with E. Gasperin, S. Gautam, A. Vañó-Viñuales.

Open problems of practice and principle

There are *fundamental* open problems in NR even in the most conservative setting. These include;

- Extreme spacetimes.
- Compact objects.
- The weak-field.

Here: focus on last of these.



The timelike outer boundary. Vañó-Viñuales. 2015.

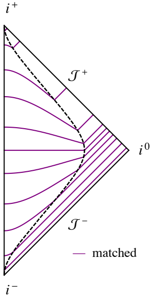
The weak-field I

The wavezone is *weak* so how is it a problem? Infinity *really* big.

- Asymptotic Flatness: Metric $\rightarrow$ Minkowski near infinity.
- Idea: draw infinity to a finite place. How could this work?
- Key complication: managing irregular terms.

Conformal approach:

- Analysis: Penrose, Friedrich.
- Numerics: Frauendiener, Hübner.



CCM Cartoon. Vañó-Viñuales. 2015.

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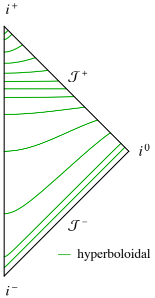


Hyperboloidal foliation. Vañó-Viñuales. 2015.

The weak-field II

A *dual-foliation* strategy:

- Difficulty: Vars/EoMs divergent.
- Observation: global inertial representation of MK metric trivially regular.
- Use nice representation. With care EoMs regular?

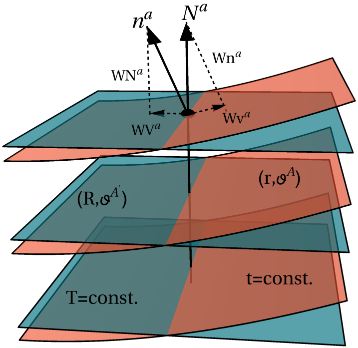


Illustration of DF setup.

The dual foliation formalism

Relationship between geometry with $X^{\underline{\mu}} = (T, X^{\underline{i}})$ or $x^\mu = (t, x^i)$?

- Parametrize the inverse Jacobian $J^{-1} = \partial_{\underline{\alpha}} x^\alpha$ as,

$$J^{-1} = \begin{pmatrix} \alpha^{-1} W (A - B^{\underline{j}} V_{\underline{j}}) & (A - B^{\underline{j}} V_{\underline{j}})\, \Pi^i + B^{\underline{j}} (\varphi^{-1})^i{}_{\underline{j}} \\ -\alpha^{-1} W V_{\underline{i}} & (\varphi^{-1})^i{}_{\underline{i}} - \Pi^i V_{\underline{i}} \end{pmatrix}.$$

- Suppose we have a system

$$\partial_T \mathbf{u} = (A \mathbf{A}^{\underline{p}} + B^{\underline{p}} \mathbf{1}) \partial_{\underline{p}} \mathbf{u} + A \mathbf{S}\,,$$

- Then in the lowercase coordinates we have

$$(1 + \mathbf{A}^V) \partial_t \mathbf{u} = \alpha W^{-1} (\mathbf{A}^{\underline{p}} (\varphi^{-1})^p{}_{\underline{p}} + (1 + \mathbf{A}^V) \Pi^p) \partial_p \mathbf{u} + \alpha W^{-1} \mathbf{S}.$$

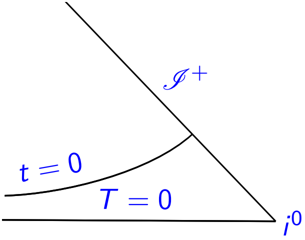
How to choose Jacobian?

The hyperboloidal initial value problem

$$T = T(t,r) = t + H(R), \qquad R = R(r) = \Omega(r)^{-1} r, \qquad \theta^{\underline{A}} = \theta^{A}\,.$$

- Height function H, compression function Ω.
- Hyperboloidal Jacobian;

$$J_{hyp} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ H'R' & R' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Rough Idea: $R' \simeq R^n$ and $H' \simeq 1 - 1/R'$, $1 < n \leq 2$ achieves desirable coordinate lightspeeds *whilst* compactifying.

Regularity of the principal part, asymptotics primer

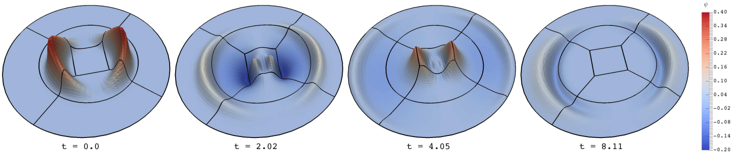
For systems with wave-equation like principal part (KG, GR in GHG) combining with the J_{hyp} gives;

$$(1+\mathbf{A}^V)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma_2 W^2 V_{\underline{i}} & {}^{(N)}\mathbf{g}^{\underline{j}}{}_{\underline{i}} & W^2 V_{\underline{i}} \\ -\gamma_2 (W^2-1) & W^2 V^{\underline{j}} & W^2 \end{pmatrix}$$

Observations:

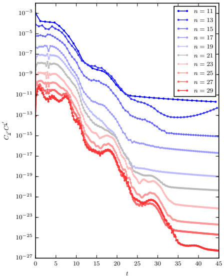
- Composite lower case principal part matrices *regular by construction*; symmetric hyperbolicity invariant.
- On the other hand $R' \sim R^n \implies W \sim \alpha \sim R^{n/2}$. Therefore need decay in sources $\mathbf{S}$ to absorb growth.

Hyperboloidal numerics with the DF-wave equation



A first numerical sanity check:

- For wave equation **S** small.
- Can even evolve *radiation field* $R\phi$. [Target for GR].
- Respectable pseudospectral convergence achieved.



Numerics with the wave equation in `bamps`.

Asymptotic flatness

GR in GHG: can decay compensate growing terms? Sufficient conditions for $(1+\mathbf{A}^V)^{-1}\mathbf{S} < \infty$ are

- (Weak) Asymptotic flatness assumption,

$$g_{\underline{\mu\nu}} = m_{\underline{\mu\nu}} + O(R^{-\epsilon}), \qquad \partial_{\underline{\alpha}} g_{\underline{\mu\nu}} = O(R^{-\epsilon}), \qquad \epsilon > 1/2.$$

- (Strong) Lightspeed condition $C_+^R = A/L - B^R$,

$$\partial_{\underline{\alpha}} C_+^R = O(R^{-1-\delta}), \qquad \delta > 0.$$

For the latter magic is needed!

Taking stock

Questions:

- What is behind the “mysterious” lightspeed condition?
- Can be the equations explicitly regularised?

Observations:

- Specific non-linear structure has not been exploited up so far.
- Relevant: Global non-linear stability of Minkowski by Lindblad & Rodnianski. Spoiler: The weak null condition!

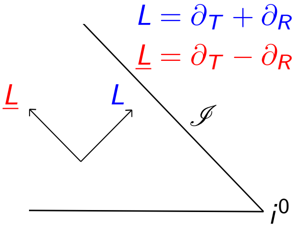
The classical null condition

For quasilinear wave-equations with quadratic nonlinearity in $\nabla\phi$, classical null condition $\implies$ global existence [Kla86,Chr86].

Example:

$$\Box\phi = \nabla^a\phi\nabla_a\phi,$$

but $m^{ab}L_aL_b = 0$, so the CNC holds. Too restrictive.



- WNC: "*asymptotic system* admits global solutions that do not grow too fast" [LinRod03].
- WNC $\overset{???}{\implies}$ Global existence [Conjecture].

The GB-model I

What is the asymptotic system? Example:

$$\Box g = 0, \quad \Box b = (\partial_T g)^2.$$

Recipe:

- Rescale $\qquad G = Rg, \qquad B = Rb.$
- Change coordinates $\quad u = T - R, \qquad s = \log(R).$
- Turn krank, collect leading order in R^{-1}.

For the GB-model this gives

$$\partial_s \partial_u G = 0, \qquad 2\partial_s \partial_u B = -(\partial_u G)^2.$$

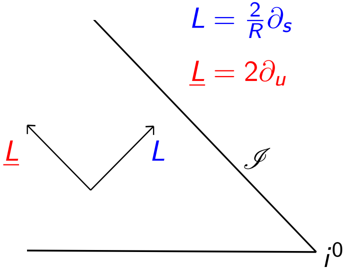
[NB. We have worked all of this out for first order systems.]

The GB-model II

$$\partial_s \partial_u G = 0, \qquad 2\partial_s \partial_u B = -(\partial_u G)^2.$$

Solution to asymptotic system

- $\partial_s \partial_u G = 0 \implies G = \mathcal{F}_g(u, \theta^A)$.
- $\partial_s \partial_u B = -\frac{1}{2}(\partial_u G)^2 \implies$
  $\partial_u B = -\frac{1}{2} s (\partial_u G)^2$.
- $B = (\ln R)\, \mathcal{F}_b(u, \theta^A)$.



Predicts the asymptotics of original fields

$$g = \frac{1}{R}\mathcal{F}_g(u, \theta^A), \qquad b = \frac{\log(R)}{R}\mathcal{F}_b(u, \theta^A)$$

GHG with constraint damping I

- Reduced Ricci

$$\mathcal{R}_{\underline{\alpha\beta}} = R_{\underline{\alpha\beta}} - \nabla_{(\underline{\alpha}}C_{\underline{\beta})} + W_{\underline{\alpha\beta}}$$

- $W_{\underline{\alpha\beta}}$ homogeneous in $C_{\underline{\alpha}}$
- The reduced EFE $\mathcal{R}_{\underline{\alpha\beta}} = 0 \quad \Longrightarrow$

$$g^{\underline{\mu\nu}}\partial_{\underline{\mu}}\partial_{\underline{\nu}}g_{\underline{\alpha\beta}} = N_{\underline{\alpha\beta}}[\partial g, \partial g] + P_{\underline{\alpha\beta}}[\partial g, \partial g] + F_{\underline{\alpha\beta}} + 2W_{\underline{\alpha\beta}},$$

GHG with constraint damping II

To apply recipe to GHG write

$$g_{ab} = m_{ab} + h_{ab}, \qquad H_{ab} = Rh_{ab}.$$

Define flat-null frame $\{L, \underline{L}, S_A\}$

- G-fields $H_{\mathcal{G}}$: $\quad H_{L\underline{L}}, H_{\underline{L}S_A}, H_\times \equiv 2H_{S_1S_2}, H_+ \equiv H_{S_1S_1} - H_{S_2S_2}.$
- B-field: $\quad H_{\underline{L}\underline{L}}.$
- U-fields $H_{\mathcal{U}}$: $\quad H_{LL}, H_{LS_A}, H_\varnothing \equiv H_{S_1S_1} + H_{S_2S_2}.$

Asymptotic constraints

$$C_U = \partial_u H_U, \quad \text{free evolution} \quad C_U \neq 0.$$

The good the bad and the ugly

It turns out that W_{ab} can be prescribed so that

$$\left(\partial_s + \frac{H_{LL}}{2}\partial_u\right)\partial_u H_{\mathcal{G}} = 0,$$

$$\left(\partial_s + \frac{H_{LL}}{2}\partial_u\right)\partial_u H_{\underline{LL}} = \frac{(\partial_u H_+)^2}{2} + \frac{(\partial_u H_\times)^2}{2},$$

$$\left(\partial_s + \frac{H_{LL}}{2}\partial_u\right)\partial_u H_{\mathcal{U}} = -\delta\partial_u H_{\mathcal{U}} - \frac{1}{2}\partial_u H_{\mathcal{U}}\partial_u H_{LL}.$$

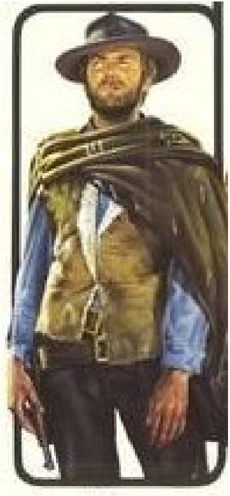


Figure: The good

The good the bad and the ugly

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$$\left(\partial_s + \frac{H_{LL}}{2}\partial_u\right)\partial_u H_{\mathcal{W}} = -\delta\partial_u H_{\mathcal{W}} - \frac{1}{2}\partial_u H_{\mathcal{W}}\partial_u H_{LL}.$$

Figure: The bad

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Figure: The Ugly

The good the bad and the ugly

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$$\left(\partial_s + \frac{H_{LL}}{2}\partial_u\right)\partial_u H_{\underline{LL}} = \frac{(\partial_u H_+)^2}{2} + \frac{(\partial_u H_\times)^2}{2},$$
$$\left(\partial_s + \frac{H_{LL}}{2}\partial_u\right) C_U = -\delta C_U - \frac{1}{2} C_U C_{LL}.$$

We have $C_+^R \simeq 1 - H_{LL}/(2R)$. But $H_{LL} \in H_{\mathcal{U}}$.
Lightspeed condition obtained. Magic!

Figure: The Ugly

The GBU-model

Consider the toy model for GR in harmonic gauge;

$$\Box g = 0, \qquad \Box b = (\partial_T g)^2, \qquad \Box u \simeq \tfrac{2}{R}\partial_T u.$$

- Asymptotics; $g \sim R^{-1}$, $b \sim R^{-1}\log(R)$, $u \sim R^{-2}$.
- "Subtract the logs" regularization; evolving

$$\begin{aligned} G &\simeq Rg, & B &\simeq Rb + \tfrac{1}{8}\log(R)\eta, \\ U &\simeq R^2 u, & \partial_u \eta &\sim (\partial_u G)^2, \end{aligned}$$

in fact gives regular equations for regular unknowns!

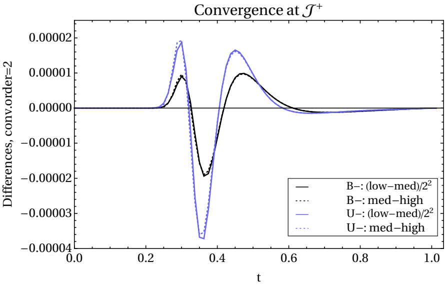
[NB. η analagous to news in GR].

Hyperboloidal numerics with the GBU-model

Second numerical sanity check:

- Again like ‘radiation field’.
- Implemented GBU-model in spherical FD code.
- Convergence despite logs.
- (Spectral numerics desirable too; patience needed!)

Can specially chosen basis functions save hassle here or for GR?



Numerics with GBU-model.

Conclusions

Motivated by need for GWs at null infinity we are developing a new regularization using compactified hyperboloids. Features include:

- Dual-foliation formalism.
- Exploiting null-structure for NR (lightspeed condition achieved!).
- Nonlinear change of variables to get *regular equations for regular unknowns*.

GR on the way - stay tuned!