Averaging in spatially homogenous Einstein-Klein-Gordon cosmology

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Meudon, 27.02.2020

Introducing myself

- 2 year Postdoc @ LESIA in GRAVITY group
- Main project: probe possibility of detection of extended mass distribution between S2 and SBH @ galactic centre.
  → celestial mechanics. newtonian, post newtonian, relativistic
- Studied Physics in Innsbruck (BSc), Vienna (MSc) and Cardiff (PhD)
- PhD thesis with Mark Hannam on black hole initial data for 3+1 simulations (under umbrella of LSC)
- MSc thesis with Mark Heinzle and first Postdoc in Vienna with David Fajman on mathematical cosmology. → spatially homogenous cosmology with various matter models.

Spatially homogenous cosmology

Friedman cosmology

inhomogenous cosmology

analytically

(generally) numerically

Spatially homogenous cosmology

Friedman cosmology	spatially homogenous cosmology	inhomogenous cosmology
analytically	qualitatively analytically	(generally) numerically

Spatially homogenous cosmology

Friedman cosmology	spatially homogenous cosmology	inhomogenous cosmology
analytically	qualitatively analytically	(generally) numerically

↓

eg exact past & future asymptotic dynamics

- D Fajman, G Heißel, M Maliborski
  On the oscillations and future asymptotics of locally rotationally symmetric Bianchi type III cosmologies with a massive scalar field
  arXiv:2001.00252, submitted for publication

- D Fajman, G Heißel, JW Jang
  Averaging with a time dependent perturbation
  drafting

- D Fajman, G Heißel
  Averaging methods in spatially homogenous Einstein-Klein-Gordon cosmology
  drafting

SH scalar field cosmology

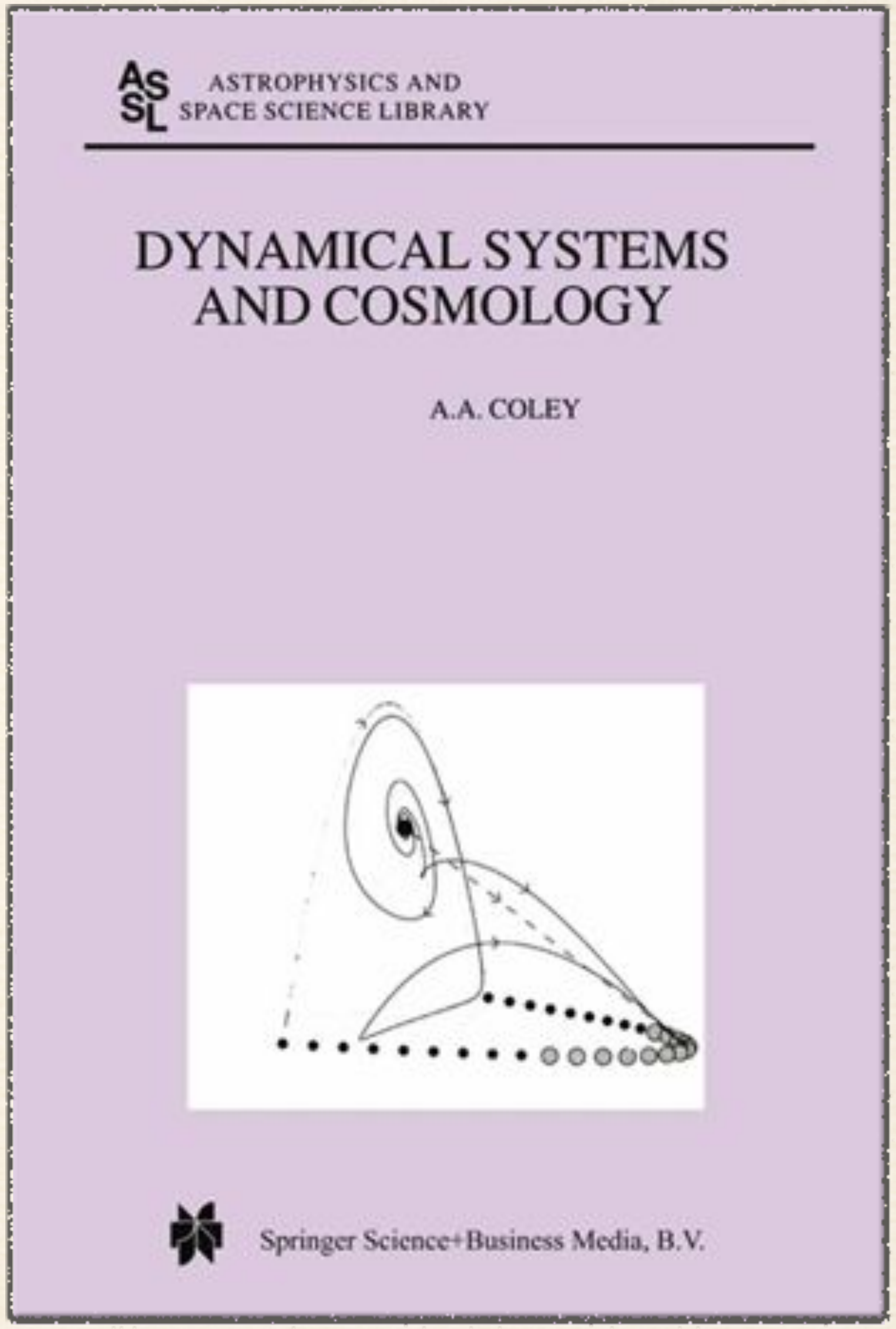
- $R_{ab} - \frac{R}{2} g_{ab} = T_{ab}$

  $T_{ab} = \nabla_a \phi \nabla_b \phi - \left(\frac{1}{2} \nabla_c \nabla^c \phi + V(\phi) \right) g_{ab}$

  $\Box_{\mathbf{g}} \phi = V'(\phi)$

- Formulate Einstein-matter eqs as dynmaical system in expansion (/Hubble) normalised variables.

  $H := \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b} \right)/3$



(2003)

- Exponential potential: $V(\phi) \propto e^{\kappa\phi}$

  → Interesting in the context of inflation.
  → $V'(\phi) \propto V(\phi) \Rightarrow$ Raychaudhuri eq $\dot{H}(t) = \ldots$ decouples
  → Asymptotics related to equilibrium points of the reduced system.
  → Dynamical systems analysis to determine asymptotics.

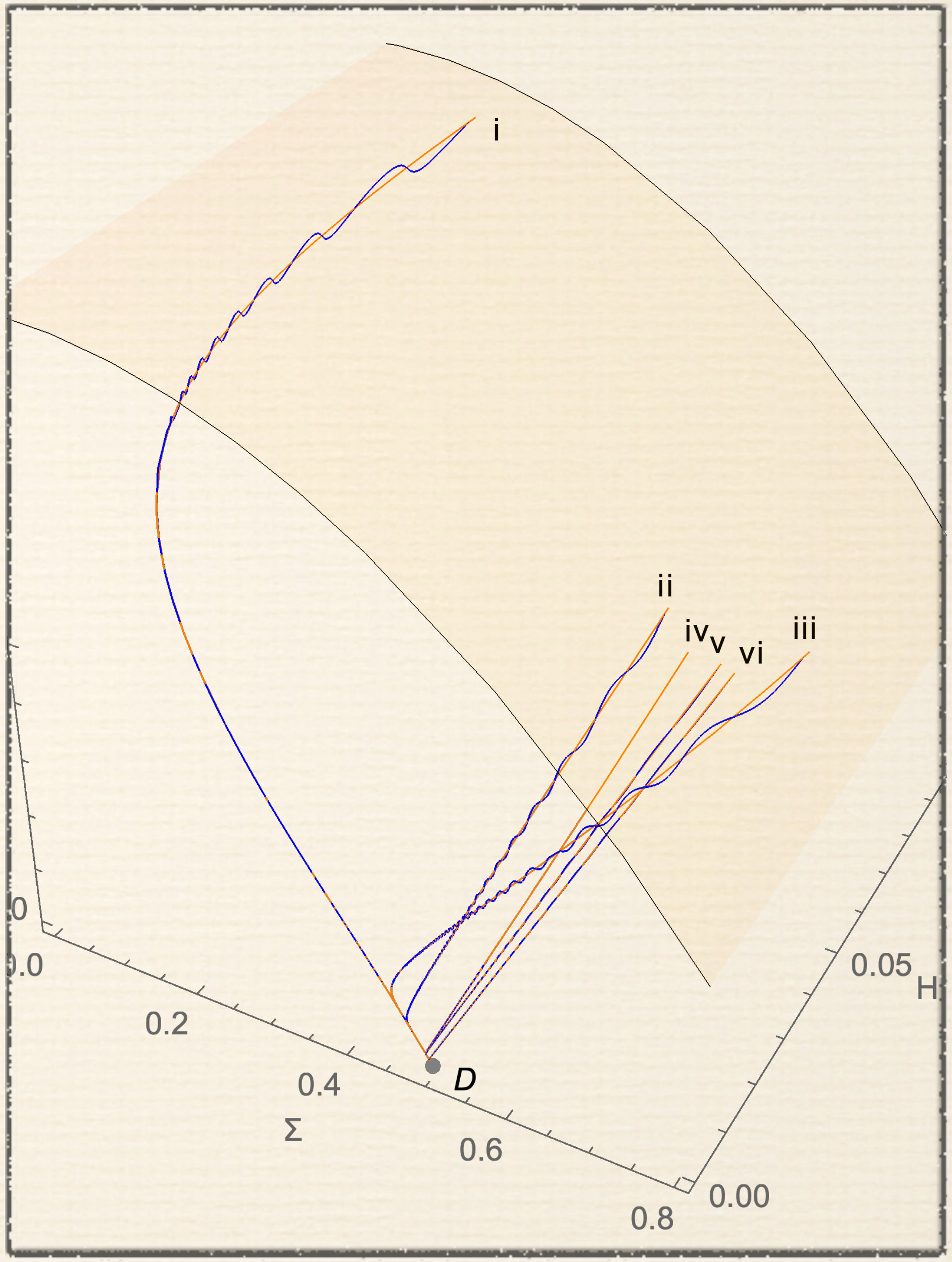
- Harmonic potential: $V(\phi) = m^2\phi^2/2$

  → Massive scalar field / Klein-Gordon field.
  → Raychaudhuri eq does *not* decouple.
  → Standard dynamical systems approach *not* a priori applicable.

LRS Bianchi III Einstein-Klein-Gordon

D Fajman, G Heißel, M Maliborski

arXiv:2001.00252
submitted for publication



- Choose formulation of Rendall & Uggla (2000)

$$\mathbf{g} = -\,\mathrm{d}t^2 + a(t)^2\mathrm{d}r^2 + b(t)^2\mathbf{g}_{H^2}$$

$$H := \frac{1}{3}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)$$ Hubble scalar

$$\Sigma_+ := \frac{1}{3H}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right)$$ shear variable

$$T^a{}_b = \mathrm{diag}(-\rho, p_1, p_2, p_3)$$

- Plug in $m = 1$ Klein-Gordon field into their equations

$$\rho = \frac{1}{2}\left(\dot{\phi}^2 + \phi^2\right)$$

$$p_i = \frac{1}{2}\left(\dot{\phi}^2 - \phi^2\right)$$

- Define

$$q := 2\Sigma_+^2 + \frac{1}{6H^2}\left(2\dot{\phi}^2 - \phi^2\right) \qquad \text{deceleration parameter}$$

$$\Omega := \frac{\rho}{3H^2} \qquad \text{rescaled energy density}$$

- Yields reduced system

$$\left.\begin{aligned} \dot{H} &= H^2\left[-(1+q)\right] \\ \dot{\Sigma}_+ &= H\left[-(2-q)\Sigma_+ + 1 - \Sigma_+^2 - \Omega\right] \end{aligned}\right\} \text{Einstein evol.}$$

$$\boxed{\ddot{\phi} + \phi = H\left[-3\dot{\phi}\right]} \qquad \text{Klein-Gordon}$$

$$1 > \Sigma_+^2 + \Omega \qquad \text{Hamiltonian constraint}$$

The Van der Pol equation

- Consider general class of Van der Pol equations

$$\ddot{\phi} + \phi = \epsilon g(\dot{\phi}, \phi), \epsilon = \text{const} \qquad \text{vs} \qquad \ddot{\phi} + \phi = H \left[-3\dot{\phi} \right], H = H(t)$$

- Apply amplitude phase (variation of constants) transformation

$$\begin{aligned} \phi &= r\sin(t-\varphi) \\ \dot{\phi} &= r\cos(t-\varphi) \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \dot{r} \\ \dot{\varphi} \end{bmatrix} = \epsilon \begin{bmatrix} \cos(t-\varphi)g(\phi,\dot{\phi}) \\ \frac{1}{r}\sin(t-\varphi)g(\phi,\dot{\phi}) \end{bmatrix} =: \epsilon \mathbf{f}^1(t,r,\varphi)$$

- Note that $\mathbf{f}^1(t, r, \varphi)$ is 2π periodic in t.

- Idea: take time average of right hand side function

$$\bar{\mathbf{f}}^1(r,\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \mathbf{f}^1(s,r,\varphi)\,\mathrm{d}s$$

- Construct averaged system

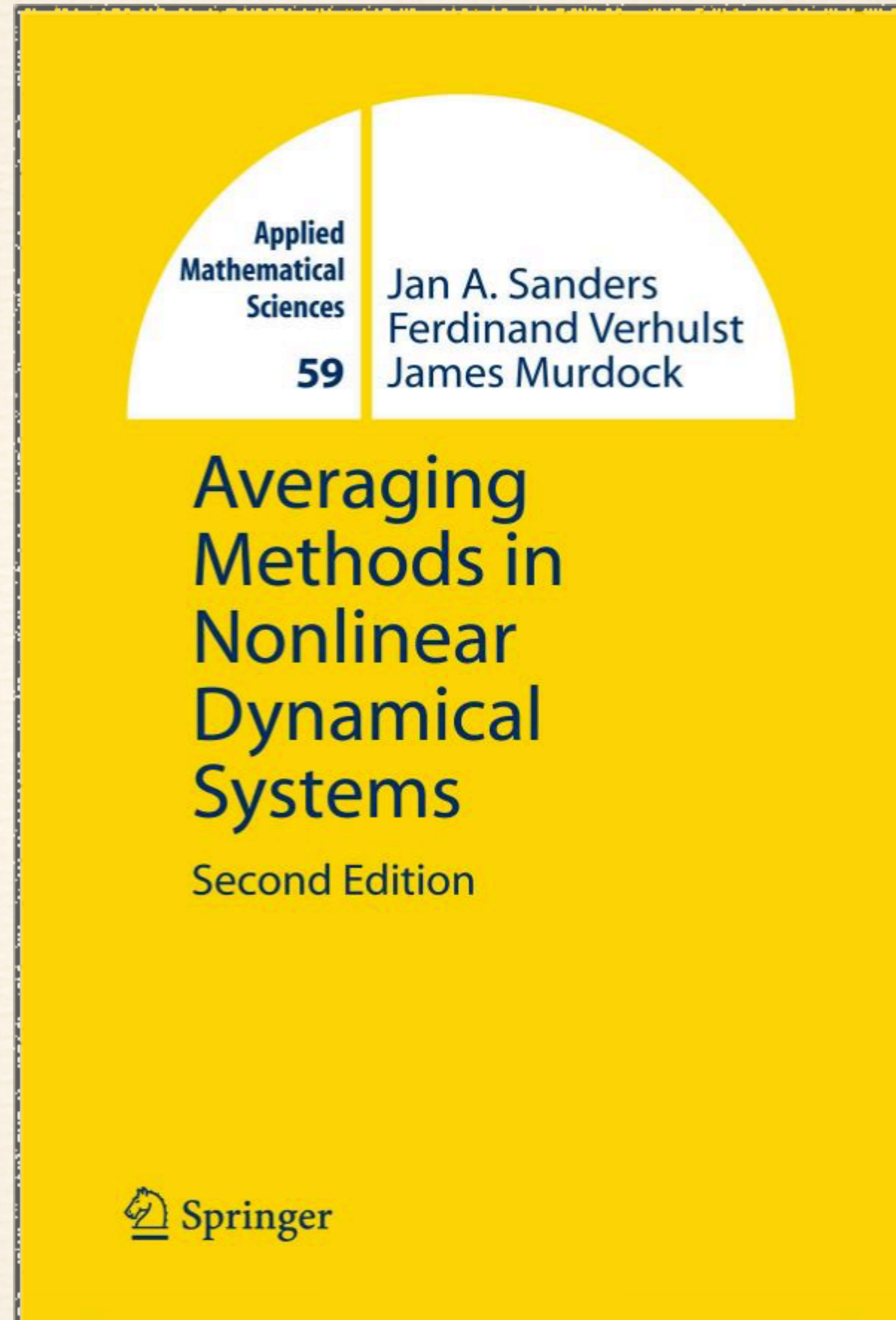
$$\begin{bmatrix} \dot{\bar{r}} \\ \dot{\bar{\varphi}} \end{bmatrix} = \epsilon\bar{\mathbf{f}}^1(\bar{r},\bar{\varphi})$$

- How good is this approximation?

Theory of averaging

… originally motivated by celestial mechanics

→ *perturbed 2 body problem, etc.*



(2007)

- **Theorem 1** (periodic averaging)

$$\boxed{\dot{\mathbf{x}} = \epsilon\, \mathbf{f}^1(\mathbf{x}, t) + \epsilon^2\, \mathbf{f}^{[2]}(\mathbf{x}, t, \epsilon),} \quad \mathbf{x}(0) = \mathbf{a}$$

• $\mathbf{f}^1, \mathbf{f}^{[2]}$ T-periodic in t
• $\mathbf{f}^1$ Lipschitz, $\mathbf{f}^{[2]}$ continuous

$$\boxed{\dot{\mathbf{z}} = \epsilon\, \bar{\mathbf{f}}^1(\mathbf{z}), \quad \mathbf{z}(0) = \mathbf{a},} \quad \bar{\mathbf{f}}^1(\mathbf{z}) = \frac{1}{T}\int_0^T \mathbf{f}^1(\mathbf{z}, s)\, \mathrm{d}s$$

Then $\boxed{\mathbf{x}(t) - \mathbf{z}(t) = \mathcal{O}(\epsilon)}$ on timescales of order ϵ^{-1}.

- **Theorem 2** (Eckhaus/Sanchez-Palencia)

$$\boxed{\dot{\mathbf{x}} = \epsilon\, \mathbf{f}^1(\mathbf{x}, t),} \quad \mathbf{x}(0) = \mathbf{a} \qquad \boxed{\dot{\mathbf{z}} = \epsilon\, \bar{\mathbf{f}}^1(\mathbf{z}),} \quad \mathbf{z}(0) = \mathbf{a}, \qquad \mathbf{f}^1 \text{ KBM}$$

If $\mathbf{z} = 0$ is an asymptotically stable equilibrium point of the linearisation…

…then $\boxed{\mathbf{x}(t) - \mathbf{z}(t) = \mathcal{O}(\epsilon)}$ for $0 \le t < \infty$.

Back to LRS Bianchi III

- Equations in $\{H, \Sigma_+, \phi, \dot{\phi}\}$

$$\dot{H} = H^2\big[-(1+q)\big]$$
$$\dot{\Sigma}_+ = H\ \big[-(2-q)\Sigma_+ + 1 - \Sigma_+^2 - \Omega\big]$$
$$\ddot{\phi} + \phi = H\ \big[-3\dot{\phi}\big]$$

$\Sigma_+^2 + \Omega < 1$ Hamiltonian constraint

- Amplitude phase (variation of constants) transformation

$$\begin{aligned}\phi &= r\sin(t-\varphi)\\ \dot{\phi} &= r\cos(t-\varphi)\end{aligned}, \qquad r \mapsto \Omega = \frac{r^2}{6H^2} \quad \Longrightarrow \quad q = 2\Sigma_+^2 + \Omega\big(3\cos(t-\varphi)^2 - 1\big)$$

- Equations in $\{H, \Sigma_+, \Omega, \varphi\}$

$$\dot{H} = H^2\big[-(1+q)\big]$$
$$\dot{\Sigma}_+ = H\ \big[-(2-q)\Sigma_+ + 1 - \Sigma_+^2 - \Omega\big]$$
$$\dot{\Omega} = H\ \big[2\Omega\big(1 + q - 3\cos(t-\varphi)^2\big)\big]$$
$$\dot{\varphi} = H\ \big[-3\sin(t-\varphi)\cos(t-\varphi)\big]$$

$$1 > \Sigma_+^2 + \Omega$$

- Writing $\mathbf{x} = [\Sigma_+, \Omega, \varphi]^{\mathrm{T}}$ this has the form

$$\begin{bmatrix} \dot{H} \\ \dot{\mathbf{x}} \end{bmatrix} = H\,\mathbf{F}^1(\mathbf{x}, t) + H^2\,\mathbf{F}^{[2]}(\mathbf{x}, t) = H \begin{bmatrix} 0 \\ \mathbf{f}^1(\mathbf{x}, t) \end{bmatrix} + H^2 \begin{bmatrix} f^{[2]}(\mathbf{x}, t) \\ \mathbf{0} \end{bmatrix}$$

analogous to a periodic perturbation problem in standard form

$$\dot{\mathbf{x}} = \epsilon\,\mathbf{f}^1(\mathbf{x}, t) + \epsilon^2\,\mathbf{f}^{[2]}(\mathbf{x}, t, \epsilon).$$

- Difference: → $H(t)$ is time dependent …
  → … and itself subject to evolution equation which is part of the system

- Idea: View this as an averaging problem with time dependent perturbation function.

- Equations in $\{H, \Sigma_+, \Omega, \varphi\}$

$$\dot{H} = H^2\big[-(1+q)\big]$$
$$\dot{\Sigma}_+ = H\ \big[-(2-q)\Sigma_+ + 1 - \Sigma_+^2 - \Omega\big]$$
$$\dot{\Omega} = H\ \big[2\Omega\big(1 + q - 3\cos(t-\varphi)^2\big)\big]$$
$$\dot{\varphi} = H\ \big[-3\sin(t-\varphi)\cos(t-\varphi)\big]$$

$$1 > \Sigma_+^2 + \Omega$$

Same core idea independently before us:
A Alho & C Uggla (2015)
A Alho et al (2015)
A Alho et al (arXiv, 2019)

- Writing $\mathbf{x} = [\Sigma_+, \Omega, \varphi]^{\mathrm{T}}$ this has the form

$$\begin{bmatrix}\dot{H}\\ \dot{\mathbf{x}}\end{bmatrix} = H\,\mathbf{F}^1(\mathbf{x},t) + H^2\,\mathbf{F}^{[2]}(\mathbf{x},t) = H\begin{bmatrix}0\\ \mathbf{f}^1(\mathbf{x},t)\end{bmatrix} + H^2\begin{bmatrix}f^{[2]}(\mathbf{x},t)\\ \mathbf{0}\end{bmatrix}$$

analogous to a periodic perturbation problem in standard form

$$\dot{\mathbf{x}} = \epsilon\,\mathbf{f}^1(\mathbf{x},t) + \epsilon^2\,\mathbf{f}^{[2]}(\mathbf{x},t,\epsilon).$$

- Difference: → $H(t)$ is time dependent …
  → … and itself subject to evolution equation which is part of the system

- Idea: View this as an averaging problem with time dependent perturbation function.

- Full eq

$$\dot{\mathbf{x}} = \epsilon\, \mathbf{f}^1(\mathbf{x}, t) + \epsilon^2\, \mathbf{f}^{[2]}(\mathbf{x}, t, \epsilon)$$

- Averaged eq

$$\dot{\mathbf{z}} = \epsilon\, \bar{\mathbf{f}}^1(\mathbf{z})$$

- Theorem

On timescales of order ϵ^{-1}

$$\mathbf{x}(t) - \mathbf{z}(t) = \mathcal{O}(\epsilon)$$

- Full eq

$$\begin{bmatrix} \dot{H} \\ \dot{\mathbf{x}} \end{bmatrix} = H \begin{bmatrix} 0 \\ \mathbf{f}^1(\mathbf{x}, t) \end{bmatrix} + H^2 \begin{bmatrix} f^{[2]}(\mathbf{x}, t) \\ \mathbf{0} \end{bmatrix}$$

- Averaged eq

$$\dot{\mathbf{z}} = H(t)\, \bar{\mathbf{f}}^1(\mathbf{z})$$

- Conjecture

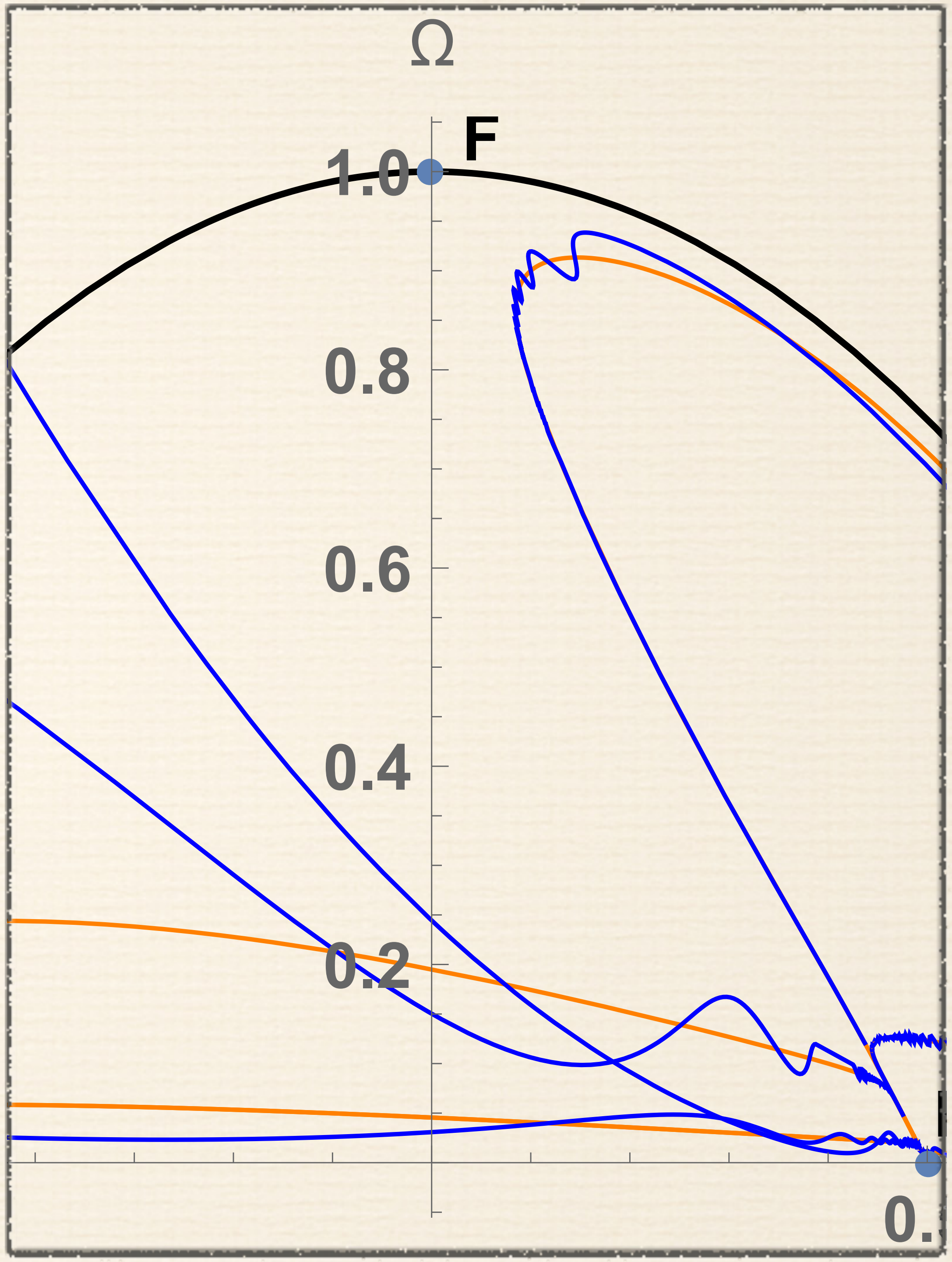
$\exists\, t_*$ such that $\forall\, t > t_*$

$$\mathbf{X}(t) - \mathbf{Z}(t) = \mathcal{O}\big(H(t)\big)$$

- Lemma. H is strictly decreasing with t and $\lim_{t\to\infty} H(t) = 0$.

Analytical support for the conjecture

Back to LRS Bianchi III



- Start with full system

$$\begin{bmatrix} \dot{H} \\ \dot{\mathbf{x}} \end{bmatrix} = H \begin{bmatrix} 0 \\ \mathbf{f}^1(\mathbf{x}, t) \end{bmatrix} + H^2 \begin{bmatrix} f^{[2]}(\mathbf{x}, t) \\ \mathbf{0} \end{bmatrix}$$

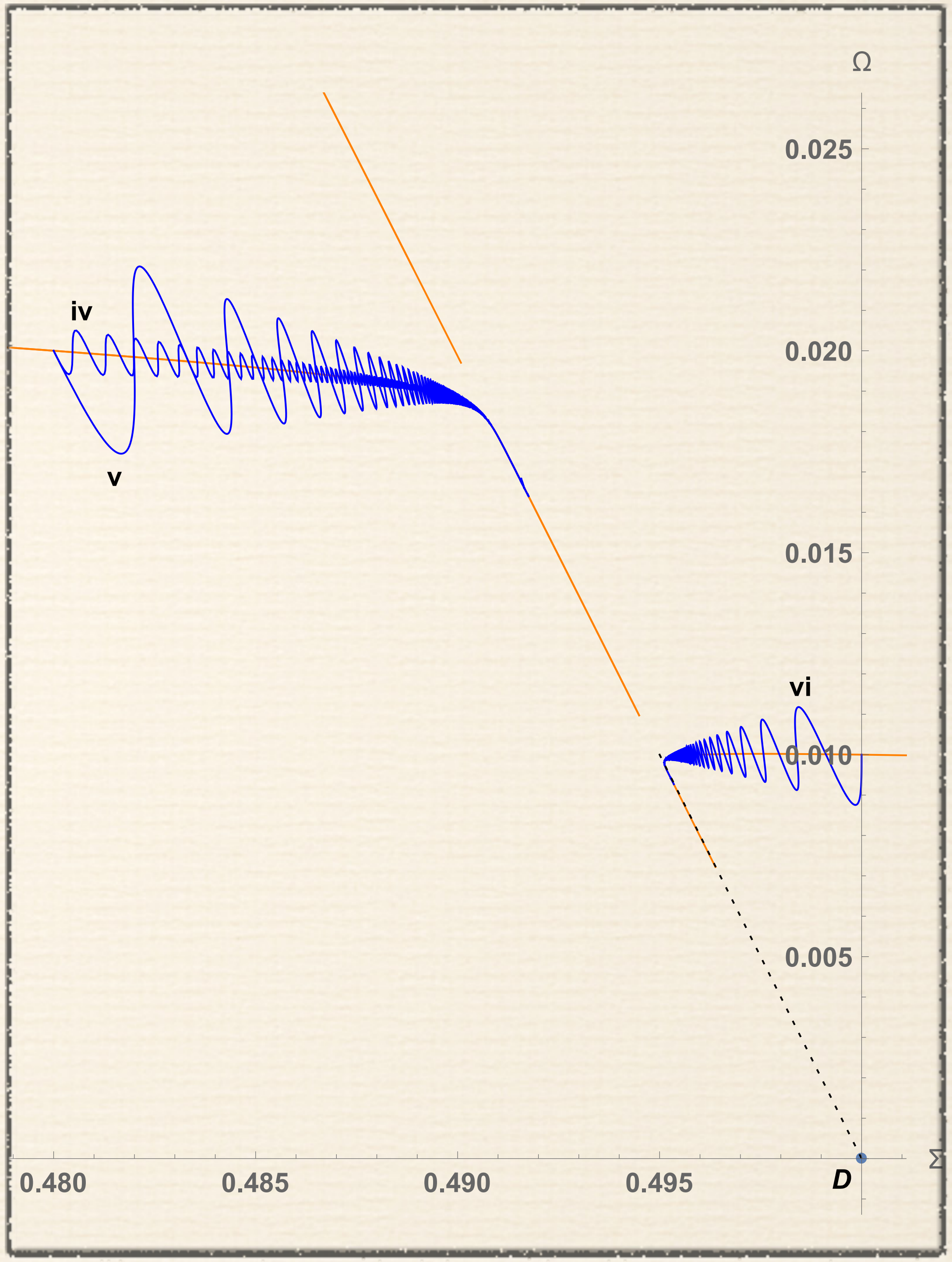
- Truncate to first order after sufficiently large $t = t_*$ $\quad \Rightarrow \quad H^2 \ll H \ll 1$

$$\begin{bmatrix} \dot{\mathscr{H}} \\ \dot{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathscr{H}\, \mathbf{f}^1(\mathbf{y}, t) \end{bmatrix} = \begin{bmatrix} 0 \\ H_*\, \mathbf{f}^1(\mathbf{y}, t) \end{bmatrix}, \text{ with } H_* = \mathscr{H}(t_*) = H(t_*)$$

- Assumption: $\mathbf{x}(t) - \mathbf{y}(t) = \mathcal{O}(H_*)$
- Average first order system $\dot{\mathbf{z}} = H_*\, \bar{\mathbf{f}}^1(\mathbf{z})$
- By standard averaging Thm: $\mathbf{X}(t) - \mathbf{Y}(t) = \mathcal{O}(H_*)$
- Together with truncation error: $\mathbf{X}(t) - \mathbf{Z}(t) = \mathcal{O}(H_*)$
- Continuum limit yields statement of conjecture: $\mathbf{X}(t) - \mathbf{Z}(t) = \mathcal{O}\big(H(t)\big)$

Future asymptotics of LRS Bianchi III E-K-G

Under the premise that the conjecture holds.



- Transform to rescaled time

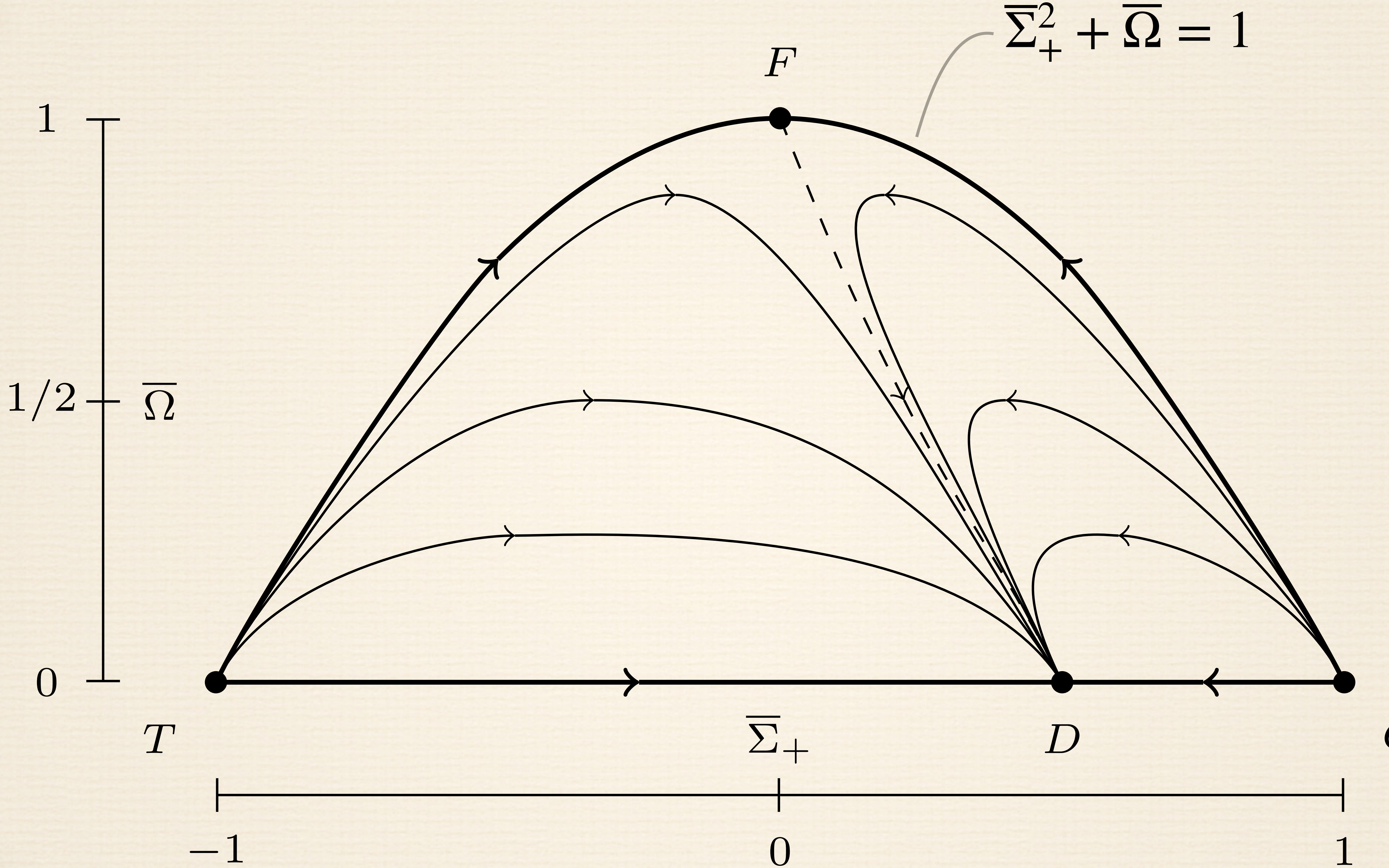
$$t \mapsto \tau := \int_{t_0}^{t} H(s)\,\mathrm{d}s \quad \Longrightarrow \quad \partial_t = H\partial_\tau$$

- System transforms to

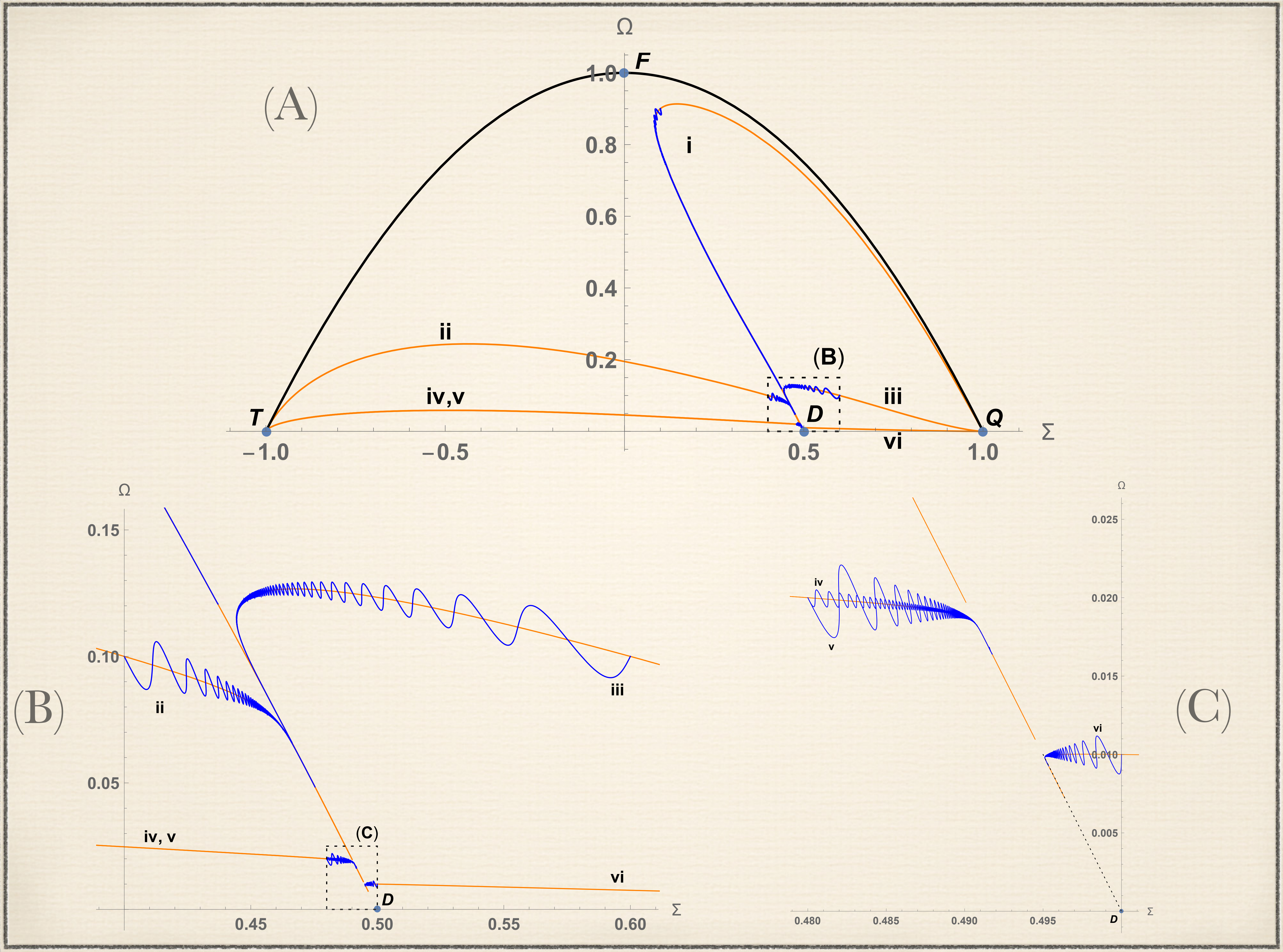
$$\begin{bmatrix} \dot{H} \\ \dot{\mathbf{x}} \end{bmatrix} = H\,\mathbf{F}^1(\mathbf{x},\tau) + H^2\,\mathbf{F}^{[2]}(\mathbf{x},\tau) \quad \mapsto \quad \begin{bmatrix} \partial_\tau H \\ \partial_\tau \mathbf{x} \end{bmatrix} = \mathbf{F}^1(\mathbf{x},\tau) + H\,\mathbf{F}^{[2]}(\mathbf{x},\tau)$$

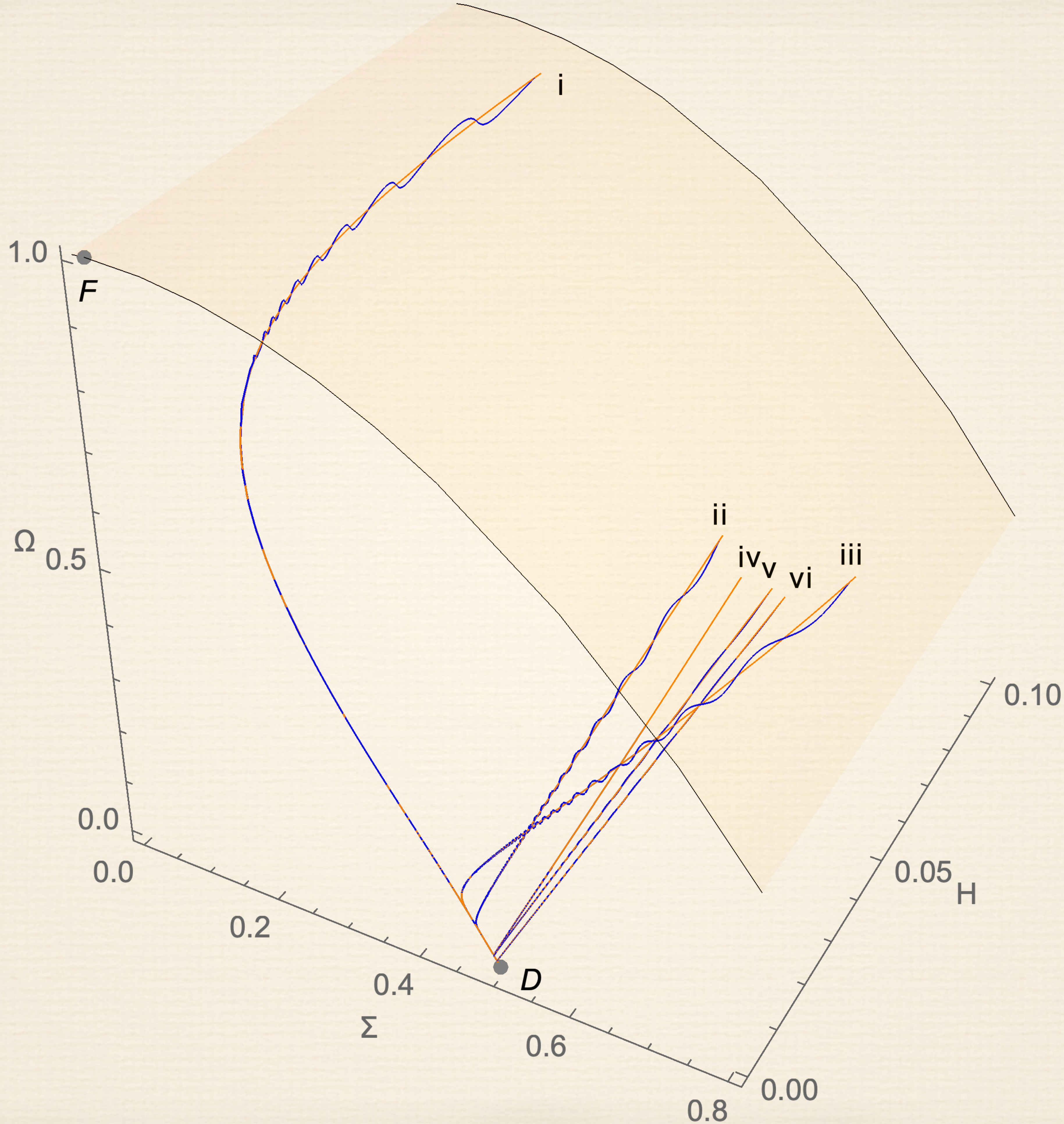
- Averaging to $\partial_\tau \overline{\mathbf{x}} = \overline{\mathbf{F}}^1(\mathbf{x},\tau)$

$$\begin{bmatrix} \partial_\tau \overline{\Sigma}_+ \\ \partial_\tau \overline{\Omega} \\ \partial_\tau \overline{\varphi} \end{bmatrix} = \begin{bmatrix} -\left(2\left(1-\overline{\Sigma}_+^2\right) - \frac{\overline{\Omega}}{2}\right)\overline{\Sigma}_+ + 1 - \overline{\Sigma}_+^2 - \overline{\Omega} \\ \overline{\Omega}\left(4\overline{\Sigma}_+^2 - \left(1-\overline{\Omega}\right)\right) \\ 0 \end{bmatrix}$$



$\overline{\Sigma}_+^2 + \overline{\Omega} = 1$
F
1
1/2
$\overline{\Omega}$
0
T
$\overline{\Sigma}_+$
D
Q
−1
0
1


(A)
Ω
F
1.0
0.8
0.6
0.4
0.2
i
ii
iii
iv,v
vi
(B)
T
D
Q
Σ
−1.0
−0.5
0.5
1.0
(B)
Ω
0.15
0.10
0.05
ii
iii
iv, v
vi
(C)
D
Σ
0.45
0.50
0.55
0.60
(C)
Ω
0.025
0.020
0.015
0.010
0.005
iv
v
vi
D
Σ
0.480
0.485
0.490
0.495



i
1.0
F
Ω
0.5
0.0
0.0
0.2
0.4
Σ
0.6
0.8
D
ii
iv
v
vi
iii
0.10
0.05
H
0.00

Exact solutions associated with equilibrium points

- First, solve Raychaudhuri eq at equilibrium points

$$\dot{H} = H^2\big[-(1+q)\big], \quad q = 2\Sigma_+^2 + \Omega\big(3\cos(t-\varphi)^2 - 1\big)$$

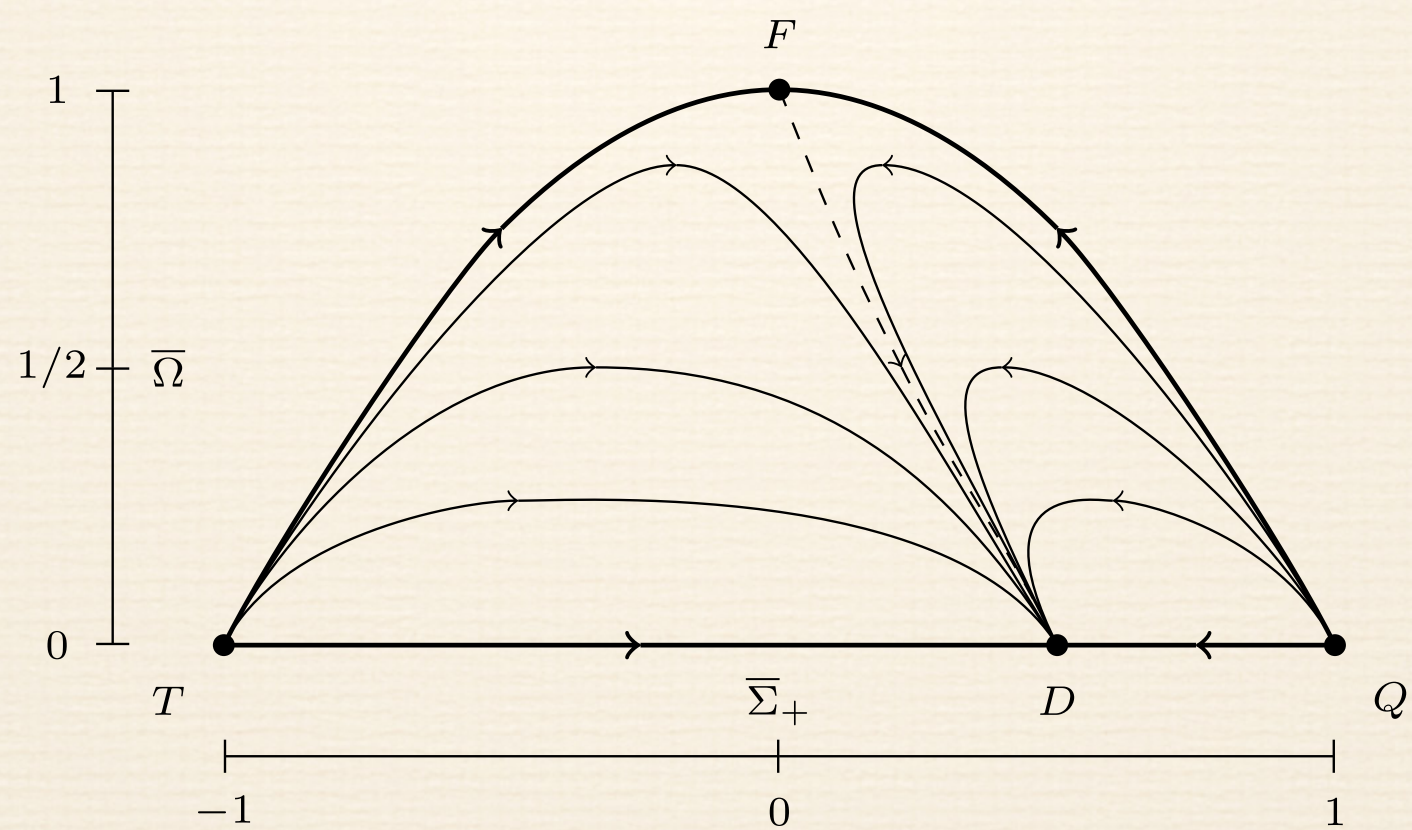
$$\overset{D}{\Longrightarrow} H(t) = \frac{2}{3t} \quad \text{for large } t$$

- Then, solve evolution eqs of the scale factors

$$\begin{aligned} \dot{a} &= aH(1-2\Sigma_+) \\ \dot{b} &= bH(1+\Sigma_+) \end{aligned} \quad \overset{D}{\Longrightarrow} \quad \begin{aligned} a(t) &= c_1 \\ b(t) &= c_2 t \end{aligned}$$

$$\mathbf{g} = -\,\mathrm{d}t^2 + a(t)^2\mathrm{d}r^2 + b(t)^2\mathbf{g}_{H^2}$$

equil. point	$a(t)$	$b(t)$	solution
T	$c_1 t$	c_2	Taub (flat LRS Kasner)
Q	$c_1 t^{-1/3}$	$c_2 t^{2/3}$	non-flat LRS Kasner
D	c_1	$c_2 t$	Bianchi III form of flat spacetime
F	$c_1 t^{2/3}$	$c_2 t^{2/3}$	Einstein-de-Sitter (flat Friedman)



- Centre manifold analysis yields

  Lemma.

  $$\overline{\Sigma}_+ \approx \frac{1}{2} - \frac{1}{2\tau} \quad \text{and} \quad \overline{\Omega}(\tau) \approx \frac{1}{\tau} \quad \text{for large } \tau$$

- Assuming that conjecture $\mathbf{X}(t) - \mathbf{Z}(t) = \mathcal{O}\big(H(t)\big)$ holds

  Lemma. $H(t) \approx \frac{2}{3t}$ for large t

  Lemma. $\tau(t) \approx \frac{2}{3}\ln t$ for large t $\left(t \mapsto \tau := \int_{t_0}^{t} H(s)\,\mathrm{d}s \right)$

  $$\Longrightarrow \mathbf{X}(t) = \mathbf{Z}(t) + \mathcal{O}\big(H(t)\big) = \begin{bmatrix} \frac{1}{2} - \frac{3}{4\ln t} \\ \frac{3}{2\ln t} \end{bmatrix} + \mathcal{O}(t^{-1})$$

- ...

$$\Longrightarrow \mathbf{X}(t) = \mathbf{Z}(t) + \mathcal{O}\big(H(t)\big) = \begin{bmatrix} \frac{1}{2} - \frac{3}{4\ln t} \\ \frac{3}{2\ln t} \end{bmatrix} + \mathcal{O}(t^{-1})$$

- **Theorem 1.** Assuming that the conjecture holds,

$$\Sigma_+(t) \approx \frac{1}{2} - \frac{3}{4\ln t} \quad \textbf{and} \quad \Omega(t) \approx \frac{3}{2\ln t} \quad \textbf{for large } t$$

- **Theorem 1.** Assuming that the conjecture holds,

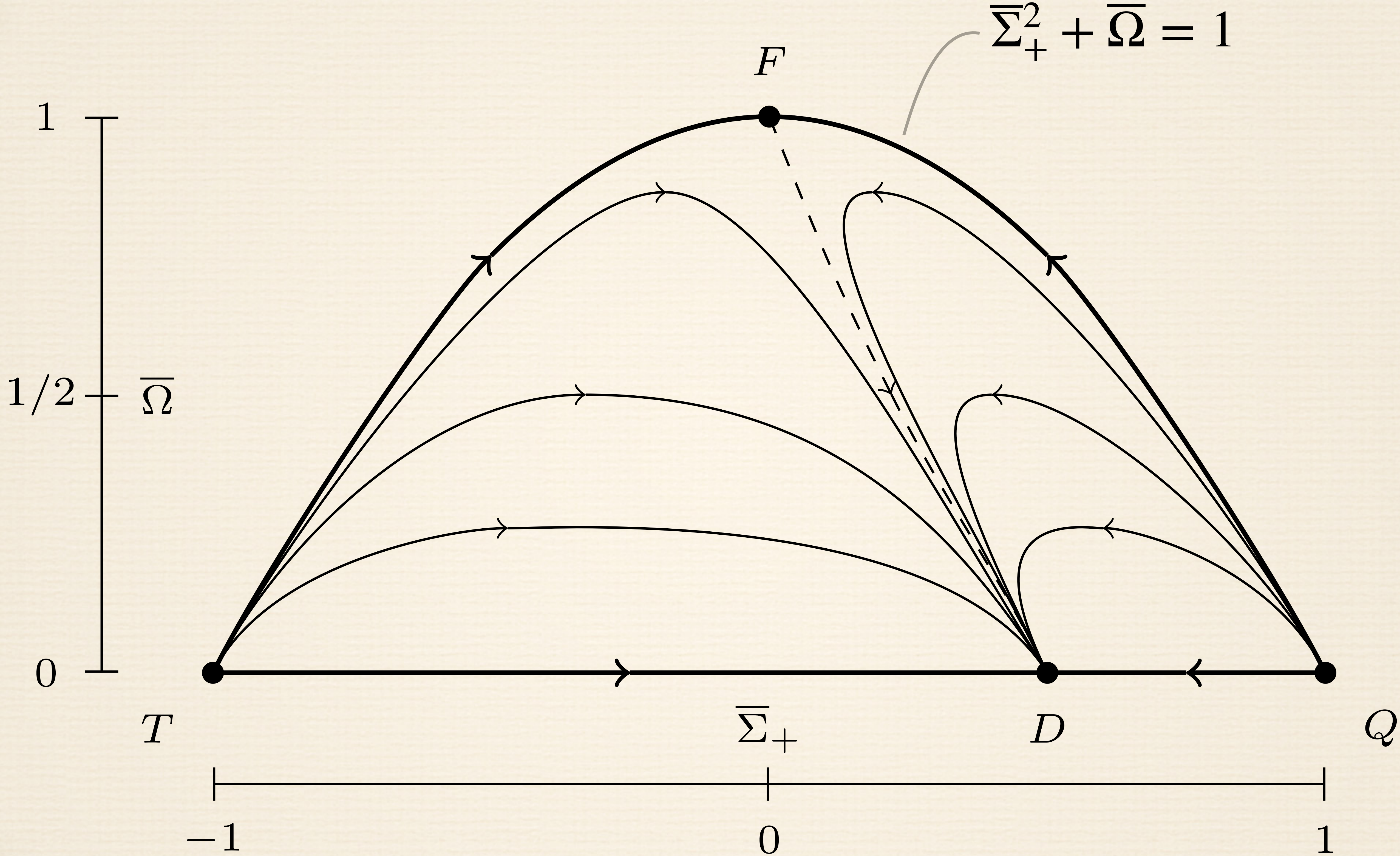
  $\Sigma_+(t) \approx \frac{1}{2} - \frac{3}{4\ln t}$ and $\Omega(t) \approx \frac{3}{2\ln t}$ for large t.

- **Theorem 2.** Assuming that the conjecture holds,

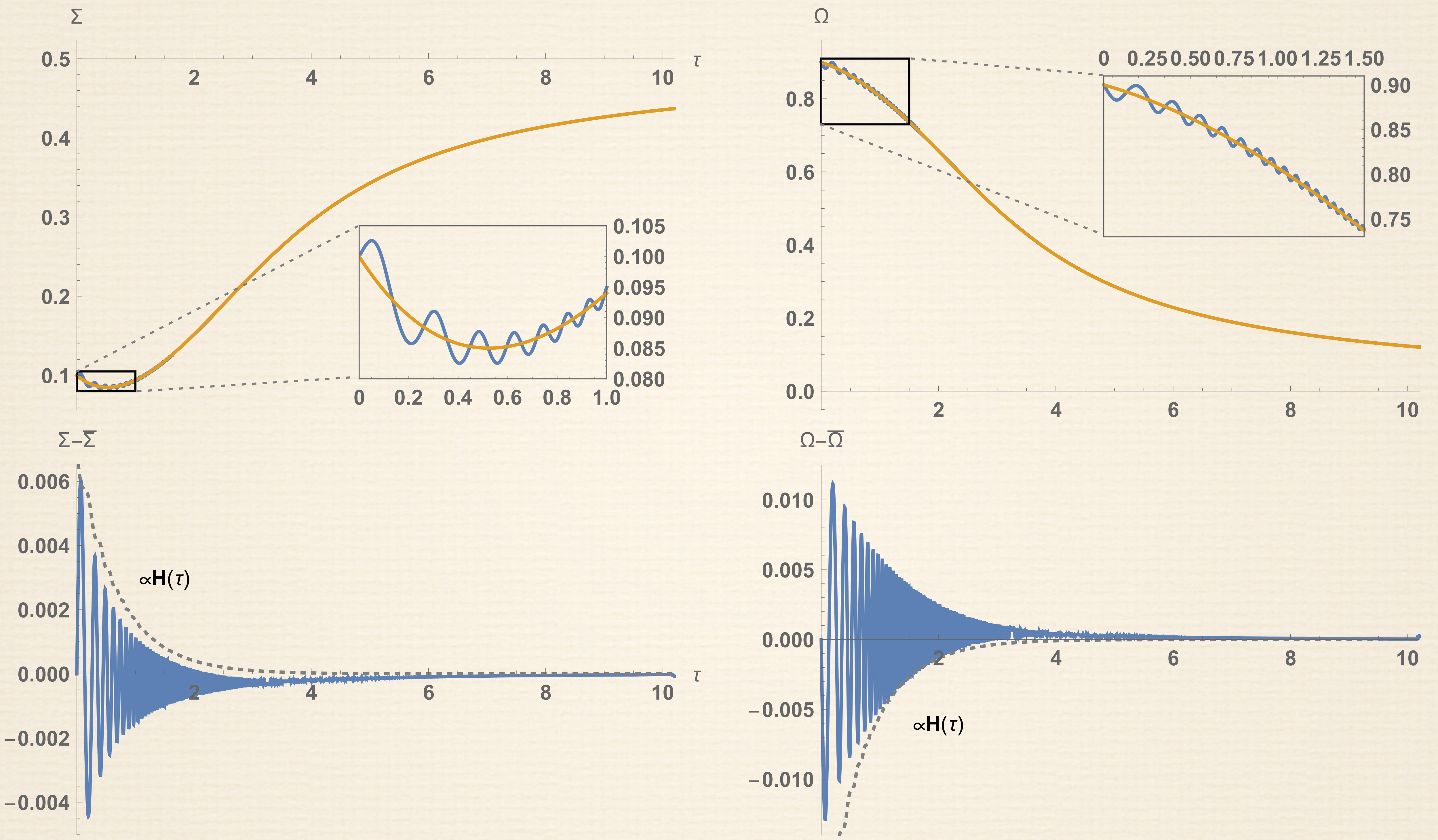
  $\mathbf{g} = -\,\mathrm{d}t^2 + a(t)^2\mathrm{d}r^2 + b(t)^2\mathbf{g}_{H^2}$

  with $a(t) \approx c_1 \ln t$ and $b(t) \approx c_2 t$.

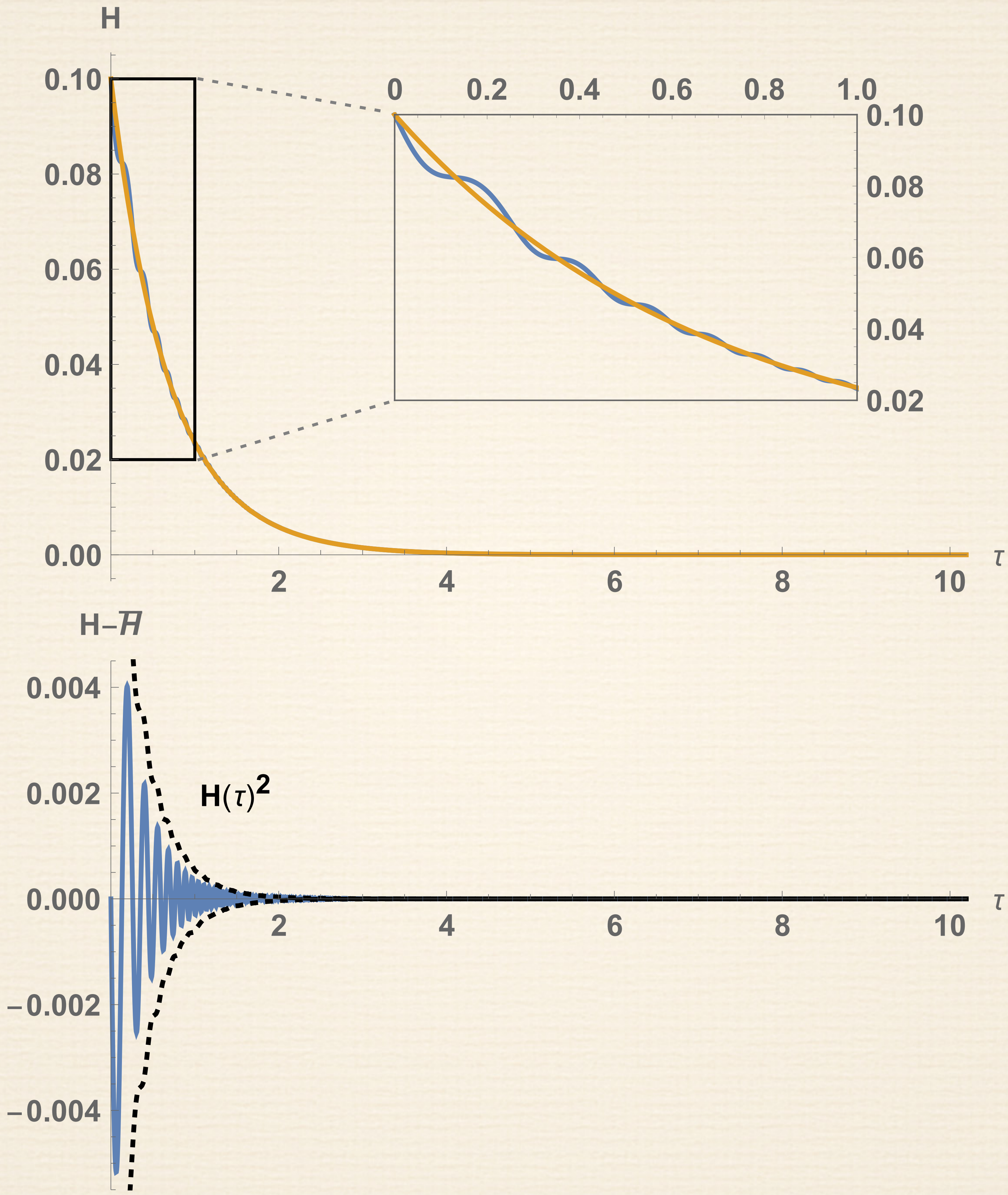
- Compare to vacuum solutions: $a(t) \approx c_1$, $b(t) \approx c_2 t$



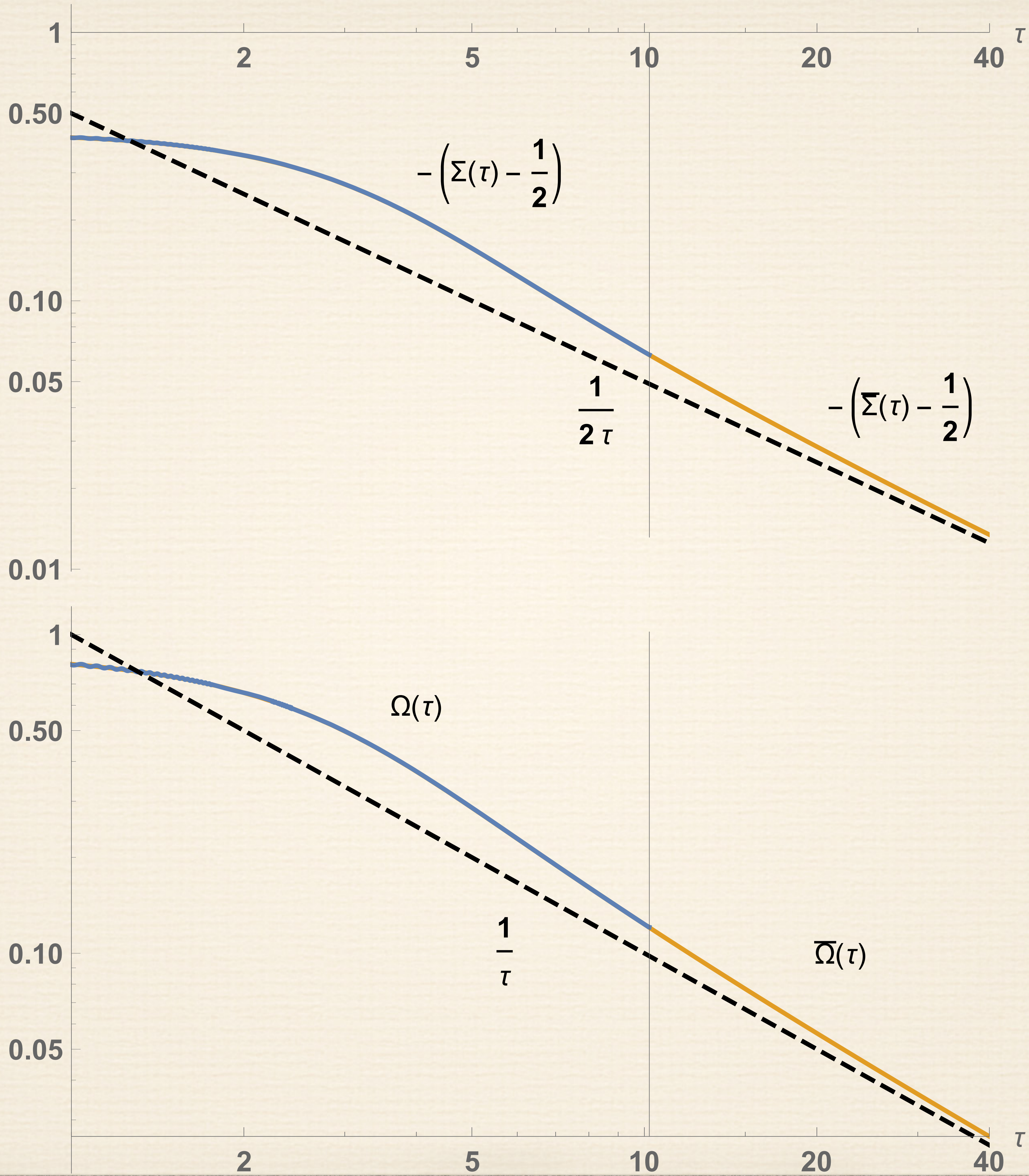
$\overline{\Sigma}_+^2 + \overline{\Omega} = 1$
F
1
1/2
$\overline{\Omega}$
0
T
$\overline{\Sigma}_+$
D
Q
−1
0
1



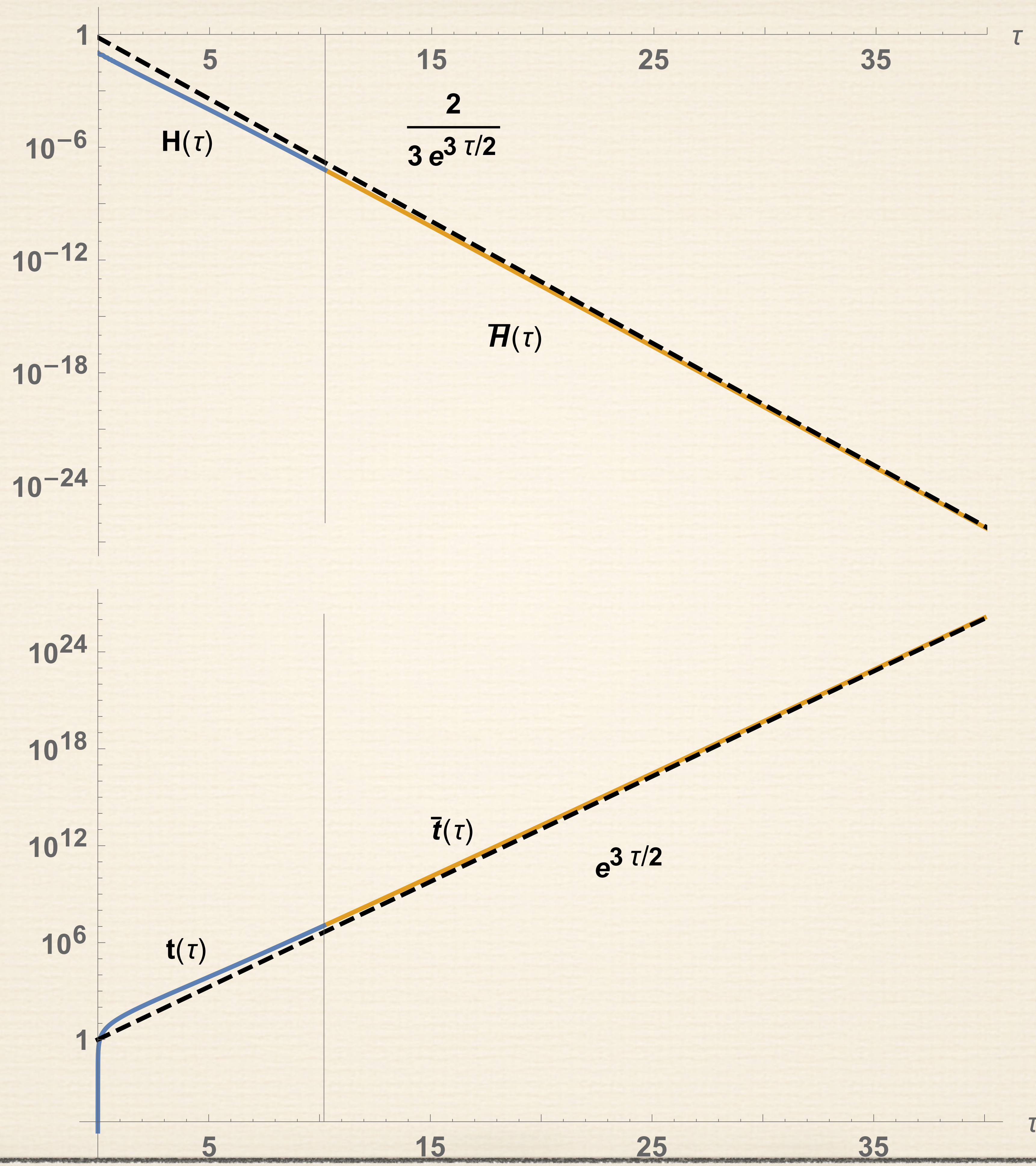
Σ
τ
Ω
Σ−Σ̅
∝H(τ)
Ω−Ω̅
∝H(τ)



H
0.10
0.08
0.06
0.04
0.02
0.00
2
4
6
8
10
τ
0
0.2
0.4
0.6
0.8
1.0
0.10
0.08
0.06
0.04
0.02
H−H̄
0.004
0.002
0.000
−0.002
−0.004
H(τ)²
2
4
6
8
10
τ



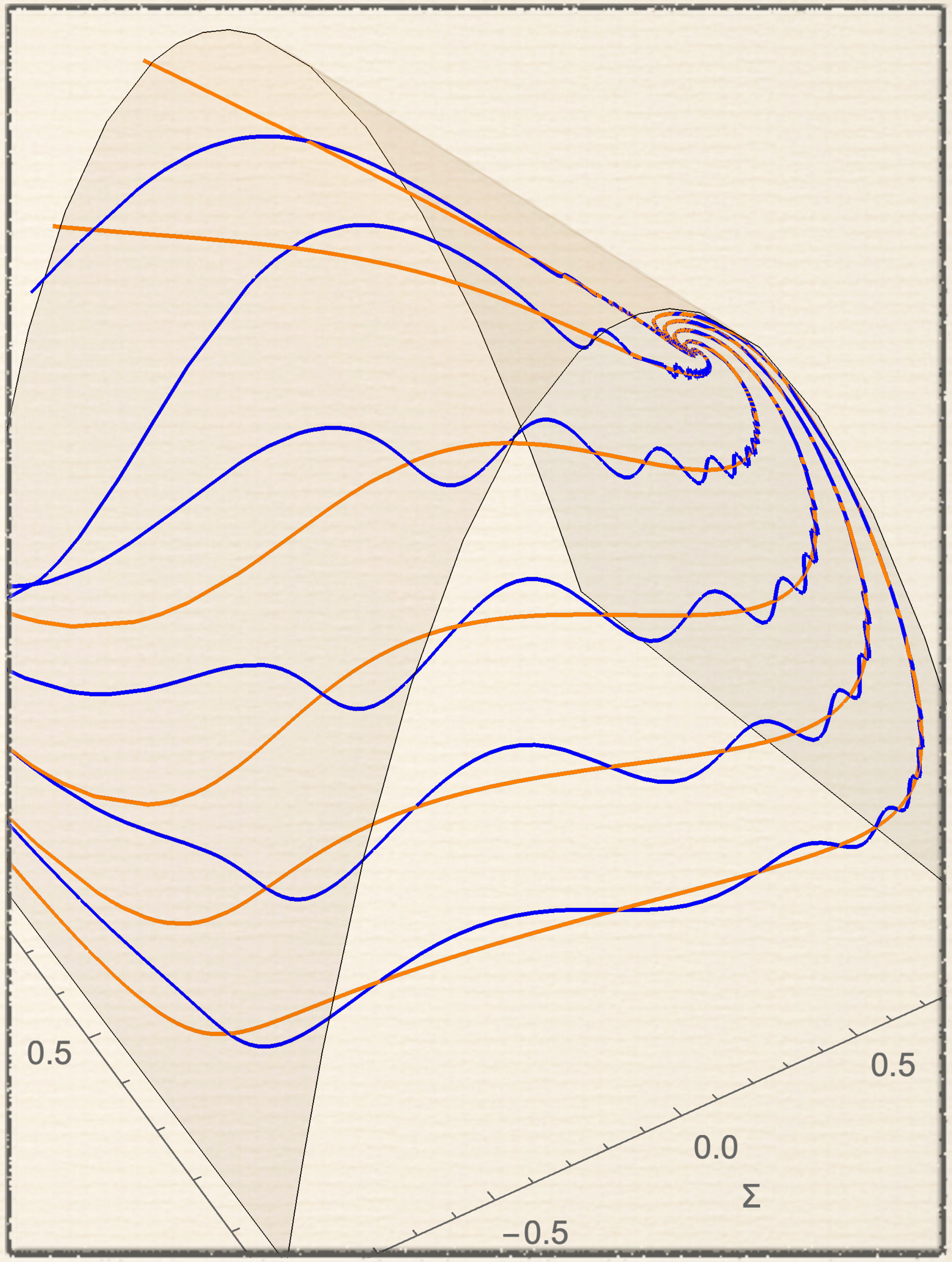
$-\left(\Sigma(\tau)-\frac{1}{2}\right)$
$\frac{1}{2\tau}$
$-\left(\overline{\Sigma}(\tau)-\frac{1}{2}\right)$
$\Omega(\tau)$
$\frac{1}{\tau}$
$\overline{\Omega}(\tau)$
τ
1
0.50
0.10
0.05
0.01
2
5
10
20
40



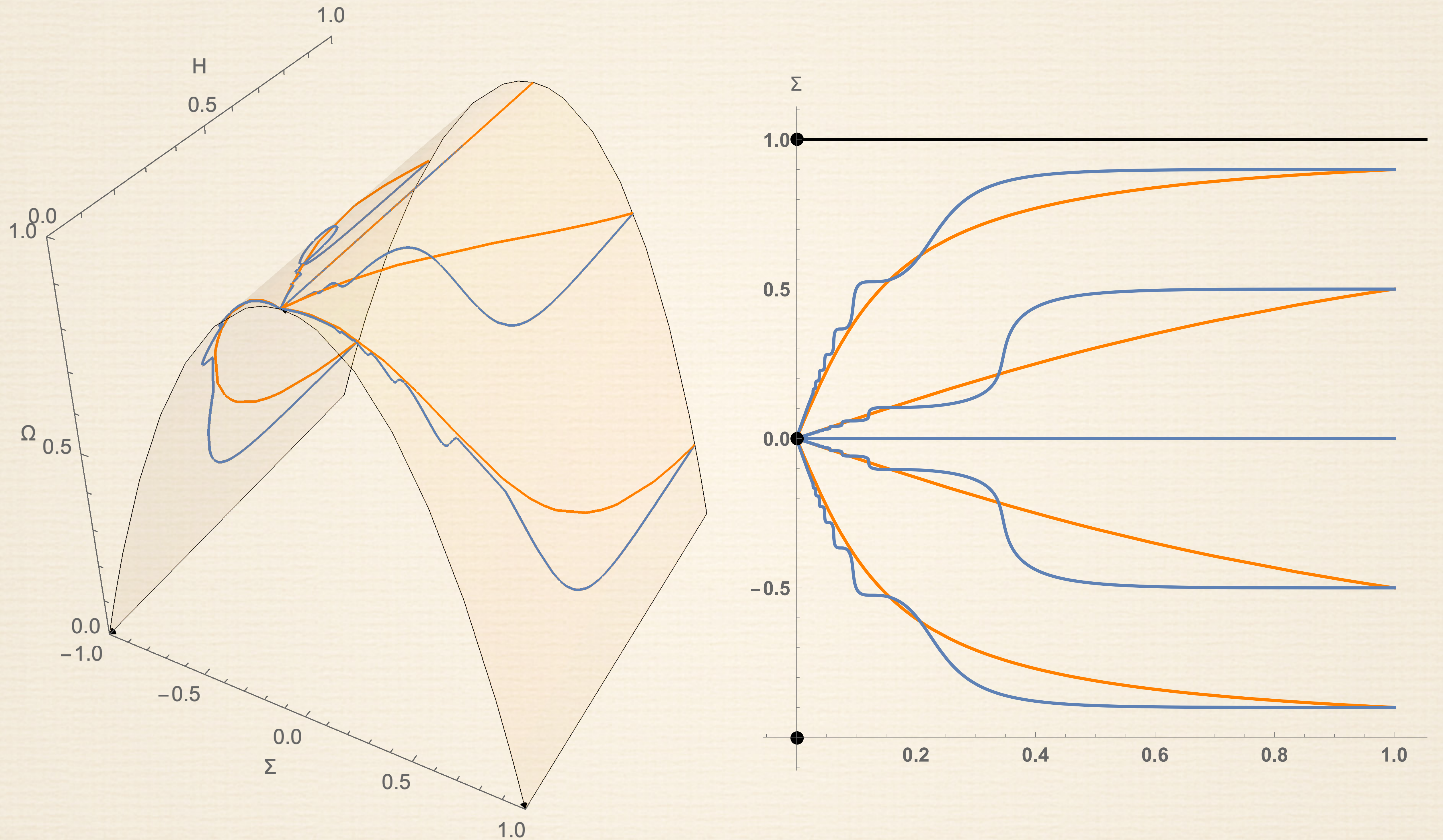
1
5
15
25
35
τ
10^{-6}
10^{-12}
10^{-18}
10^{-24}
H(τ)
$\frac{2}{3\,e^{3\,\tau/2}}$
$\bar{H}(\tau)$
10^{24}
10^{18}
10^{12}
10^{6}
1
t(τ)
$\bar{t}(\tau)$
$e^{3\,\tau/2}$
5
15
25
35
τ

LRS Bianchi I & II

D Fajman, G Heißel

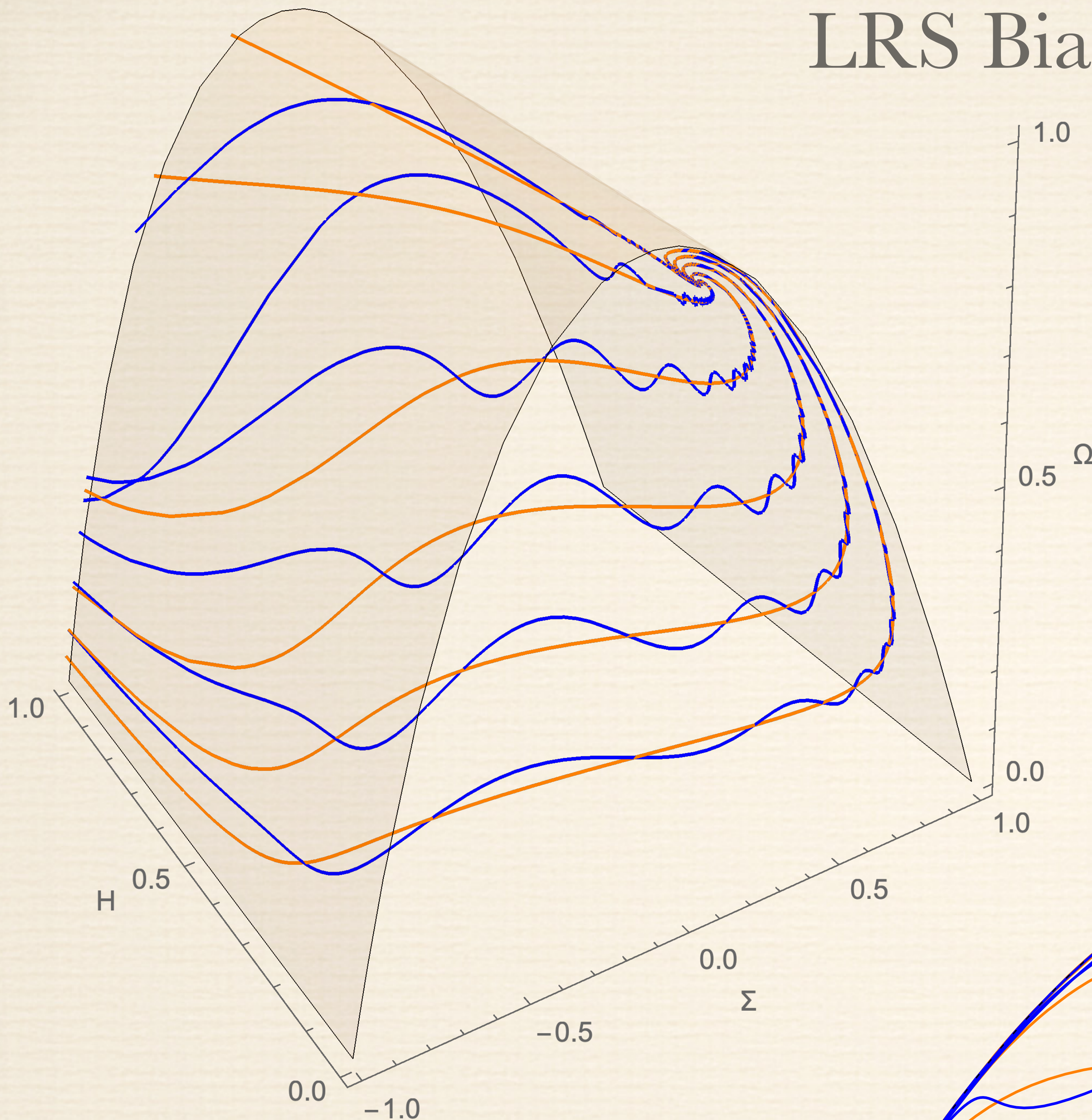


LRS Bianchi I

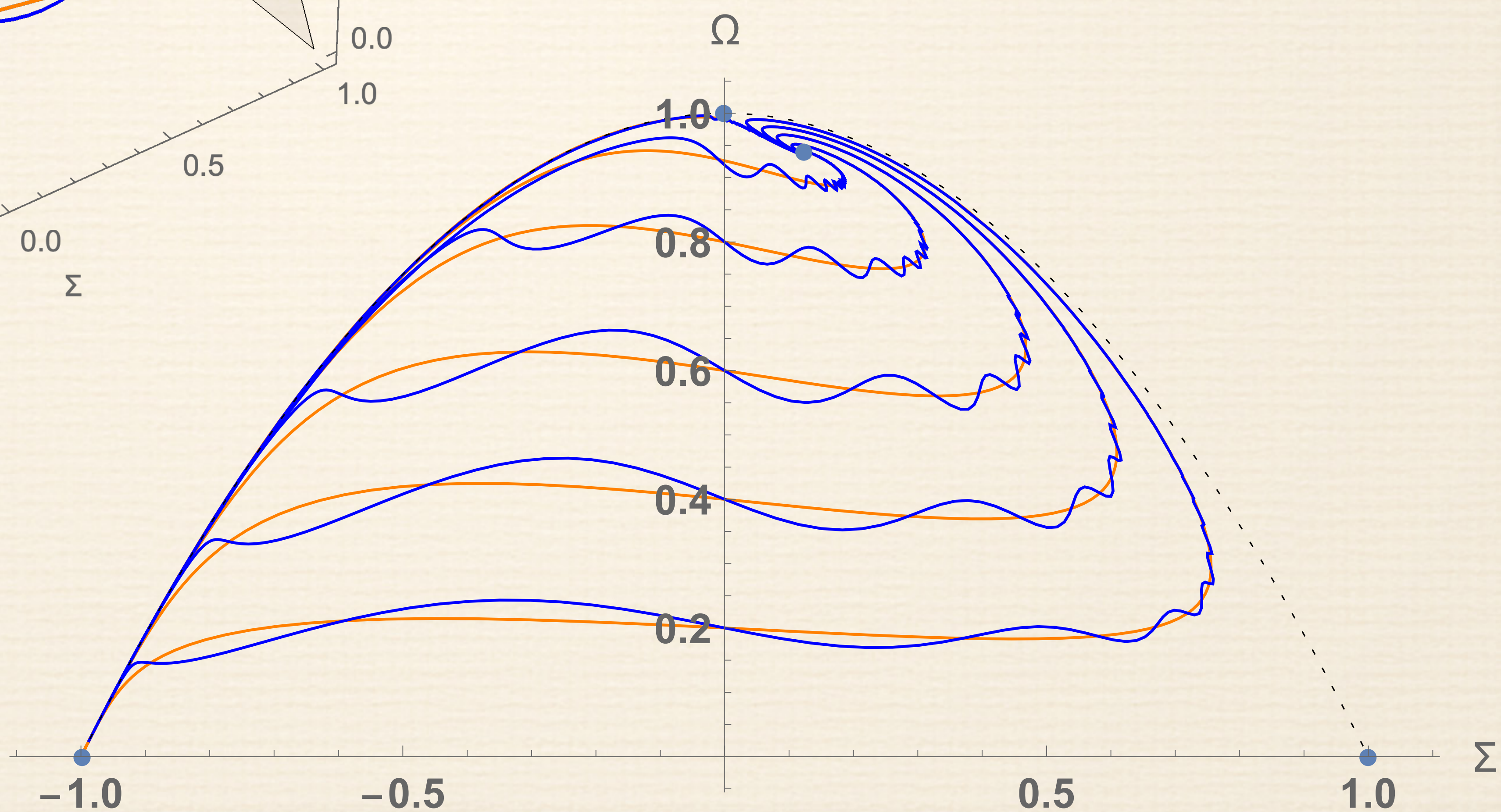


Asymptotics: Einstei-De-Sitter $\quad a(t), b(t) \propto t^{2/3}$

LRS Bianchi II



Asymptotics
Collins-Stewart (dust)
$a(t) \propto \sqrt{t}, b(t) \propto t^{3/4}$



Thm: Averaging with time dependent perturbation

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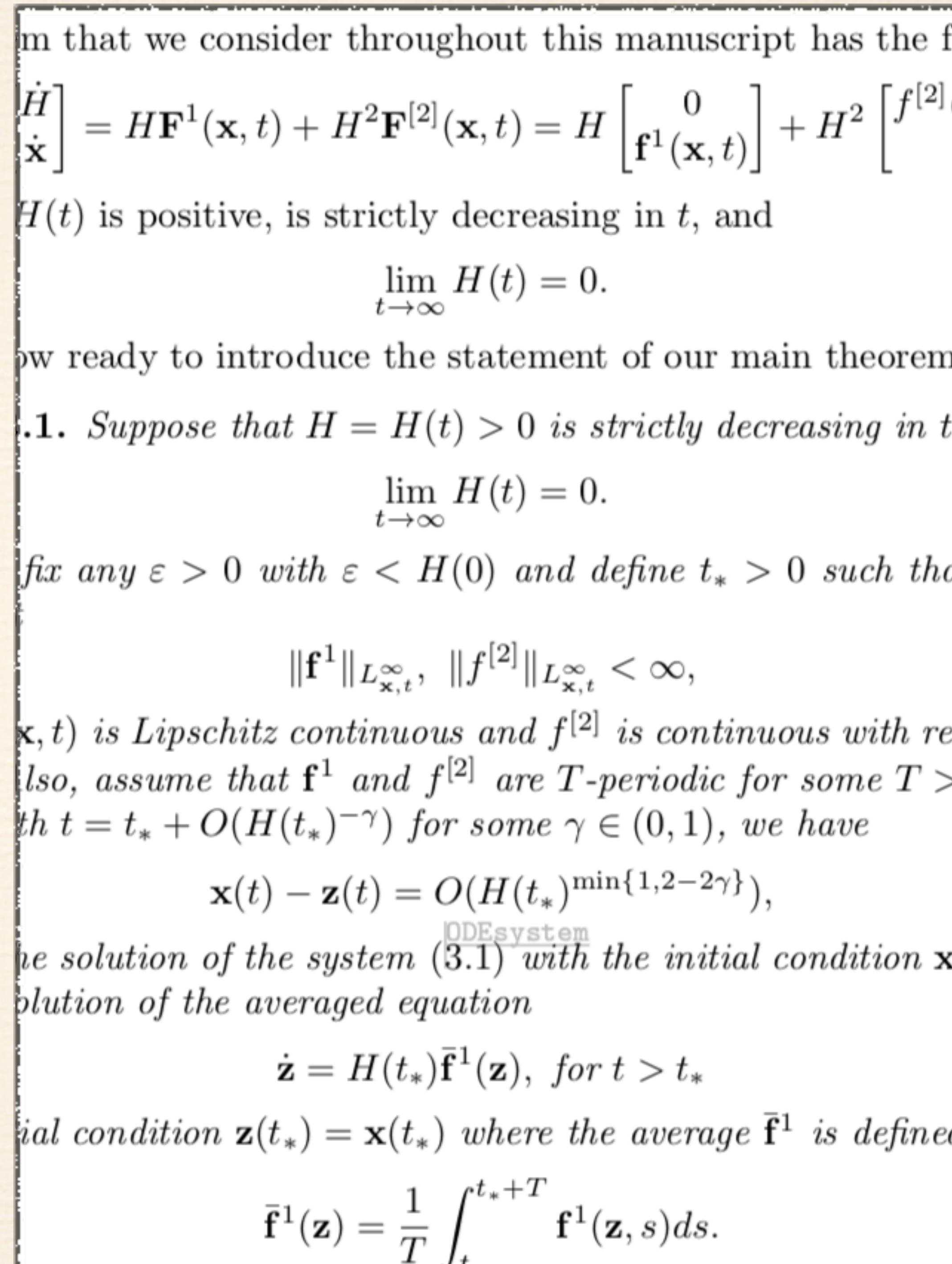

m that we consider throughout this manuscript has the fo

$$\begin{bmatrix} \dot{H} \\ \dot{\mathbf{x}} \end{bmatrix} = H\mathbf{F}^1(\mathbf{x},t) + H^2\mathbf{F}^{[2]}(\mathbf{x},t) = H\begin{bmatrix} 0 \\ \mathbf{f}^1(\mathbf{x},t) \end{bmatrix} + H^2\begin{bmatrix} f^{[2]}(\\ 0 \end{bmatrix}$$

$H(t)$ is positive, is strictly decreasing in t, and

$$\lim_{t\to\infty} H(t) = 0.$$

ow ready to introduce the statement of our main theorem:

.1. *Suppose that $H = H(t) > 0$ is strictly decreasing in t,*

$$\lim_{t\to\infty} H(t) = 0.$$

fix any $\varepsilon > 0$ with $\varepsilon < H(0)$ and define $t_* > 0$ such that

$$\|\mathbf{f}^1\|_{L^\infty_{\mathbf{x},t}},\ \|f^{[2]}\|_{L^\infty_{\mathbf{x},t}} < \infty,$$

$\mathbf{x}, t)$ is Lipschitz continuous and $f^{[2]}$ is continuous with res
lso, assume that $\mathbf{f}^1$ and $f^{[2]}$ are T-periodic for some $T >$
th $t = t_* + O(H(t_*)^{-\gamma})$ for some $\gamma \in (0,1)$, we have

$$\mathbf{x}(t) - \mathbf{z}(t) = O(H(t_*)^{\min\{1,2-2\gamma\}}),$$

he solution of the system (3.1) with the initial condition $\mathbf{x}($
olution of the averaged equation

$$\dot{\mathbf{z}} = H(t_*)\bar{\mathbf{f}}^1(\mathbf{z}),\ \text{for } t > t_*$$

ial condition $\mathbf{z}(t_*) = \mathbf{x}(t_*)$ where the average $\bar{\mathbf{f}}^1$ is defined

$$\bar{\mathbf{f}}^1(\mathbf{z}) = \frac{1}{T}\int_{t_*}^{t_*+T} \mathbf{f}^1(\mathbf{z},s)ds.$$

$$\begin{bmatrix} \dot{H} \\ \dot{\mathbf{x}} \end{bmatrix} = H \begin{bmatrix} 0 \\ \mathbf{f}^1(\mathbf{x}, t) \end{bmatrix} + H^2 \begin{bmatrix} f^{[2]}(\mathbf{x}, t) \\ \mathbf{0} \end{bmatrix} \qquad (3.1)$$

Theorem 3.1 (Local-in-time asymptotic). *Suppose that $H = H(t) > 0$ is strictly decreasing in t, and*

$$\lim_{t \to \infty} H(t) = 0.$$

Choose and fix any $\varepsilon > 0$ with $\varepsilon < H(0)$ and define $t_* > 0$ such that $\varepsilon = H(t_*)$. Suppose that

$$\|\mathbf{f}^1\|_{L^\infty_{\mathbf{x},t}}, \ \|f^{[2]}\|_{L^\infty_{\mathbf{x},t}} < \infty,$$

and that $\mathbf{f}^1(\mathbf{x}, t)$ is Lipschitz continuous and $f^{[2]}$ is continuous with respect to $\mathbf{x}$ for all $t \geq t_*$. Also, assume that $\mathbf{f}^1$ and $f^{[2]}$ are T-periodic for some $T > 0$. Then for all $t > t_*$ with $t = t_* + O(H(t_*)^{-\gamma})$ for some $\gamma \in (0, 1)$, we have

$$\boxed{\mathbf{x}(t) - \mathbf{z}(t) = O(H(t_*)^{\min\{1, 2-2\gamma\}}),}$$

where $\mathbf{x}$ is the solution of the system (3.1) with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{z}(t)$ is the solution of the averaged equation

$$\dot{\mathbf{z}} = H(t_*)\bar{\mathbf{f}}^1(\mathbf{z}), \ \textit{for } t > t_*$$

with the initial condition $\mathbf{z}(t_*) = \mathbf{x}(t_*)$ where the average $\bar{\mathbf{f}}^1$ is defined as

$$\bar{\mathbf{f}}^1(\mathbf{z}) = \frac{1}{T} \int_{t_*}^{t_*+T} \mathbf{f}^1(\mathbf{z}, s) ds.$$

Theorem 3.2 (Global-in-time asymptotic). *Assume the same assumptions of Theorem 3.1. Then we have*

$$\lim_{\tau \to \infty} \|\mathbf{x}(\tau) - \mathbf{z}(\tau)\| = 0.$$

- Sufficient to prove LRS Bianchi I & II asymptotics
- Insufficient to prove LRS Bianchi III asymptotics

Thank you!