The Weyl BMS group and Einstein's equations

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Mainly based on arXiv:2104.05793/2104.12881 with Laurent Freidel, Daniele Pranzetti and Simone Speziale

Motivation and outline

Asymptotic symmetries are crucial to study:

- Gravitational waves

  in asymptotically flat spacetimes → Bondi-Metzner-Sachs (BMS) group, implications for waveforms

- Infrared sector of gauge theories

  interplay among asymptotic symmetries, soft theorems and memory effects (the Strominger's triangle)

- the holographic nature of gravity

  e.g., Brown-Henneaux in AdS_3 with Dirichlet boundary conditions and conformal transformations in CFT_2

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In this talk:

i. review the original, extended and generalized BMS groups

   Bondi-Sachs gauge, boundary conditions, and asymptotic symmetry group

ii. introduce the Weyl BMS (or BMSW) group

   new boundary conditions, properties of the BMSW group, action on the asymptotic phase space

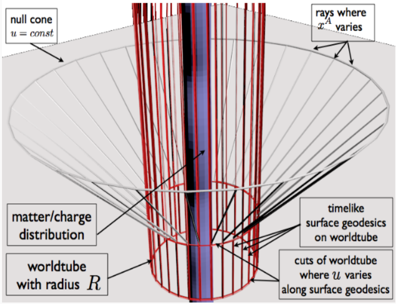
iii. the BMSW charge algebra and the asymptotic Einstein's equations

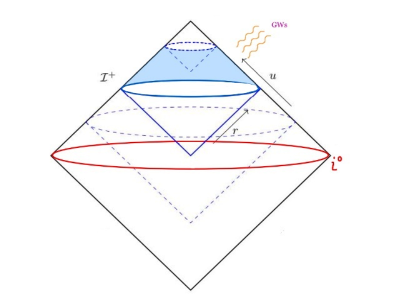
   new charge bracket, its algebra, obtaining asymptotic Einsteins' equations

iv. rediscovering BMSW: pushing the extended corner symmetry to infinity

The Weyl BMS group

The Bondi-Sachs gauge





[Madler&Winicour, 1609.01731]

Bondi coordinates: $x^\mu = (u, r, x^A)$.

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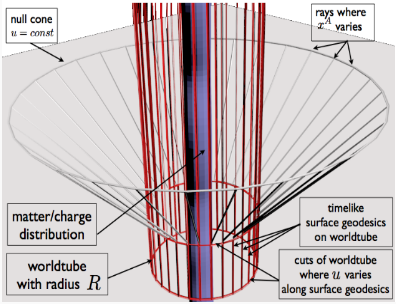

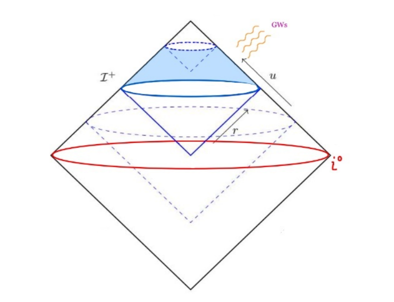

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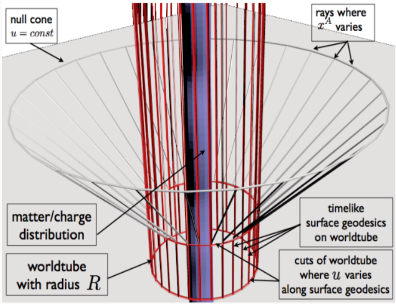

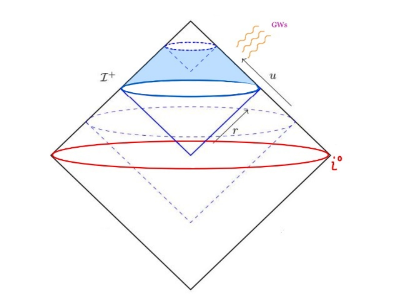

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Bondi-Sachs metric:

$$\boxed{ds^2 = -2e^{2\beta}du(Fdu + dr) + r^2 q_{AB}(dx^A - U^A du)(dx^B - U^B du),}$$

where β, F, U^A, and q_{AB} are functions of (u, r, x^A).

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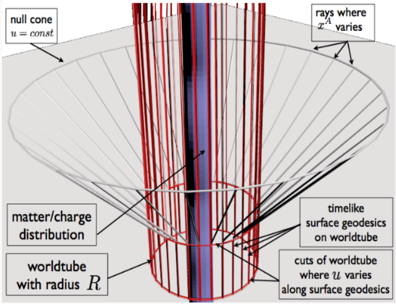

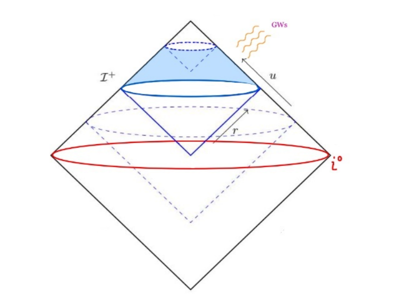

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What is the radial asymptotic behaviour of these quantities?

The boundary conditions (1/2)

The *ur* and *uA* components obey the fall-off conditions:

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$$\begin{cases} g_{uu} = -1 + \mathcal{O}(r^{-1}), & q_{AB} = \mathring{q}_{AB} + \mathcal{O}(r^{-1}) & \text{(original BMS)}, \\ g_{uu} = \mathcal{O}(r), & q_{AB} = e^{2\phi(u)}\mathring{q}_{AB} + \mathcal{O}(r^{-1}) & \text{(extended BMS)}, \\ g_{uu} = \mathcal{O}(1), & q_{AB} = \bar{q}_{AB} + \mathcal{O}(r^{-1}) & \text{(generalized BMS)}, \end{cases}$$

o-BMS: $\mathring{q}_{AB}$ is the round metric on S^2 with Ricci scalar $\mathring{R} = 2$ [Bondi-Metzner-Sachs, 1962]
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Remark 1: $\partial_u \bar{q}_{AB} = 0$ implies that $g_{uu} = \mathcal{O}(1)$.

Enough to describe MPM spacetimes [Blanchet et al, 2021]

Remark 2: relaxing bcs → divergences → phase-space renormalization!

in AdS/CFT adding boundary action counter-terms [deHaro-Solodukhin-Skenderis, 2001], [Compère-Marolf, 2008]

in g-BMS adding boundary Lagrangian (and associated symplectic potential) [Compère et al., 2018]

Additional investigation of this issue in [Freidel-Geiller-Pranzetti, 2020]

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$$g_{uu} := -2Fe^{2\beta} \quad \Longrightarrow \quad F = \bar{F} - \frac{M}{r} + \mathcal{O}(r^{-2})\,,$$

$$g_{ur} := -e^{2\beta} \quad \Longrightarrow \quad \beta = \frac{\bar{\beta}}{r^2} + \mathcal{O}(r^{-3})\,,$$

$$g_{uA} := -r^2 q_{AB} U^B \Longrightarrow U^A = \frac{\bar{U}^A}{r^2} - \frac{2}{3r^3}\bar{q}^{AB}\left(P_B + C_{BC}\bar{U}^C + \partial_B\bar{\beta}\right) + \mathcal{O}(r^{-4})\,,$$

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The asymptotic Einstein's equations read as

$$\bar{F} = \frac{1}{4}\bar{R}, \quad \bar{\beta} = -\frac{1}{32} C_{AB} C^{AB}, \quad \bar{U}^A = -\frac{1}{2}\bar{D}_B C^{AB}, \quad \partial_u \bar{q}_{AB} = 0,$$

together with the Bondi mass and angular momentum loss formulae ($N_{AB} \equiv \partial_u C_{AB}$)

$$\partial_u M = -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{4}\bar{D}_A \bar{D}_B N^{AB} + \frac{1}{2}\bar{\Delta}\bar{F},$$

$$\partial_u P_A = \{\text{terms linear and quadratic in the shear } C_{AB} \text{ and news } N_{AB}\}$$

also known as flux-balance laws! PN expanded in ([Nichols, 2017-18], [Compère-RO-Seraj, 2019]).

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AIM: We want to derive these equations using asymptotic symmetries!

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gives

$$\boxed{\xi^u = \tau, \quad \xi^r = -rW + \frac{r}{2} \left[D_A \left(I^{AB} \partial_B \tau \right) + U^A \partial_A \tau \right], \quad \xi^A = Y^A - I^{AB} \partial_B \tau}$$

Here τ, W, and Y^A are functions of (u, x^A), and $I^{AB} = \int_r^{+\infty} dr' e^{2\beta} q^{AB} / r'^2$. Moreover, we allow the scale structure to vary:

$$\boxed{\delta_\xi \sqrt{\bar{q}} = \left(D_A Y^A - 2W \right) \sqrt{\bar{q}}}$$

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$$\mathcal{L}_\xi g_{uu} = \mathcal{O}(1), \quad \mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_\xi g_{uA} = \mathcal{O}(1), \quad \mathcal{L}_\xi g_{AB} = \mathcal{O}(r^2)$$

gives

$$\boxed{\tau = T + uW, \quad \partial_u W = 0 = \partial_u T, \quad \partial_u Y^A = 0}$$

$T(x^A)$: super-translations; $W(x^A)$: Weyl rescaling of S^2; $Y^A(x^B)$: diffeos of S^2.

The boundary symmetry algebra

The leading order BMSW vector fields are

$$\boxed{\bar{\xi}_{(T,W,Y)} := T\partial_u + Y^A\partial_A + W\left(u\partial_u - r\partial_r\right)}$$

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we obtain the $\mathfrak{bmsw}$ Lie algebra: $\left(\text{diff}(S^2) \oplus \mathbb{R}_W\right) \oplus \mathbb{R}_T$.

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Comparison with previous literature:

	background structure	restriction	parametrisation
bmsw	$\emptyset$	$\emptyset$	(T, W, Y)
generalized bms	scale structure	$\delta\sqrt{q} = 0$	$(T, \frac{1}{2}D_A Y^A, Y)$
extended bms	conformal structure	$\delta[q_{AB}] = 0$	$(e^{\phi}t, \frac{1}{2}(D_A Y^A - w), Y)$
original bms	round sphere structure	$\delta q_{AB} = 0$	$(T, \frac{1}{2}D_A Y^A, Y)$

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	group	algebra
bmsw	$(\mathrm{Diff}(S^2) \ltimes \mathbb{R}_W) \ltimes \mathbb{R}_T$	$(\mathrm{diff}(S^2) \oplus \mathbb{R}_W) \oplus \mathbb{R}_T$
generalized bms	$\mathrm{Diff}(S^2) \ltimes \mathbb{R}_T$	$\mathrm{diff}(S^2) \oplus \mathbb{R}_T$
extended bms	$(\text{loc-CKV}(S^2)_{\mathring{q}} \ltimes \mathbb{R}_T) \times \mathbb{R}_W$	$(\mathrm{Vir} \oplus \mathrm{Vir} \oplus \mathbb{R}_T) \oplus \mathbb{R}_W$
original bms	$\text{glob-CKV}(S^2)_{\mathring{q}} \ltimes \mathbb{R}_T$	$\mathrm{SL}(2,\mathbb{C}) \oplus \mathbb{R}_T$

Action on the asymptotic phase space

The asymptotic phase space is parametrized by $\Phi^i = (\bar{F}, \bar{\beta}, \bar{U}^A, \bar{q}_{AB}, M, P_A, C_{AB})$.

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$$\begin{aligned}\delta_{(\tau,Y)}\Phi^i &= \delta_{\bar{\xi}_{(\tau,Y)}}\Phi^i + \Delta_\xi \Phi^i \\ &= [\tau\partial_u + \mathcal{L}_Y - s\dot{\tau}]\Phi^i + L^A_{\Phi^i}\bar{D}_A\tau + \tilde{L}^A_{\Phi^i}\bar{D}_A\dot{\tau} + Q^{AB}_{\Phi^i}\bar{D}_A\bar{D}_B\tau + \tilde{Q}^{AB}_{\Phi^i}\bar{D}_A\bar{D}_B\dot{\tau}\end{aligned}$$

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A special class of functionals with no quadratic anomaly: the covariant functionals.
For instance, the covariant mass $\mathcal{M} := M + \frac{1}{8}C_{AB}N^{AB}$ transforms as

$$\delta_{(\tau,Y)}\mathcal{M} = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{M} + M^A\partial_A\tau, \quad M^A = \frac{1}{2}\bar{D}_B N^{AB} + \bar{D}^A\bar{F}$$

In particular, the covariant functionals

$$\boxed{\mathcal{P}_A := P_A - 2\bar{D}_A\bar{\beta} - \frac{1}{2}C_{AB}\bar{U}^B, \quad \mathcal{M} := M + \frac{1}{8}C_{AB}N^{AB}, \quad M^A = \frac{1}{2}\bar{D}_B N^{AB} + \bar{D}^A\bar{F}}$$

are the leading Weyl scalars ψ_1, $\mathrm{Re}(\psi_2)$, and ψ_3 respectively.

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$$\delta_{(\tau,Y)}\mathcal{M} = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{M} + M^A\partial_A\tau, \quad M^A = \frac{1}{2}\bar{D}_B N^{AB} + \bar{D}^A\bar{F}$$

In particular, the covariant functionals

$$\boxed{\mathcal{P}_A := P_A - 2\bar{D}_A\bar{\beta} - \frac{1}{2}C_{AB}\bar{U}^B, \quad \mathcal{M} := M + \frac{1}{8}C_{AB}N^{AB}, \quad M^A = \frac{1}{2}\bar{D}_B N^{AB} + \bar{D}^A\bar{F}}$$

are the leading Weyl scalars ψ_1, $\mathrm{Re}(\psi_2)$, and ψ_3 respectively.

Manifold of flat vacua is defined by $\mathcal{M} = 0$, $M^A = 0$, $\dot{N}_{AB} = 0$.
Vacuum sector – orbits of the BMSW group: $|\tau, Y\rangle = \hat{e}_{(\tau,Y)}|0\rangle$.

The Weyl BMS, its charge algebra and Einstein's equations

Canonical analysis

Nomenclature:
$\{d, i_\xi, \mathcal{L}_\xi\}$, spacetime differential, contraction and Lie derivative: $\mathcal{L}_\xi = di_\xi + i_\xi d$
$\{\delta, I_\xi, \delta_\xi\}$, field space differential, contraction and variation: $\delta_\xi = \delta I_\xi + I_\xi \delta$.

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(assume that L and θ_L have no anomalies for the sake of presentation)

Noether's theorems say that

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$$\Omega = \int_\Sigma \delta\theta_L, \quad Q_\xi = \int_{S^2} q_\xi, \quad \mathcal{F}_\xi = \int_{S^2} (i_\xi \theta_L + q_{\delta_\xi})$$

obey the fundamental canonical relation (see e.g., [Lee-Wald, 1990], [Iyer-Wald, 1994])

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Contracting again (using antisymmetry in the second step),

$$-I_\chi I_\xi \Omega \approx \delta_\chi Q_\xi - I_\chi \mathcal{F}_\xi \approx -\left(\delta_\xi Q_\chi - I_\xi \mathcal{F}_\chi\right)$$

Charge bracket

Using only the antisymmetry of the symplectic form Ω:

$$I_\chi I_\xi \Omega \approx \delta_\xi Q_\chi - I_\xi \mathcal{F}_\chi$$

suggesting the definition of the charge bracket (generalizes [Barnich-Troessaert, 2011])

$$\boxed{\{Q_\xi, Q_\chi\}_L := \delta_\xi Q_\chi - I_\chi \mathcal{F}_\xi + \int_{S^2} i_\xi i_\chi L}$$

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$$[\![\xi, \chi]\!] := [\xi, \chi]_{Lie} + \delta_\chi \xi - \delta_\xi \chi$$

s.t. the commutator of two field space variations is still a symmetry transformation

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This is the flux-balance relation, equivalent to (asymptotic) Einstein's equations.

Obtaining the asymptotic Einstein's equations (1/2)

Interplay among:
geometric data – phase-space data – dynamics

$$\{Q_\xi, Q_\chi\}_L + Q_{[\![\xi,\chi]\!]} \approx 0 \Longleftrightarrow \delta_\xi Q_\chi + Q_{[\![\xi,\chi]\!]} \approx I_\chi \mathcal{F}_\xi + \int_{S^2} i_\chi i_\xi L$$

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BMSW generators (ξ, χ)	$\{Q_\xi, Q_\chi\} + Q_{[\![\xi,\chi]\!]} = 0$	**Einstein's equations**
(∂_u, ξ_T)	$2\mathsf{E}_M - \frac{1}{4}\bar{\Delta}\mathsf{E}_{\bar{F}} = 0$	$\xi_T^\mu G_\mu{}^r = 0$
(ξ_T, ∂_u)	$2\mathsf{E}_M + \bar{D}^A\dot{\mathsf{E}}_{\bar{U}_A} + \frac{1}{4}\bar{\Delta}\mathsf{E}_{\bar{F}} = 0$	$\xi_T^u G_u{}^r - \xi_T^r G_u{}^u = 0$
(∂_u, ξ_W)	$\bar{D}^A\mathsf{E}_{\bar{U}_A} + u\left(2\mathsf{E}_M - \frac{1}{4}\Delta\mathsf{E}_{\bar{F}}\right) = 0$	$\xi_W^\mu G_\mu{}^r = 0$
(ξ_W, ∂_u)	$-\bar{D}^A\mathsf{E}_{\bar{U}_A} + u\left(2\mathsf{E}_M + \bar{D}^A\dot{\mathsf{E}}_{\bar{U}_A} + \frac{1}{4}\Delta\mathsf{E}_{\bar{F}}\right) = 0$	$\xi_W^u G_u{}^r - \xi_W^r G_u{}^u = 0$
(∂_u, ξ_Y)	$\mathsf{E}_{\bar{P}_A} + 2\bar{D}_A\dot{\mathsf{E}}_{\bar{\beta}} - 2\bar{F}\mathsf{E}_{\bar{U}_A} - \frac{1}{2}\bar{U}_A\mathsf{E}_{\bar{F}} = 0$	$\xi_Y^\mu G_\mu{}^r = 0$
(ξ_Y, ∂_u)	$0 = 0$	$0 = 0$

-original BMS: 1 flux-balance (energy);
-generalized BMS: 3 flux-balances (energy, angular mom) – importance of diff(S^2);
-BMSW: 8 flux-balances – importance of the Weyl rescalings.

Obtaining the asymptotic Einstein's equations (2/2)

Interplay among:
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$$\boxed{\{Q_\xi, Q_\chi\}_L + Q_{[\![\xi,\chi]\!]} = \int_{S^2} i_\xi C_\chi \approx 0 \Longleftrightarrow \delta_\xi Q_\chi + Q_{[\![\xi,\chi]\!]} \approx I_\chi \mathcal{F}_\xi + \int_{S^2} i_\chi i_\xi L}$$

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-BMSW: 8 flux-balances – importance of the Weyl rescalings;
-extension of the BMSW?

Rediscovering BMSW: pushing extended corner symmetry to scri

Extended corner symmetry

Corner symmetry group: surface diffeomorphisms “plus” surface boosts

[Donnelly-Freidel, 2016], [Donnelly-Freidel-Moosavian-Speranza, 2020]

$$\mathfrak{g}_{S^2} = \mathsf{diff}(S^2) \oplus \mathfrak{sl}(2, \mathbb{R})$$

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Extended corner symmetry includes surface translations (see also [Ciambelli-Leigh, 2021])

$$\boxed{\mathfrak{g}^{ext}_{S^2} = \big(\text{diff}(S^2) \oplus \mathfrak{sl}(2, \mathbb{R})\big) \oplus \mathbb{R}^2}$$

To prove this, consider the following metric around the corner S^2:

$$ds^2 = h_{ab}dx^a dx^b + \gamma_{AB}(d\sigma^A - U^A_a dx^a)(d\sigma^B - U^B_b dx^b)$$

One defines $Y^A = \xi^A|_{x^a=0}$, $W_a{}^b = \partial_a \xi^b|_{x^a=0}$, $T^a = \xi^a|_{x^a=0}$ and the associated charges

$$P_A = \frac{1}{2}\gamma_{AB}\epsilon^{ab}(\partial_a + U^A_a \partial_A)U^B_b, \quad N_b{}^a = \frac{1}{2}h_{bc}\epsilon^{ca}$$

$$Q_a = \frac{1}{2}\epsilon^{cb}(\partial_b + U^A_b \partial_A)h_{ac} - U^B_a P_B - D_C(N_a{}^b U^C_b),$$

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Pushing these charges to scri, one gets (after renormalization) the $\mathfrak{bmsw}$ algebra

$$\boxed{\mathfrak{g}^{ext}_{S^2} = \big(\text{diff}(S^2) \oplus \mathfrak{sl}(2, \mathbb{R})\big) \oplus \mathbb{R}^2 \quad \xrightarrow{r \to \infty} \quad \mathfrak{bmsw} = \big(\text{diff}(S^2) \oplus \mathbb{R}_W\big) \oplus \mathbb{R}_T}$$

The factor $\mathfrak{sl}(2, \mathbb{R})$ is typical of GR; it might change in modified theories of gravity.
Deformation/extension of $\text{diff}(S^2)$? [Rojo-Prochazka-Sachs, 2021]

Conclusions

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Recap:

- new asymptotic symmetries in GR: the BMSW group;
- derivation of (asymptotic) Einstein's equations from first principles;
- phase-space renormalization (not discussed here, only mentioned!);
- vacua labelled by diff(S), in addition to super-translations and Weyl labels

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Follow-ups:

- relax also $\partial_u \bar{q}_{AB} = 0$;
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Other interesting directions:

- study asymptotically spatially flat FLRW
  [Bonga-Prabhu, 2020], [Rojo-Heckelbacher, 2020, 2021], [Rojo-Heckelbacher-RO, 2022]
- investigate the "triangle" in the cosmological setting
  (adiabatic modes, squeezed limits)
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THANK YOU FOR YOUR ATTENTION!