Isochrony and self-gravitating systems dynamics

Alicia Simon-Petit, Jérôme Perez, Guillaume Duval

June 28, 2018

References

2018, Communication in Mathematical Physics,
Alicia Simon-Petit, Jérôme Perez, Guillaume Duval.
Isochrony in 3D radial potentials.
Accepted, preprint: https://arxiv.org/abs/1804.11282.

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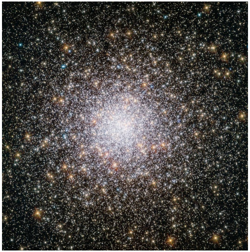

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Gravitational dynamics

Collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla\psi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

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- Distribution function $f(\mathbf{r}, \mathbf{v}, t)$.

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- Distribution function $f(\mathbf{r}, \mathbf{v}, t)$.
- Gravitation with potential:

$$\psi = -G \cdot \left(\mathcal{G} * \int f \mathrm{d}\mathbf{v} \right),$$

solving Poisson eq: $\Delta\psi = 4\pi G \underbrace{\int f \mathrm{d}\mathbf{v}}_{\text{density, } \rho}$.

Gravitational dynamics

Collisionless Boltzmann equation:

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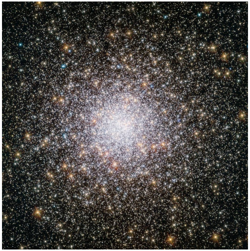
solving Poisson eq: $\Delta\psi = 4\pi G \underbrace{\int f \mathrm{d}\mathbf{v}}_{\text{density, } \rho}$.

- No collision

since $t \ll T_{\text{2-body relax}}$ in SGS.

Globular clusters



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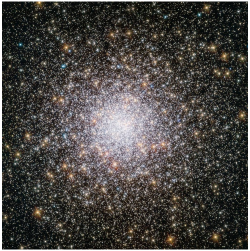

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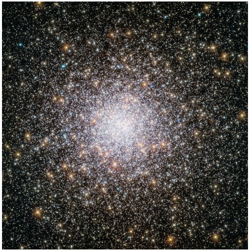
Which ψ ?

Globular clusters



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Which ψ ?

1. From observations: mass-luminosity relation $\nu \propto \rho$ and Poisson's equation $\Delta\psi = \rho$.

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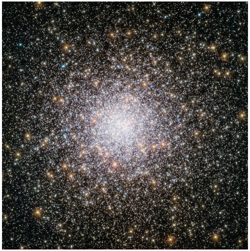

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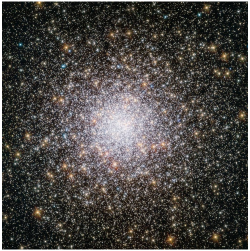
Which ψ ?

1. From observations: mass-luminosity relation $\nu \propto \rho$ and Poisson's equation $\Delta\psi = \rho$.
2. From theoretical models of GCs' evolution.

Globular clusters



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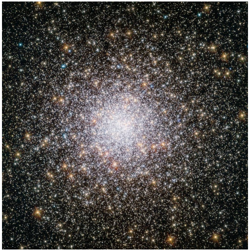
Which ψ ? $\Delta\psi = \rho$ with

Globular clusters



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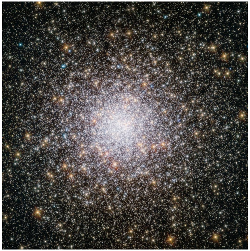
Which ψ ? $\Delta\psi = \rho$ with

- $\rho = \text{cst} \implies \psi_{\text{ha}}(r) = \frac{1}{2}\omega^2 r^2,$

Globular clusters



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Which ψ ? $\Delta\psi = \rho$ with

- $\rho = \text{cst} \implies \psi_{\text{ha}}(r) = \frac{1}{2}\omega^2 r^2,$
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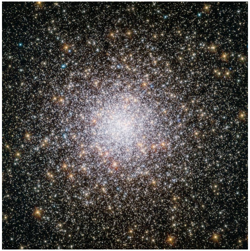

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Which ψ ? $\Delta\psi = \rho$ with

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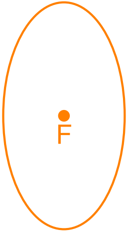

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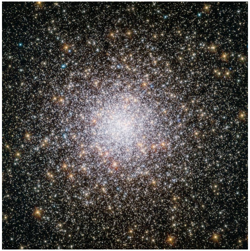

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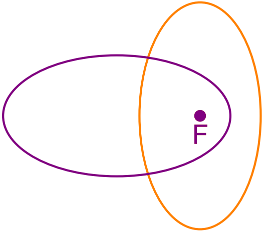

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- $\rho = \text{cst} \implies \psi_{\text{ha}}(r) = \frac{1}{2}\omega^2 r^2$, $\quad T = \frac{2\pi}{\omega}$
- $\rho = \text{“}\delta(\mathbf{r}_0)\text{”}M \implies \psi_{\text{ke}}(r) = -\frac{GM}{r}$, $T^2 \propto a^3 \propto |E|^{-3}$, (Kep. 3^{rd} law)

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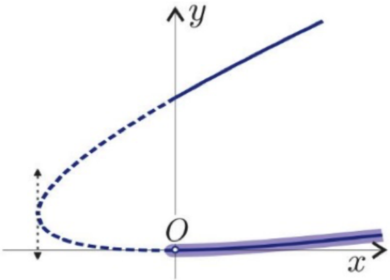

Isochrone potentials

Energy:

$$E = m\xi = \frac{1}{2}m\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{\Lambda^2}{2r^2}\right] + m\psi$$

provides the radial equation of motion in the trajectory plane given by Λ :

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{\Lambda^2}{2r^2} - (\xi - \psi) = 0 \qquad \text{(M)}$$

Isochrone potentials

Radial equation of motion in the trajectory plane given by Λ :

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{\Lambda^2}{2r^2} - (\xi - \psi) = 0 \tag{M}$$

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- For a given ψ, the set of radially periodic orbits is given by

$$\Theta_\psi = \left\{(\xi,\Lambda) \in \mathbb{R}^2,\ \text{s.t. } r(\cdot) \text{ solving (M) and periodic}\right\}.$$

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- If $(\xi, \Lambda) \in \Theta_\psi$, then the period

$$\tau_r(\xi, \Lambda) = 2\int_{r_{min}}^{r_{max}} \frac{\mathrm{d}r}{\sqrt{2\left(\xi - \psi(r)\right) - \frac{\Lambda^2}{r^2}}} < \infty.$$

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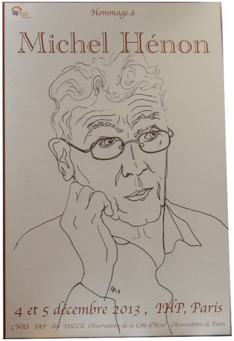
Look for isochrone potentials

Find all $\psi(r)$, s.t. $\Theta_\psi \neq \emptyset$ and $\forall\,(\xi, \Lambda) \in \Theta_\psi, \tau_r(\xi, \Lambda) \equiv \tau_r(\xi)$.

Isochrone geometric property

Idea : $\begin{pmatrix} r \\ \psi \end{pmatrix} \to \begin{pmatrix} x = 2r^2 \\ Y(x) = x\psi(x) \end{pmatrix}$,

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \xi - \frac{\Lambda^2}{2r^2} - \psi(r). \qquad \text{(M)}$$



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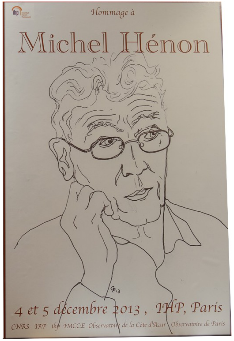
Idea : $\begin{pmatrix} r \\ \psi \end{pmatrix} \to \begin{pmatrix} x = 2r^2 \\ Y(x) = x\psi(x) \end{pmatrix}$,

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2 = \xi x - \Lambda^2 - Y(x). \qquad \text{(M)}$$



Isochrone geometric property

Idea : $\begin{pmatrix} r \\ \psi \end{pmatrix} \to \begin{pmatrix} x = 2r^2 \\ Y(x) = x\,\psi(x) \end{pmatrix}$,

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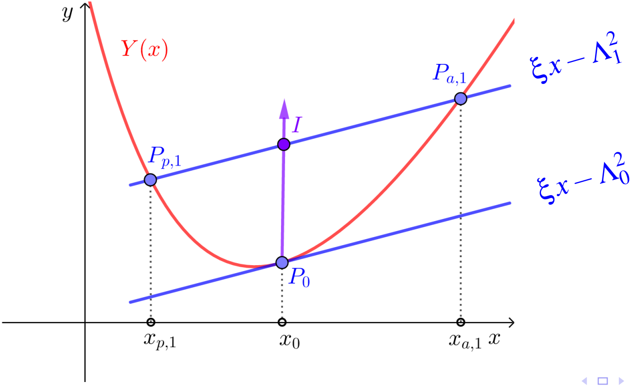
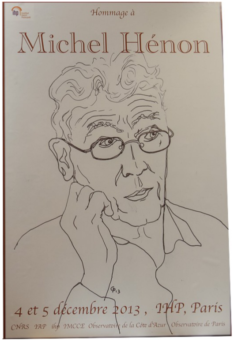
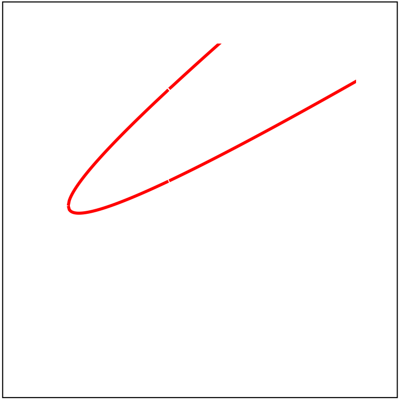


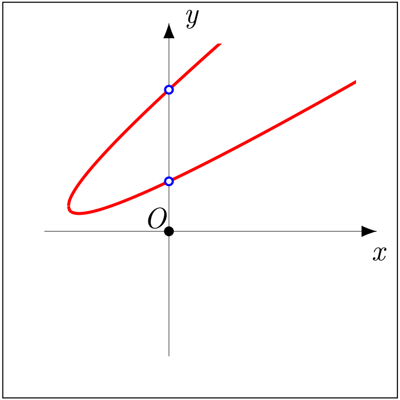


Isochrone parabolas



$$Y(x) = x\psi(x)$$

Isochrone parabolas



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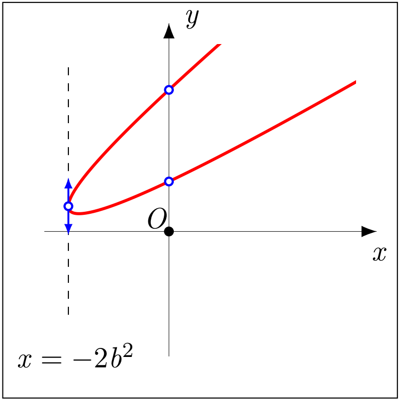

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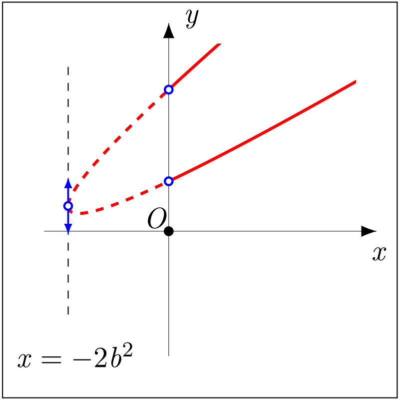

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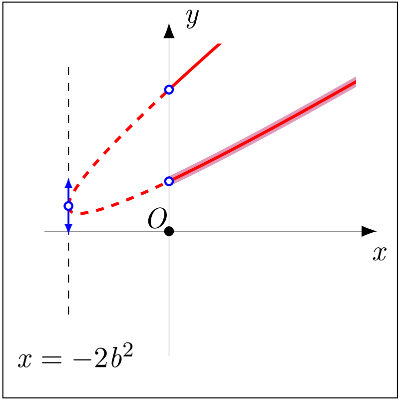

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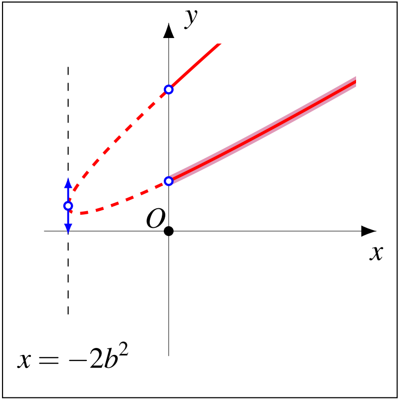
Isochrone parabolas

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2 = \xi x - \Lambda^2 - Y(x). \qquad \text{(M)}$$



ε−transvection: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad Y(x) = x\,\psi(x)$

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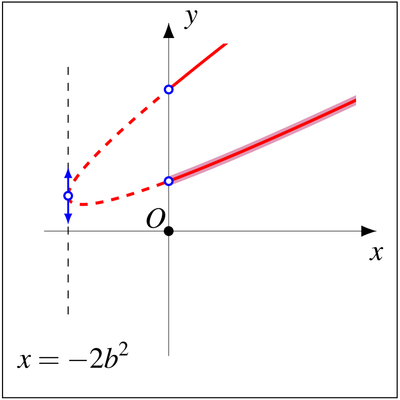

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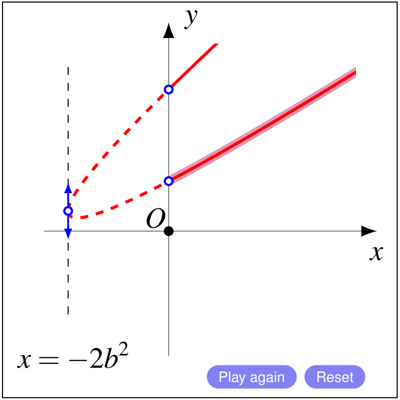

$\varepsilon-$transvection: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad Y(x) = x\,\psi(x)$

Isochrone parabolas

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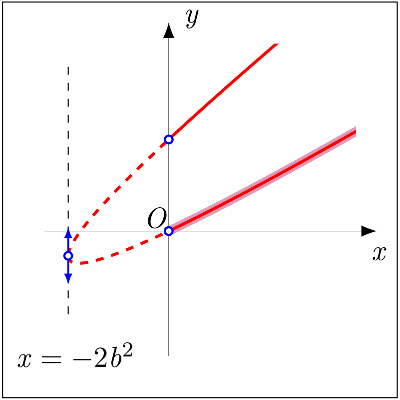
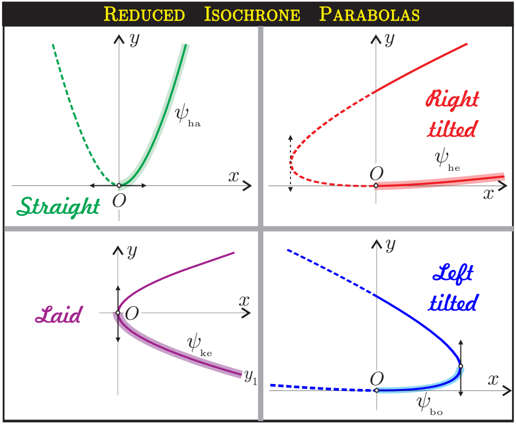


λ−translation: $y \to y+\lambda,\ \ Y(x) = x\,\psi(x)$

Isochrone potentials



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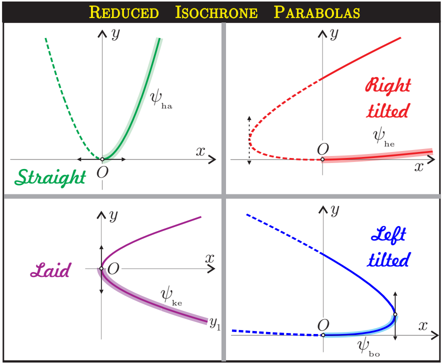

Theorem

There are four types of isochrone potentials: ψ_{ke}, ψ_{ha}, ψ_{he} *and* ψ_{bo}*.*
Any isochrone potential is in the group orbit of $\{\psi_{\text{ke}}, \psi_{\text{ha}}, \psi_{\text{he}}, \psi_{\text{bo}}\}$
under the group action of $\mathbb{A} = \{\varepsilon - \text{transvections}, \Lambda - \text{translations}\}$*.*

Isochrone potentials

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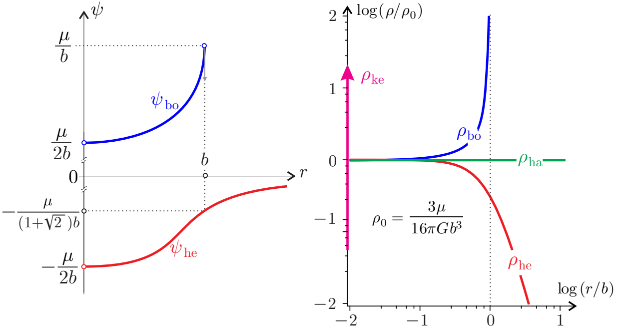

Hénon, bounded potentials. Mass densities.

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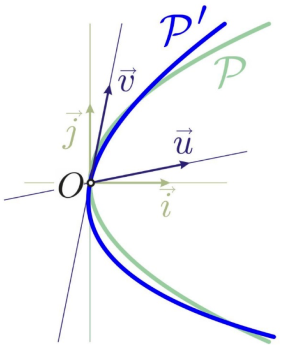

From harmonic to Kepler

From harmonic to Kepler

Harmonic

$$\mathcal{H}_{\mathrm{ha}} = \frac{1}{2}\left(p_q^2 + \frac{p_\theta^2}{q^2}\right) + \underbrace{\frac{1}{2}\omega q^2}_{\psi_{\mathrm{ha}}(q)=\frac{1}{2}\omega q^2}$$

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Canonical transformation:

$$(q, \theta, p_q, p_\theta) \to (x, \theta, p_x, p_\theta)$$

with $\begin{vmatrix} \dfrac{\partial x}{\partial q} & \dfrac{\partial x}{\partial p_q} \\ \dfrac{\partial p_x}{\partial q} & \dfrac{\partial p_x}{\partial p_q} \end{vmatrix} = 1.$

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$$\mathcal{H}_{\mathrm{ha}} = \frac{4x}{l}\left[\frac{1}{2}\left(p_x^2 + \frac{p_\theta^2}{4x^2}\right) + \frac{\omega^2 l^2}{8}\right].$$

From harmonic to Kepler

Harmonic

$$\psi_{\mathrm{ha}}(q) = \frac{1}{2}\omega q^2$$

Kepler

$$\psi_{\mathrm{ke}}(r) = -\frac{\mu}{r}$$

Canonical transformation:

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$$\mathcal{H}_{\mathrm{ke}} = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{\mu}{r}$$

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$$\mathcal{H}_{\mathrm{ke}} = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{\mu}{r}$$

Noting

$$\boxed{-\frac{\omega^2 l^2}{8}} = \frac{1}{2}\left(p_x^2 + \boxed{\frac{p_\theta^2}{4x^2}}\right) \boxed{-\frac{l\mathcal{H}_{\mathrm{ha}}}{4x}}$$

and setting $= \mathcal{H}_{\mathrm{ke}} = \xi_{\mathrm{ke}}$ $\quad p_\varphi = \frac{p_\theta}{2}$ $\quad = -\frac{\mu}{x}$

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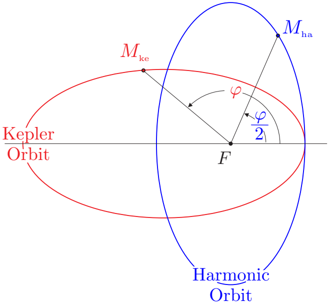

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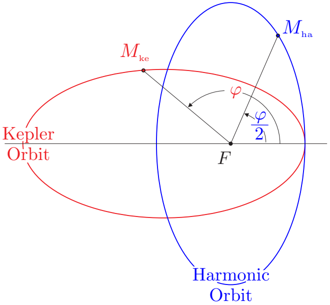

Aka : Goursat, Darboux, Levi-Civita, Bohlin transformation ($z \mapsto \frac{1}{2}z^2$), etc.

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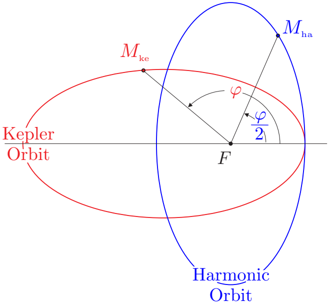
$$\psi_{\mathrm{ke}}(r) = -\frac{\mu}{r}$$

$$\mathcal{H}_{\mathrm{ke}} = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{\mu}{r}$$



Aka : Goursat, Darboux, Levi-Civita, Bohlin transformation ($z \mapsto \frac{1}{2}z^2$), etc.

$$\xi x \xleftrightarrow[\text{exchange}]{\text{Total}} Y(x) = x\psi(x)$$

Bolsts

Partial exchange $\xi x \leftrightarrow Y(x)$ preserving isochrony ?

Bolsts

Partial exchange $\xi x \leftrightarrow Y(x)$ preserving isochrony ?

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2 + \Lambda^2 = \xi x - Y(x), \qquad \frac{1}{16}\left(\frac{dx'}{dt'}\right)^2 + (\Lambda')^2 = \xi' x' - Y'(x').$$

Linear exchanges between ξx and y:

$$\xi x - y = \xi' x' - y',$$

Bolsts

Partial exchange $\xi x \leftrightarrow Y(x)$ preserving isochrony ?

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2+\Lambda^2=\xi x-Y(x), \qquad \frac{1}{16}\left(\frac{dx'}{dt'}\right)^2+(\Lambda')^2=\xi' x'-Y'(x').$$

Linear exchanges between ξx and y:

$$\begin{pmatrix} \xi' x' \\ y' \end{pmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{pmatrix} \xi x \\ y \end{pmatrix}$$

Bolsts

Partial exchange $\xi x \leftrightarrow Y(x)$ preserving isochrony ?

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2+\Lambda^2=\xi x-Y(x), \qquad \frac{1}{16}\left(\frac{dx'}{dt'}\right)^2+(\Lambda')^2=\xi' x'-Y'(x').$$

Linear exchanges between ξx and y:

$$\begin{pmatrix} \xi' x' \\ y' \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} \xi x \\ y \end{pmatrix}$$

Bolsts

Partial exchange $\xi x \leftrightarrow Y(x)$ preserving isochrony ?

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2 + \Lambda^2 = \xi x - Y(x), \qquad \frac{1}{16}\left(\frac{dx'}{dt'}\right)^2 + (\Lambda')^2 = \xi' x' - Y'(x').$$

Linear exchanges between ξx and y:

$$\begin{pmatrix} \xi' x' \\ y' \end{pmatrix} = \begin{bmatrix} \alpha & \beta \\ \alpha - 1 & \beta + 1 \end{bmatrix} \begin{pmatrix} \xi x \\ y \end{pmatrix}$$

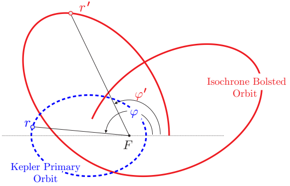
Bolsts

Theorem

When $\alpha\beta\xi' \neq 0$, the image of a keplerian PRO *by $B_{\alpha,\beta}$ is an isochrone orbit, with, given $\chi = \dfrac{p\alpha|\xi|}{\mu\beta}$, the non-linear relations*

$$\left(r'\right)^2 = \frac{\alpha\xi r^2 - \mu\beta r}{\xi'},$$

$$\varphi'(\varphi) = \frac{\varphi}{2} + \frac{\chi}{\sqrt{(1+\chi)^2 - e^2}} \arctan\left[\sqrt{\frac{1+\chi-e}{1+\chi+e}} \tan\left(\frac{\varphi}{2}\right)\right].$$



*i*Bolsts

Partial exchange $\xi x \leftrightarrow Y(x)$ preserving isochrony ?

$$\frac{1}{16}\left(\frac{dx}{dt}\right)^2 + \Lambda^2 = \xi x - Y(x)\,, \qquad \frac{1}{16}\left(\frac{dx'}{dt'}\right)^2 + (\Lambda')^2 = \xi' x' - Y'(x')$$

Symmetric linear exchanges between ξx and y:

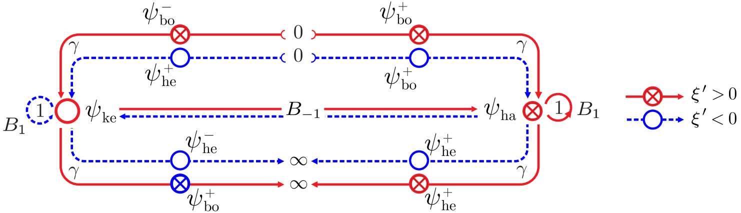
$$\begin{pmatrix} \xi' x' \\ y' \end{pmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} \gamma+1 & \gamma-1 \\ \gamma-1 & \gamma+1 \end{bmatrix}}_{B_\gamma} \begin{pmatrix} \xi x \\ y \end{pmatrix},$$

where B_γ invertible $\Leftrightarrow \gamma = \alpha + \beta \neq 0$.

Keplerian $\boldsymbol{i}\,\mathrm{Bolst}$ group orbit

Proposition

Any isochrone potential is in the group orbit of the kepler potential under the action of the $\boldsymbol{i}\,\mathrm{Bolst}$ *group* $\mathbb{B} = \left\{B_\gamma, \gamma \in \mathbb{R}^*\right\}$.



i Bolsts

Additive representation of $\mathbb{B}$:

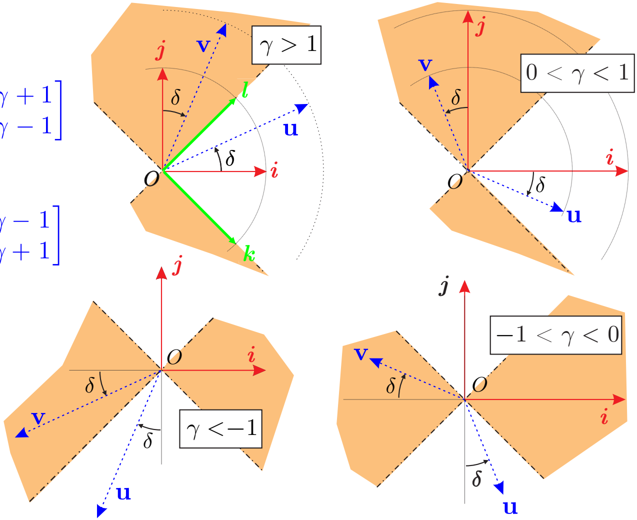
$$B_\chi = e^\chi \begin{bmatrix} \cosh(\chi) & \sinh(\chi) \\ \sinh(\chi) & \cosh(\chi) \end{bmatrix} \quad \text{when } \gamma > 0,$$

with $\gamma = e^{2\chi}$.

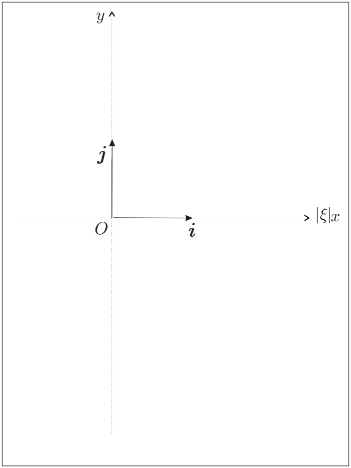
$\boldsymbol{i}$Bolsts action

$$\mathbf{u}_{\mathcal{R}} = \frac{1}{2}\begin{bmatrix}\gamma+1\\ \gamma-1\end{bmatrix}$$

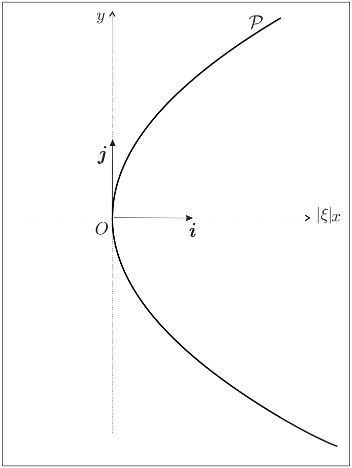
$$\mathbf{v}_{\mathcal{R}} = \frac{1}{2}\begin{bmatrix}\gamma-1\\ \gamma+1\end{bmatrix}$$



i Bolsts action



$\boldsymbol{i}$ Bolsts action



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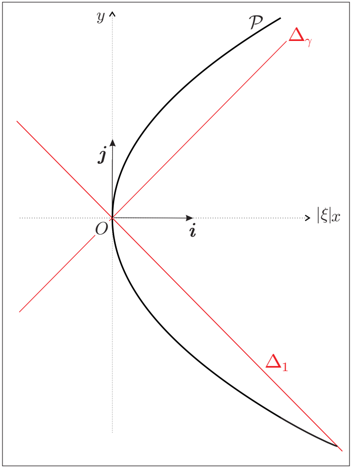

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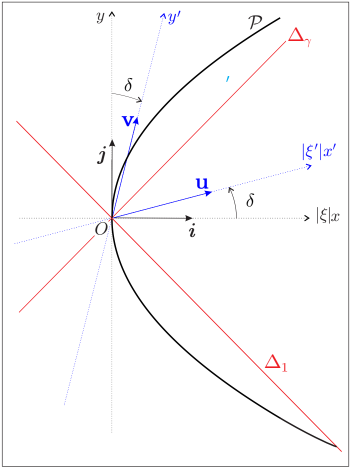
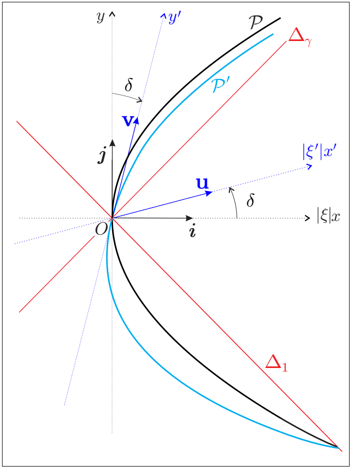


$$\tan\delta = \left|\frac{\gamma+1}{\gamma-1}\right|$$

i Bolsts action



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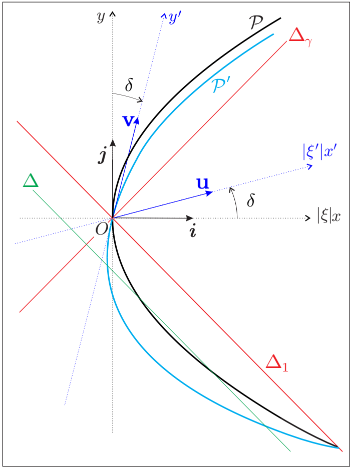
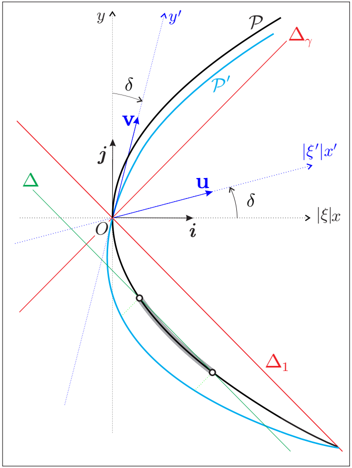


iBolsts action



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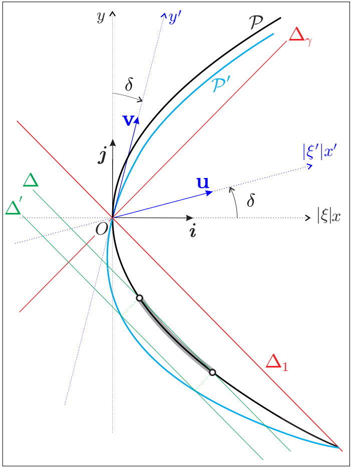
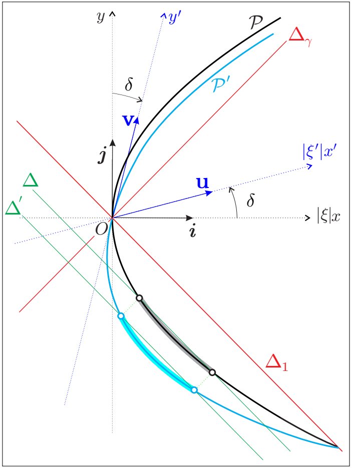


$$\Lambda' = \sqrt{|\gamma|}\,\Lambda$$

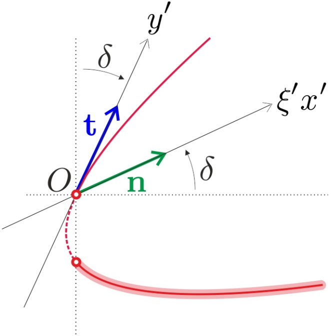
i Bolsts action



Reference frames

Definition

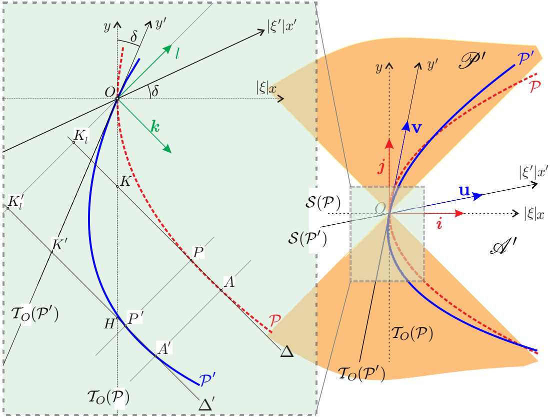
The reference frame of a given parabola $\mathcal{P}$ is the frame $(O, \mathbf{t}, \mathbf{n})$ where the tangent to the parabola at the origin is $\mathcal{T}_O(\mathcal{P}) = \mathbb{R}\mathbf{t}$ and the symmetry axis is $\mathcal{S}(\mathcal{P}) = \mathbb{R}\mathbf{n}$.



Isochrony is keplerian in reference frames

Theorem

An orbit is isochrone $\Leftrightarrow$ *it is the* $\boldsymbol{i}$ Bolsted *image of a keplerian orbit.*



$\boldsymbol{i}$Bolsts ...

Consider $\mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ and $\mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$ the two eigenvectors of the $\boldsymbol{i}$Bolst B_γ such that

$$B_\gamma(\mathbf{k}) = \mathbf{k} \text{ and } B_\gamma(\mathbf{l}) = \gamma\mathbf{l}.$$

With the affine coordinates system $(w_1 = \xi x, w_2 = y)$ and setting $\mathbf{w}' = B_\gamma(\mathbf{w})$, then

$$\left\{ \begin{array}{c} \xi' x' - y' = \xi x - y \\ \xi' x' + y' = \gamma(\xi x + y) \end{array} \right. \implies (\xi' x')^2 - y'^2 = \gamma\left[(\xi x)^2 - y^2\right].$$

Isochrone relativity and special relativity

- Einstein principle of special relativity: the laws of physics are written in the same way in all galilean frames;
- The length of any space-time interval, $c^2dt^2 - x^2$, is conserved through changes of galilean frames.

Isochrone relativity and special relativity

- Isochrone principle of relativity: the laws of motion are written in the same way in all reference frames;
- The length of the *"isochrone interval"*, $\xi x - y$, is conserved through changes of reference frames.

Isochrone relativity and special relativity

Isochrone principle of relativity

In the canonical frame $\mathcal{R}_O$, with proper time $d\tau = \xi dt$, a keplerian orbit (ξ, Λ^2) in the affine coordinates $(\xi x, y)$ verifies

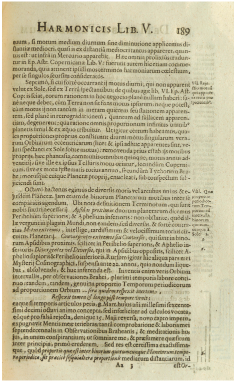
$$\frac{1}{16}\left[\frac{d}{d\tau}(\mathbf{w}|i)\right]^2 = (\mathbf{w}|i-j) + (\mathbf{w}_\Lambda|j)$$

where $\mathbf{w}_\Lambda = -\Lambda^2 j$.
In the bolsted frame $\mathcal{R}'_O$ with affine coordinates $(\xi' x', y')$ and proper time $d\tau' = \xi' dt'$, the bolsted orbital differential equation reads

$$\frac{1}{16}\left[\frac{d}{d\tau'}(\mathbf{w}'|\mathbf{u})\right]^2 = \ (\mathbf{w}'|\mathbf{u}-\mathbf{v}) + (\mathbf{w}'_\Lambda|\mathbf{v})$$

with $\mathbf{w}'_\Lambda = -\Lambda^2 \mathbf{v}$.

Contents



HARMONICIS LIB. V. 189

suum, si motum medium diurnum sine diminutione applicemus distantiæ mediocri, quasi is ex distantiâ mediocri tantus appareret, quantus est: ut infrà in Mercurio apparebit. Hæc omnia prolixiùs traduntur in Ep. Astr. Copernicanæ Lib. V: fuerunt autem hîc etiam commemoranda, quia attinent ipsissimos terminos harmoniarum cœlestium, per se singulos seorsim consideratos.

Septimò, si cui fortè occurrant ij motus diurni, qui non apparent velut ex Sole, sed ex Terrâ spectantibus; de quibus agit lib. VI. Ep. Ast. Cop: is sciat, eorum rationem in hoc negocio planè nullam haberi: sanè neque debet, cùm Terra non sit fons motus ipsorum; neque potest, cùm motus ij non tantùm in meram quietem seu stationem apparentem, sed planè in retrogradationem, quantum ad fallacem apparentiam, degenerent; qua ratione omnis proportionum infinitas omnib' planetis simul & ex æquo tribuitur. Ut igitur certum habeamus, quales proportiones proprias constituant diurni motus singularum verarum Orbitarum eccentricarum (licet & ipsi adhuc apparentes sint, veluti spectanti ex Sole fonte motus): removenda prius est ab ijs motibus proprijs, hæc phantasia, communis omnibus quinque, motus annui adventitij; sive ille ex ipsius Telluris motu oriatur, secundùm Copernicum; sive ex motu systematis totius annuo, secundùm Tychonem Brahe; motusque cuique Planetæ proprij, enucleati, sub conspectum subjiciendi sunt. VII. Rejectio motuũ qui apparẽt ex terra prospicientib'.

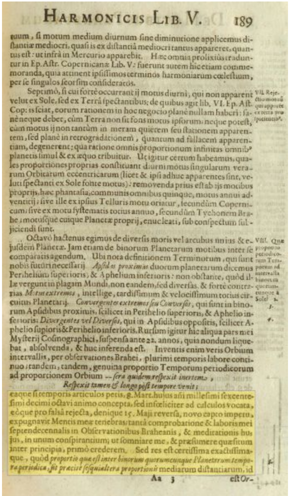
Octavò hactenus egimus de diversis moris vel arcubus unius & ejusdem Planetæ. Jam etiam de binorum Planetarum motibus inter se comparatis agendum. Ubi nota definitionem Terminorum, qui sunt nobis futuri necessarij. *Apsidas proximas* duorum planetarum dicemus Perihelium superioris, & Aphelium inferioris: non obstante, quòd illæ vergunt in plagam Mundi, non eandem, sed diversas, & fortè contrarias. *Motus extremos*, intellige, tardissimum & velocissimum totius circuitus Planetarij. *Convergentes extremos seu Conversos*, qui sunt in binorum Apsidibus proximis, scilicet in Perihelio superioris, & Aphelio inferioris: *Divergentes vel Diversos*, qui in Apsidibus oppositis, scilicet Aphelio supioris & Perihelio inferioris. Rursum igitur hic aliqua pars mei Mysterij Cosmographici, suspensa ante 22. annos, quia nondum liquebat, absolvenda, & huc inferenda est. Inventis enim veris Orbium intervallis, per observationes Brahei, plurimi temporis labore continuo; tandem, tandem, genuina proportio Temporum periodicorum ad proportionem Orbium -- *sera quidem respexit inertem*, VIII. Quæ propotio periodicorum Temporum ad intervalla binorum quorumcunque à Sole? 2. 3. 4.

Respexit tamen & longo pòst tempore venit;

eaque si temporis articulos petis, 8. Mart. hujus añi millesimi sexcentesimi decimi octavi animo concepta, sed infœliciter ad calculos vocata, eòque pro falsâ rejecta, denique 15. Maji reversa, novo capto impetu, expugnavit Mentis meæ tenebras; tantâ comprobatione & laboris mei septendecennalis in Observationibus Braheanis, & meditationis hujus, in unum conspirantium; ut somniare me, & præsumere quæsitum inter principia, primò crederem. Sed res est certissima exactissimaque, quòd *proportio quæ est inter binorum quorumcunque Planetarum tempora periodica, sit præcisè sesquialtera proportionis* mediarum distantiarum, id 5.

Aa 3 est *Or-*

Kepler's third law



HARMONICIS LIB. V. 189

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VII. Rejectio motuũ qui apparent ex terra spectantibus.

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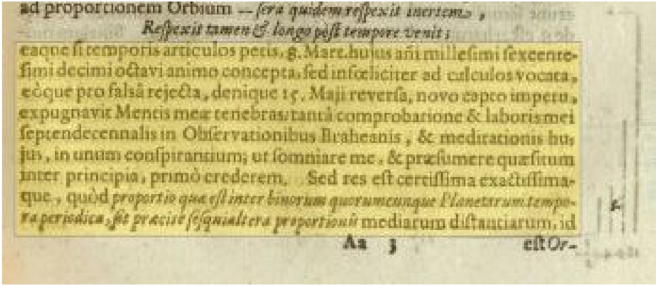
VIII. Quæ proportio periodicorum Temporum ad intervalla binorum quorumcunque à Sole? 1. 2. 3. 4.

Aa 3 est *Or-*

J. Kepler, *Harmonices Mundi*, 1619

Kepler's third law



ad proportionem Orbium — *sera quidem respexit inertem,*
Respexit tamen & longo post tempore venit;
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Aa 3 est *Or-*

J. Kepler, *Harmonices Mundi*, 1619

Isochrone semi-major axes

In $\psi_{\mathrm{ke}}(r) = -\dfrac{\mu}{r}$, define $a = \dfrac{1}{2}\,(r_a + r_p)$.

In $\psi_{\mathrm{he}}(r) = -\dfrac{\mu}{b + \sqrt{b^2 + r^2}}$, define $a = \dfrac{1}{2}\left(\sqrt{b^2 + r_a^2} + \sqrt{b^2 + r_p^2}\right)$.

In $\psi_{\mathrm{bo}}(r) = \dfrac{\mu}{b + \sqrt{b^2 - r^2}}$, define $a = \dfrac{1}{2}\left(\sqrt{b^2 - r_a^2} + \sqrt{b^2 - r_p^2}\right)$.

In ψ_{ha}^{R}, a homogeneous box of radius R, define $a = (1/2)^{2/3}\,R$.

Kepler's third law for isochrones

Theorem

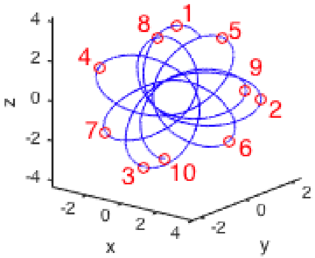
For any radially period orbit in an isochrone potential, the square of the radial period is proportional to the cube of the isochrone semi major axis:

$$\tau_r^2 = \frac{4\pi^2}{\mu} a^3$$

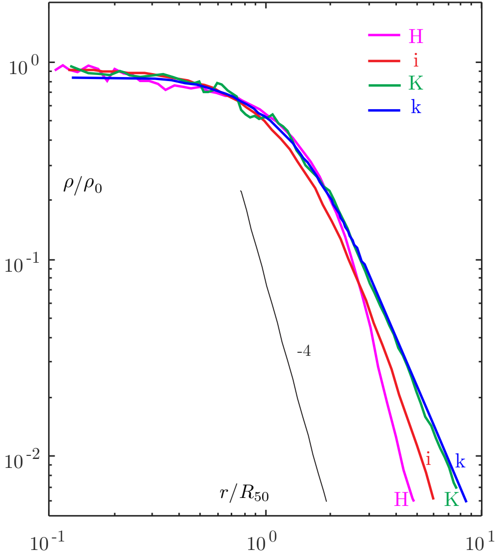
where μ is the mass parameter of ψ_{ke}, ψ_{he}, ψ_{bo} and $\mu = \omega^2 R^3$ for ψ_{ha}^R.

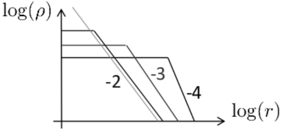
Contents

Apocenter precession over 10 periods



Mass density analysis





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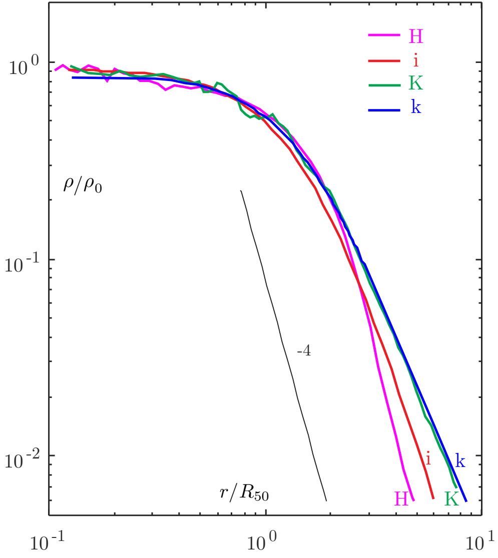
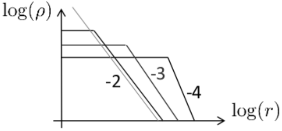
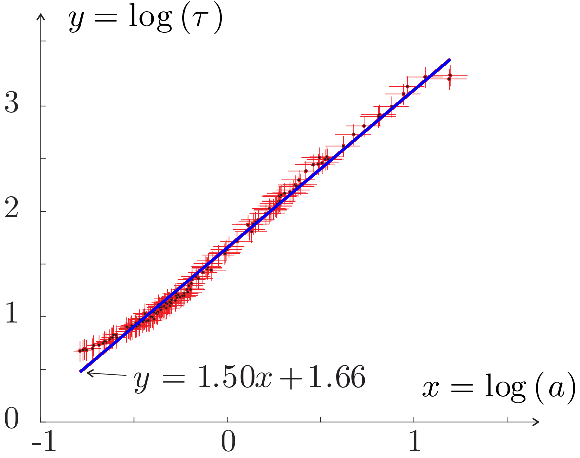




H: Hénon sphere ($t = 100T_d$),
i: isochrone model,
K: computed King model,
k: King model ($W_0 = 9$).

Isochrone analysis of a gravitational collapse



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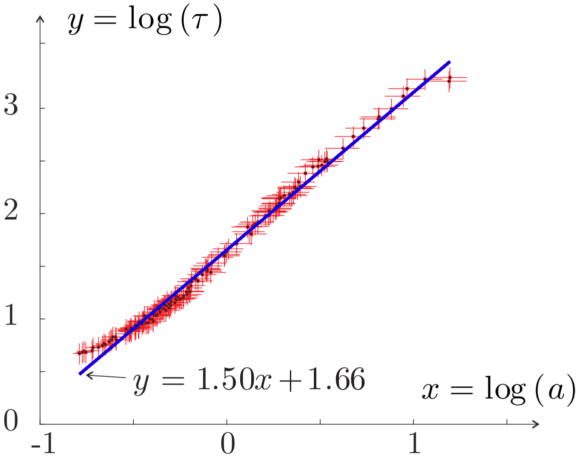
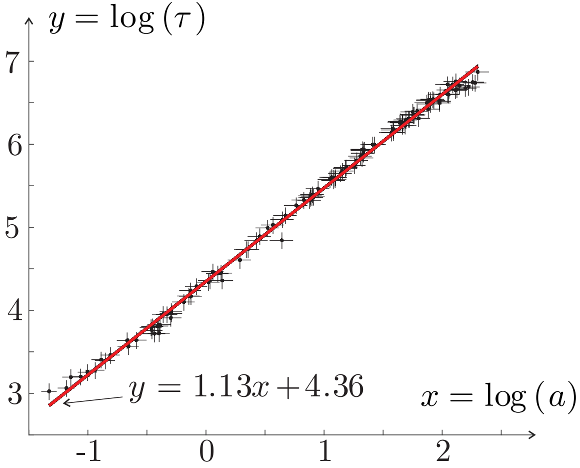
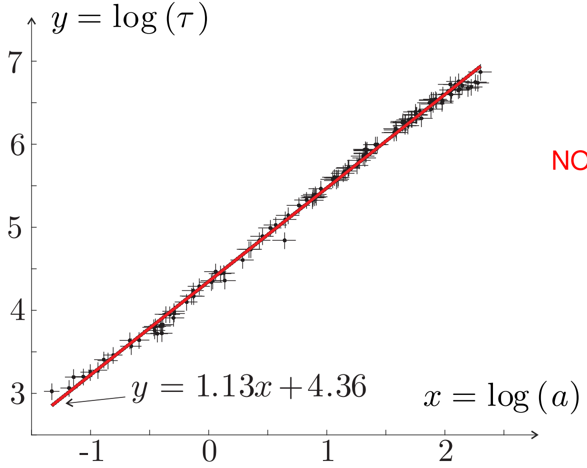


Isochrone !
$\tau^2 \propto a^3$

Isochrone analysis of a King system



Isochrone analysis of a King system



NOT isochrone !

Conclusion

- Geometrical characterization and classification of the completed isochrone set.
- Generalization of Bohlin transformation: $(\xi_{\mathrm{iso}}, \psi_{\mathrm{iso}}) \overset{B_\gamma}{\leftrightarrow} (\xi_{\mathrm{ke}}, \psi_{\mathrm{ke}})$
- Isochrone relativity: any isochrone is keplerian in his reference frame.
- Consequences: generalized Kepler's Third Law, Bertrand's theorem.
- Isochrone analysis: SGS are dynamically isochrone after gravitational collapse.

References

Alicia Simon-Petit, Jérôme Perez, Guillaume Duval. *Isochrony in 3D radial potentials.* In: Communication in Mathematical Physics. (Accepted, preprint: https://arxiv.org/abs/1804.11282).

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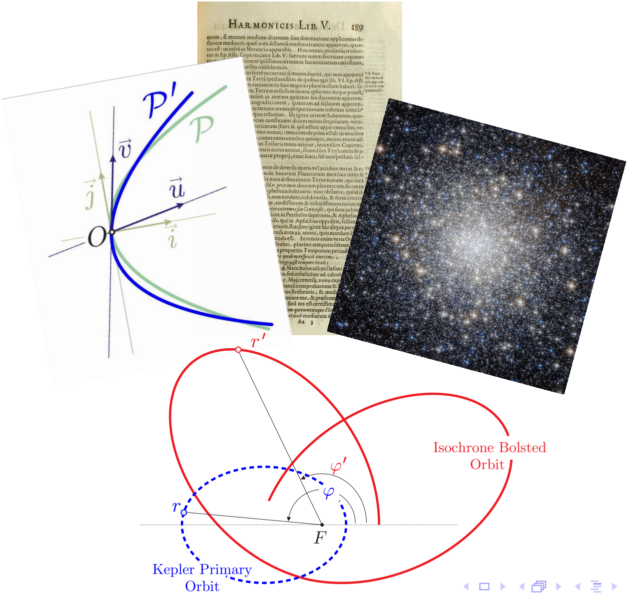
Thank you !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