Propagation of axial black hole perturbations in scalar-tensor gravity

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Introduction

- Modified gravity theories: predictions different from GR
- Relevant sector: gravitational waves emitted by black holes
- Propagation is harder to study than in GR due to more involved coupling terms
- Features of propagation can be used to rule out some theories or backgrounds

Outline

1. Modified gravity: Horndeski theory
    - Necessity for modified gravity
    - Importance of black holes
2. Gravitational waves in modified gravity
    - Perturbation setup
    - Schrödinger equations
    - Effective metric
3. Application to different black holes solutions
    - Stealth solution
    - EGB solution

Modified gravity: Horndeski theory

Motivation for beyond-GR theories

Heuristic approach

- Design new tests of GR beyond a null hypothesis check
- EFT of some high energy theory

Issues of GR

- Singularities (Big Bang, black holes)
- Cosmic expansion

$\Rightarrow$ Important to look for extensions of GR

$\Rightarrow$ Need to develop tests of these modified theories

Various theories of modified gravity

Lovelock's theorem for gravity

- Fourth dimensional spacetime
- Only field is the metric
- Second order derivatives in equations

$\Rightarrow$ GR is the only possible theory

General procedure to construct a modified gravity theory:

Break one of Lovelock's hypotheses → Make sure the theory is not pathological → Take experimental constraints into account

Cubic shift-symmetric Horndeski theory

Breaking of Lovelock's hypotheses

Add a *scalar field* ϕ coupled to $g_{\mu\nu}$ to the action

$$\phi_\mu = \nabla_\mu \phi\,, \quad \phi_{\mu\nu} = \nabla_\mu \nabla_\nu \phi\,, \quad X = \phi_\mu \phi^\mu$$

$$S = \int \mathrm{d}^4 x\,\sqrt{-g}\Big[FR + P + Q\Box\phi + 2F_X(\phi_{\mu\nu}\phi^{\mu\nu} - \Box\phi^2) + GE^{\mu\nu}\phi_{\mu\nu} + \frac{1}{3}G_X\big(\Box\phi^3 - 3\Box\phi\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu{}^\mu\big)\Big]$$

- **Most general** theory with second order equations of motion and cubic terms
- **Shift-symmetric**: functions F, P, Q and G depend on X and not ϕ
- Quadratic Horndeski: $G = 0$. GR limit: $F = 1$, other functions 0

Tests of modified gravity

Where to look for traces of modified gravity?

Black holes
- New solutions
- Different dynamics

Large scale structures
- Different growth rate
- Screenings

Cosmology
- Primordial GWs
- CMB

smaller → larger

BHs: change in theory implies change of background + change of perturbations

$\Longrightarrow$ **very interesting test system**

New black holes in Horndeski: stealth solution

Metric sector: mimic GR

$$ds^2 = -(1-\mu/r)\,dt^2 + (1-\mu/r)^{-1}\,dr^2 + r^2\,d\Omega^2$$

Scalar sector

$$\phi = qt + \psi(r)$$

$$X = -q^2 \Rightarrow \psi'(r) = q\frac{\sqrt{r\mu}}{r-\mu}$$

Properties

- Metric sector: similar to Schwarzschild, time-dependant scalar field
- $X = \text{cst} \Rightarrow$ functions of X reduced to constants

New black holes in Horndeski: EGB theory[1]

Einstein-Gauss-Bonnet Lagrangian:

$$S = \int \mathrm{d}^D x \sqrt{-g}\left(R + \alpha' \underbrace{\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)}_{\text{Gauss-Bonnet term } \mathcal{G}}\right)$$

Compactification procedure

$$\mathrm{d}s_D{}^2 = \mathrm{d}s^2 + e^{2\phi}\,\mathrm{d}\Sigma^2 \quad \text{and} \quad \alpha' = \frac{\alpha}{D-4}$$

Take $D \longrightarrow 4$: get motivated choice of parameters of Horndeski given by

$$F(X) = 1 - 2\alpha X \quad P(X) = 2\alpha X^2\,, \quad Q(X) = -4\alpha X\,, \quad G(X) = -4\alpha \ln(X)$$

[1] *Lu, H. and Pang, Y. 2020.*

New black holes in Horndeski: EGB solution[2]

Metric sector

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$A(r) = 1 - \frac{M(r)}{r}, \quad M(r) = \frac{2\mu}{1 + \sqrt{1 + 4\alpha\mu/r^3}}$$

Scalar sector

$$\phi = \psi(r)$$

$$\psi'(r) = \frac{-1 + \sqrt{A}}{r\sqrt{A}}$$

Properties

- One horizon at $r = r_h = 1/2(\mu + \sqrt{\mu^2 - 4\alpha})$
- Constant α verifies $0 \leq \alpha \leq r_h^2$

[2] *Lu, H. and Pang, Y. 2020.*

Gravitational waves in modified gravity

Axial modes

Perturbations of the metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \quad \phi = \bar{\phi} + \delta\phi$$

$$\bar{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -A(r)\,\mathrm{d}t^2 + \mathrm{d}r^2/B(r) + C(r)\,\mathrm{d}\Omega^2\,, \quad \bar{\phi} = \psi(r)$$

⇒ Separate the variables in $h_{\mu\nu}$ with Fourier transform and spherical harmonics + fix gauge

Axial modes: odd-parity perturbations

$$h_{\mu\nu} = \begin{pmatrix} & & \frac{1}{\sin\theta}\,h_0(r)\,\partial_\varphi Y_{\ell m} & -\sin\theta\,h_0(r)\,\partial_\theta Y_{\ell m} \\ & & \frac{1}{\sin\theta}\,h_1(r)\,\partial_\varphi Y_{\ell m} & -\sin\theta\,h_1(r)\,\partial_\theta Y_{\ell m} \\ \text{sym} & \text{sym} & & \\ \text{sym} & \text{sym} & & \end{pmatrix} e^{-i\omega t}\,, \quad \delta\phi = 0$$

Polar modes

Polar modes: even-parity perturbations

$$h_{\mu\nu} = \begin{pmatrix} A\,H_0(r)\,Y_{\ell m} & H_1(r)\,Y_{\ell m} & & \\ \text{sym} & B^{-1}\,H_2(r)\,Y_{\ell m} & & \\ & & K(r)\,Y_{\ell m} & \\ & & & \sin^2(\theta)\,K(r)\,Y_{\ell m} \end{pmatrix} e^{-i\omega t},$$

$$\delta\phi = \delta\varphi(r)\,Y_{\ell m} e^{-i\omega t}$$

Difficulties of polar modes study

- Coupling between scalar mode and gravitational mode
- More free functions

$\Rightarrow$ focus on axial modes for now

Resulting equations

10 perturbed Einstein's equations $\Rightarrow$ 2 first-order equations for h_0 and h_1

First-order system

- Change variables: $Y_1 = h_0\,, \quad \omega Y_2 = h_1 + \Psi h_0$
- Use λ with $2\lambda = \ell(\ell+1) - 2$

$$\frac{\mathrm{d}Y}{\mathrm{d}r} = \begin{pmatrix} C'/C + i\omega\Psi & -i\omega^2 + 2i\lambda\Phi/C \\ -i\Gamma & \Delta + i\omega\Psi \end{pmatrix} Y$$

$\Rightarrow$ This system describes the dynamics of one degree of freedom

Canonical form

Change of time coordinate

$$t_* = t - \int \mathrm{d}r\, \Psi(r)$$

New system: $$\frac{\mathrm{d}Y}{\mathrm{d}r} = \begin{pmatrix} C'/C & -i\omega^2 + 2i\lambda\Phi/C \\ -i\Gamma & \Delta \end{pmatrix} Y$$

Quadratic case $G = 0$

$$\Gamma = \frac{F + 2q^2 F_X/A}{B\mathcal{F}} + \psi^2\,, \quad \Phi = \frac{\mathcal{F}}{F - 2XF_X}\,, \quad \Psi = \frac{2qF_X\psi'}{\mathcal{F}}\,, \quad \Delta = -\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\sqrt{B/A}\,\mathcal{F}\right)$$

$$\mathcal{F} = -2q^2F_X + A(F - 2XF_X)$$

Schrödinger equation for a general metric

Constraint equation

$$\frac{\mathrm{d}Y_2}{\mathrm{d}r} = -i\Gamma Y_1 + \Delta Y_2$$

Dynamical equation

$$\frac{\mathrm{d}Y_1}{\mathrm{d}r} = \frac{C'}{C} Y_1 + i\left(\frac{2\lambda\Phi}{C} - \omega^2\right) Y_2$$

- Inject the constraint in the dynamical equation
- Renormalize Y_2 to remove $\mathrm{d}/\mathrm{d}r$ term: $Y_2 = N\mathcal{Y}$
- Change coordinate $\mathrm{d}r/\mathrm{d}r_* = n$

$$\frac{\mathrm{d}^2\mathcal{Y}}{\mathrm{d}r_*^2} + \left[\frac{\omega^2}{c_*^2} - V(r)\right]\mathcal{Y} = 0\,, \quad n^2\Gamma c_*^2 = 1$$

Physical interpretation

Wave propagation equation at $c = 1$ for $n = 1/\sqrt{\Gamma}$, scattering by potential $V_{c=1}$

Case of GR

Canonical functions in GR ($F = 1, G = 0$)

$$\Psi = 0\,, \quad \Phi = A\,, \quad \Gamma = 1/AB\,, \quad \Delta = -\frac{\mathrm{d}}{\mathrm{d}r}\ln\left(\sqrt{B/A}\right)$$

First-order system in GR: $\dfrac{\mathrm{d}Y}{\mathrm{d}r} = \begin{pmatrix} C'/C & -i\omega^2 + 2i\lambda A/C \\ -i/(AB) & -(A'/A + B'/B)/2 \end{pmatrix} Y$

⇒ Idea: identify a value for A, B, C that recreates cubic Horndeski perturbations

Comparison of canonical systems

Cubic Horndeski

$$\tilde{Y} = \alpha Y\,, \quad \mathrm{d}\tilde{Y}\big/\mathrm{d}r = \tilde{M}\tilde{Y}$$

$$\tilde{M} = \begin{pmatrix} C'/C + \alpha'/\alpha & -i\omega^2 + 2i\lambda\Phi/C \\ -i\Gamma & \Delta + \alpha'/\alpha \end{pmatrix}$$

GR on arbitrary background

$$M = \begin{pmatrix} \tilde{C}'/\tilde{C} & -i\omega^2 + 2i\lambda\tilde{A}/\tilde{C} \\ -i/(\tilde{A}\tilde{B}) & -(\tilde{A}'/\tilde{A} + \tilde{B}'/\tilde{B})/2 \end{pmatrix}$$

⇒ "equivalence" between cubic Horndeski and GR with a new background:

$$\tilde{A} = \alpha\Phi\,, \quad \frac{1}{\tilde{B}} = \alpha\Phi\Gamma\,, \quad \tilde{C} = \alpha C \quad \text{with} \quad \alpha = \mathcal{F}\sqrt{\Gamma B/A}$$

With this choice: $V_{c=1} = \dfrac{2\lambda\tilde{A}}{\tilde{C}} + \dfrac{\tilde{C}^2\tilde{C}'^2}{2C} - \dfrac{1}{2}\tilde{D}(\tilde{D}\tilde{C}')'\,, \quad D = \sqrt{\tilde{A}\tilde{B}/\tilde{C}}$

Intermezzo: massless spin-2 in GR

Consider massless spin 2 in GR: obtain propagation equation via NP formalism[3]

Propagation equation

$$\frac{\mathrm{d}^2 Z_2}{\mathrm{d}r_*^2} + (\omega^2 - V_{s=2})Z_2 = 0$$

$$V_{s=2} = \frac{2\lambda\tilde{A}}{\tilde{C}} + \frac{\tilde{C}^2\tilde{C}'^2}{2C} - \frac{1}{2}\tilde{D}(\tilde{D}\tilde{C}')' \,, \quad D = \sqrt{\tilde{A}\tilde{B}/\tilde{C}}$$

Correspondance with cubic Horndeski

- $V_{s=2} \longrightarrow$ massless spin 2 in GR with background $\tilde{A}$, ...
- $V_{c=1} \longrightarrow$ axial perturbations at $c = 1$ in cubic Horndeski with A, ...
- Both potentials are **equal**

[3] *Arbey, A. et al. 2021.*

Effective propagation metric

Propagation of axial perturbations in cubic Horndeski with background A, B, C

↔

Propagation of massless spin 2 in GR with background $\tilde{A}$, $\tilde{B}$, $\tilde{C}$

Effective metric for axial perturbations

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}\, dx^\mu\, dx^\nu = \mathcal{F}\sqrt{\frac{\Gamma B}{A}}\left(-\Phi\, dt_*{}^2 + \Gamma\Phi\, dr^2 + C\, d\Omega^2\right)$$

Specific case of quadratic Horndeski

Disformal transformation

$$\tilde{g}_{\mu\nu} = c(X)g_{\mu\nu} + d(X)\phi_\mu\phi_\nu$$

⇒ is it possible to find c and d such that $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ are linked this way? (restricting to quadratic Horndeski)

- $c = \mathcal{F}\sqrt{\Gamma B/A} = \sqrt{F(F - 2XF_X)}$
- $\Phi = A - q^2 d/c$ so $d = 2cF_X/(F - 2XF_X)$
- Other relations for $\Phi\Gamma$ and C are satisfied

Disformal effective metric in quadratic Horndeski

Link between background and perturbations

$$\tilde{g}_{\mu\nu} = \sqrt{F(F - 2XF_X)}\left(g_{\mu\nu} + \frac{2F_X}{F - 2XF_X}\phi_\mu\phi_\nu\right)$$

$\Rightarrow$ axial modes propagate in a metric disformally linked to the background metric!

- If matter is coupled to $\tilde{g}_{\mu\nu}$, it will see axial modes propagating as in GR:

$$S = S_{\text{Horn}}[g_{\mu\nu}, \phi] + S_m[\tilde{g}_{\mu\nu}, \phi]$$

- In GR both metrics are the same: $\tilde{g}_{\mu\nu} = g_{\mu\nu} \longrightarrow$ no problem for Schwarzschild
- Corresponds to a DHOST theory with $F_2 = \operatorname{sign} F$ and $A_1 = A_2 = 0$

Summary of results

- Axial modes in cubic Horndeski propagate in an effective metric given by

$$\mathrm{d}\tilde{s}^2 = \mathcal{F}\sqrt{\frac{\Gamma B}{A}}\left(-\Phi\,\mathrm{d}{t_*}^2 + \Gamma\Phi\,\mathrm{d}r^2 + C\,\mathrm{d}\Omega^2\right)$$

- In the case quadratic Horndeski ($G = 0$), this effective metric is a disformal transformation of the background metric :

$$\tilde{g}_{\mu\nu} = \sqrt{F(F - 2XF_X)}\left(g_{\mu\nu} + \frac{2F_X}{F - 2XF_X}\phi_\mu\phi_\nu\right)$$

$\Rightarrow$ study the effective metric in order to understand the behaviour of axial modes

Consequences for stability

Constraints on F

No change of signature between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$:

$$\Gamma > 0 \quad \text{and} \quad \Phi > 0$$

Recover results from the literature[a]

[a] *Takahashi, K. and Motohashi, H. 2021.*

Change of light cone

- Causality of $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ might be different
- Similar study for a scalar perturbation by Babichev et al[a]

[a] *Babichev, E. et al. 2018.*

⇒ compute the effective metric for several existing solutions

Application to different black holes solutions

Effective metric for stealth Schwarzschild

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \qquad A(r) = 1 - \frac{\mu}{r}$$

$$\mathrm{d}\tilde{s}^2 = \sqrt{1+\zeta}\left(-\frac{1}{1+\zeta}\left(1-\frac{r_g}{r}\right)\mathrm{d}t_*{}^2 + \left(1-\frac{r_g}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right),$$

$$\zeta = 2q^2F_X = \mathrm{cst}\,, \quad r_g = (1+\zeta)\mu$$

Properties

- Corresponds to Schwarzschild BH with $R = (1+\zeta)^{1/4}r$ and $T = (1+\zeta)^{-1/4}t_*$
- Horizon at $R = (1+\zeta)^{5/4}\mu$, corresponding to $r = r_g \neq \mu$
- The horizon seen by axial perturbations is **displaced**[a]

[a] *Tomikawa, K. and Kobayashi, T. 2021; Langlois, D., Noui, K., and Roussille, H. 2021.*

Effective metric for EGB

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \qquad A(r) = 1 - \frac{2\mu/r}{1+\sqrt{1+4\alpha\mu/r^3}}$$

$$\mathrm{d}\tilde{s}^2 = -\frac{1}{r^2}\sqrt{\frac{A^{1/2}\gamma_1^3\gamma_2}{\gamma_3^3}}\,\mathrm{d}t_*{}^2 + \frac{1}{r^2}\sqrt{\frac{\gamma_1\gamma_2^3}{A^{5/2}\gamma_3^5}}\,\mathrm{d}r^2 + \sqrt{\frac{\gamma_1\gamma_2}{A^{1/2}\gamma_3}}\,\mathrm{d}\Omega^2$$

- γ_1 and γ_3 are nonzero functions
- γ_2 has a zero at $r_2 = \sqrt[3]{2\alpha\mu}$
- A is zero at r_h only

Behaviour at the coordinate singularities

At $r = r_h$

$$\mathrm{d}\tilde{s}^2 \sim -c_1 (r - r_h)^{1/4}\,\mathrm{d}{t_*}^2 + \frac{c_2}{(r - r_h)^{5/4}}\,\mathrm{d}r^2 + \frac{c_3}{(r - r_h)^{1/4}}\,\mathrm{d}\Omega^2$$

$\Rightarrow$ the Ricci scalar is singular at $r = r_h$: **curvature singularity** at the horizon

At $r = r_2$

$$\mathrm{d}\tilde{s}^2 \sim -c_4 (r - r_2)^{1/2}\,\mathrm{d}{t_*}^2 + c_5 (r - r_2)^{3/2}\,\mathrm{d}r^2 + c_6 (r - r_2)^{1/2}\,\mathrm{d}\Omega^2$$

$\Rightarrow$ the Ricci scalar is singular at $r = r_2$: another **curvature singularity**

Property

The axial modes propagate in a metric with naked singularities

Conclusion

- Study of propagation of axial gravitational perturbations in cubic Horndeski theory
- Computation of the effective metric in which perturbations propagate
- Disformal link between this metric and the background: useful for coupling with matter
- Structure of effective metric computed for two background solutions
- **New behaviour** is found: horizon displaced, naked singularities
- Does not necessarily mean these theories are pathological yet

Thank you for your attention!