Analytic description of dark matter clustering: beyond perfect fluid approximation

Giulia Cusin

9^{th} March 2017

LUTH – Observatoire de Paris – Paris

UNIVERSITÉ DE GENÈVE

Our goal

Mathematical description of dark matter (DM)

- dark matter usually described as a perfect fluid with zero pressure
- baryonic matter is assumed to follow the velocity distribution of DM
- DM as perfect fluid: no generation of rotational velocity (i.e. vorticity)

From the observational side ...

- vorticity is produced in our universe (galaxies rotate etc)
- recently it has been measured to be correlated on scales $20h^{-1}$Mpc

Taylor et Jagannathan [1603.02418]

How to solve this mismatch? How to go beyond the perfect fluid description?

Outline

(1) Dark matter

- what is CDM and WDM
- standard description CDM: generation vorticity

(2) How to go beyond perfect fluid description: possibilities...

(3) What we do: analytic method followed

(4) Results for vorticity power spectrum

What is dark matter

Standard paradigm to describe evolution observed universe

ΛCDM

$$\Omega_{0DE} \simeq 0.7$$
$$\Omega_{0DM} \simeq 0.25$$
$$\Omega_{0b} \simeq 0.05$$

CDM: thermal relics mainly cold

Relics: particle species which are decoupled from primordial plasma

Thermal: in thermal equilibrium before decoupling

Cold: non-relativistic at decoupling

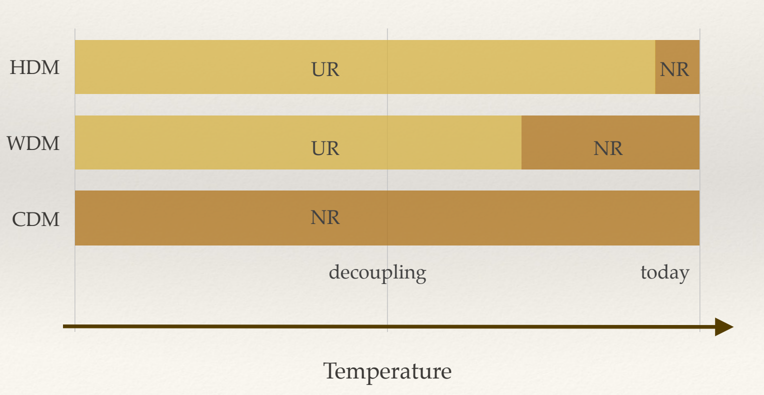
(vs Hot/warm: relativistic at decoupling)

Time line dark matter

- early times: primordial plasma with particle species in thermal equilibrium
- particle specie decouples when $\Gamma_* \ll H_*$ (rate interaction lower rate expansion universe)
- particle specie with mass m non relativistic when $T < m$ (sloppily)

e.g. neutrinos: decouple when weak interactions decouple (~ 1 MeV), non relativistic much later (mass is $10^{-(1-3)}$ eV)

Time line dark matter



Depending on prevalence CDM or WDM : $\neq$ scenarios structure formation

Standard interpretation: baryonic matter clusters in the DM potential wells

- DM mainly warm: particles with big kinetic energy, they tend to escape from potential wells and make distribution uniform.
  Cosmic structure created with a top-down scenario

- DM mainly cold: particles with smaller kinetic energy. They stay in the potential wells: small structures formed $\rightarrow$ bigger ones
  Bottom-up scenario

This second scenario seems to be the preferred one by current observations: dominant component of DM is cold

Standard description CDM

CDM perfect fluid, pressureless: density and velocity (divergence) fields

continuity equation $$\partial_\eta \delta + \nabla_{\mathbf{x}} \left((1+\delta) \, \mathbf{v} \right) = 0$$

Euler equation $$(\partial_\eta + v^i \partial_i) v_j + \mathcal{H} v_j + \partial_i \Phi = 0$$

$\delta \equiv$ overdensity, $\mathbf{v} \equiv$ peculiar velocity, $\Phi \equiv$ gravitational potential

Taking the curl of the second equation $\mathbf{w} \equiv \nabla_{\mathbf{x}} \wedge \mathbf{v}$

$$\frac{\partial \mathbf{w}}{\partial \eta} + \mathcal{H} \mathbf{w} - \nabla_{\mathbf{x}} \wedge [\mathbf{v} \wedge \mathbf{w}] = 0 \quad \rightarrow \text{homogeneous!}$$

If initial vorticity is vanishing, in this description there is no way to generate it.

How to go beyond the standard description of DM as perfect fluid

How to go beyond the perfect fluid approximation

- Vlasov equation: exact description!
- Linearize Vlasov? Not possible way...
- Truncation Boltzmann hierarchy!

DM description in terms of one-particle phase-space distribution function

$f(\eta, \mathbf{x}, \mathbf{p})$ distribution function

$(\mathbf{x}, \mathbf{p})$ comoving coord, conjugate momenta

$f(\eta, \mathbf{x}, \mathbf{p}) d^3\mathbf{p}\, d^3\mathbf{x}$ prob. having particle with momentum $\mathbf{p}$ and coord. $\mathbf{x}$

If interactions are absent: distribution function is conserved in phase space

$$\boxed{\frac{df}{d\eta} = \left(\frac{\partial f}{\partial \eta}\right)_{\mathbf{x}} + \frac{d\mathbf{x}}{d\eta} \cdot \nabla_{\mathbf{x}} f + \frac{d\mathbf{p}}{d\eta} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0}$$ Vlasov equation

Vlasov equation exactly describes the evolution of DM particles when interactions are negligible: no other assumption introduced

Example: background distribution function

Background distribution $f(\eta, p)$ in an homogeneous and isotropic universe

$$P^i = \frac{1}{a} p^i \qquad \text{physical momentum } P^i \text{, comoving } p^i$$

$$f(\eta, P) = \left(\exp \frac{\sqrt{P^2 + m^2}}{T(a)} \pm 1 \right)^{-1} = \left(\exp \frac{\sqrt{\left(\frac{p}{a}\right)^2 + m^2}}{T(a)} \pm 1 \right)^{-1}$$

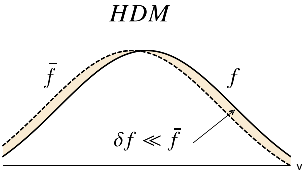
$\pm$ depending on the spin of particles

After decoupling at T_*, $df/d\eta = 0 \rightarrow f$ written in terms of comoving momenta does not depend on a

$$f(p) = \left(\exp \frac{\sqrt{p^2 + m_*^2}}{T_* a_*} \pm 1 \right)^{-1} \qquad m_* \equiv a_* m$$

Hot dark matter example: linearized Vlasov

Let us try to repeat what is usually done for HDM (e.g. neutrinos)



$$f(\eta, \mathbf{x}, \mathbf{p}) = \bar{f}(\eta, p) + \delta f(\eta, \mathbf{x}, \mathbf{p})$$

$\rightsquigarrow$ linear Vlasov for δf

$$\Psi(\eta, \mathbf{k}, \mathbf{n}, p) \propto \delta f = \sum_\ell (-)^\ell \Psi_\ell(\eta, k, p) P_\ell(\mu)$$

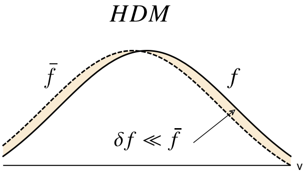
$\rightsquigarrow$Boltzmann hierarchy for Ψ_ℓ

$\ell = 1$ perfect fluid approximation

$\ell = 2$ velocity dispersion included

Hot dark matter example: linearized Vlasov

Let us try to repeat what is usually done for HDM (e.g. neutrinos)



$$f(\eta,\mathbf{x},\mathbf{p}) = \bar{f}(\eta,p) + \delta f(\eta,\mathbf{x},\mathbf{p})$$

Can we do the same for CDM?

$\rightsquigarrow$ linear Vlasov for δf

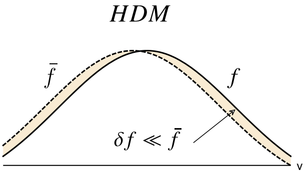
$$\Psi(\eta,\mathbf{k},\mathbf{n},p) \propto \delta f = \sum_\ell (-)^\ell \Psi_\ell P_\ell(\mu)$$

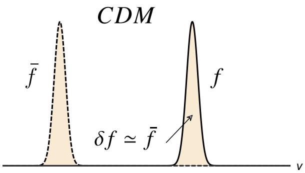
$\rightsquigarrow$Boltzmann hierarchy for Ψ_ℓ

$\ell = 1$ perfect fluid approximation

$\ell = 2$ velocity dispersion included

Let us try to repeat what is usually done for HDM (e.g. neutrinos)





$$f(\eta, \mathbf{x}, \mathbf{p}) = \bar{f}(\eta, p) + \delta f(\eta, \mathbf{x}, \mathbf{p})$$

$\rightsquigarrow$ linear Vlasov for δf

$$\Psi(\eta, \mathbf{k}, \mathbf{n}, p) \propto \delta f = \sum_{\ell} (-)^{\ell} \Psi_{\ell} P_{\ell}(\mu)$$

$\rightsquigarrow$Boltzmann hierarchy for Ψ_{ℓ}

$\ell = 1$ perfect fluid approximation

$\ell = 2$ velocity dispersion included

$\bar{f} \sim$ Dirac delta!

δf can not be treated as small quantity

We can not perturb Vlasov equation!

(Non)-relativistic kinetic theory

Solving directly Vlasov equation (perturbed) seems not to work for CDM

Beyond: which other route can be followed?

We take one step backward and we consider how the Euler and continuity equations describing DM as a perfect fluid are derived

$\rightsquigarrow$ (non)-relativistic kinetic theory

Starting point (newtonian framework)

Newtonian dynamics of a test particle in an expanding background

$$H^2 = \frac{8\pi G}{3}\bar{\rho}(\eta) \qquad \text{evolution background}$$

$$\Delta_{\mathbf{x}}\Phi = 4\pi G a^2 \delta\rho(\eta, \mathbf{x}) \qquad \text{Poisson}$$

$$\frac{d\mathbf{p}}{d\eta} = -ma\nabla_{\mathbf{x}}\Phi \qquad \text{evolution particle momentum}$$

$(\eta, \mathbf{x})$ comoving coordinates, Φ newtonian potential, $\mathbf{p} \equiv mad\mathbf{x}/d\eta$ comoving momentum, $\rho(\eta, \mathbf{x}) = \bar{\rho}(\eta) + \delta\rho(\eta, \mathbf{x})$

(Non)-relativistic kinetic theory: from single-particle to continuous description

Single-particle description → continuous one in terms of Eulerian fields

$$n_{\text{com}}(\eta, \mathbf{x}) \equiv \int d^3p f(\eta, \mathbf{x}, \mathbf{p})$$ comoving number density

$$\rho_{\text{com}}(\eta, \mathbf{x}) = \int d^3p \sqrt{m^2 + \left(\frac{p}{a}\right)^2} f(\eta, \mathbf{x}, \mathbf{p}) \simeq m \int d^3p f(\eta, \mathbf{x}, \mathbf{p}) \qquad \rho = a^{-3} \rho_{\text{com}}$$

$$v^i(\eta, \mathbf{x}) \equiv \frac{1}{n_{\text{com}}(\eta, \mathbf{x})} \int d^3p \, \frac{dx^i}{d\eta} f(\eta, \mathbf{x}, \mathbf{p})$$ peculiar velocity

$$v_i v_j + \sigma_{ij} \equiv \frac{1}{n_{\text{com}}} \int d^3p \frac{dx^i}{d\eta} \frac{dx^j}{d\eta} f(\eta, \mathbf{x}, \mathbf{p})$$ velocity dispersion tensor

$$\dots$$

we can define other macroscopic quantities using higher order momenta

For an observable $\mathcal{A}(\mathbf{x}, \mathbf{p})$ in phase space we define an average over momenta

$$\langle \mathcal{A}(\mathbf{x}) \rangle_p \equiv \frac{\int d^3p \mathcal{A}(\mathbf{x}, \mathbf{p}) f(\eta, \mathbf{x}, \mathbf{p})}{\int d^3p f(\eta, \mathbf{x}, \mathbf{p})}$$

It follows

$$v^i \equiv \left\langle \frac{dx^i}{d\eta} \right\rangle_p \qquad \sigma^{ij} \equiv \left\langle \frac{dx^i}{d\eta} \frac{dx^j}{d\eta} \right\rangle_p - \left\langle \frac{dx^j}{d\eta} \right\rangle_p \left\langle \frac{dx^j}{d\eta} \right\rangle_p$$

Vlasov equation: continuity equation in phase space

$$\boxed{\left(\frac{\partial f}{\partial \eta} \right)_{\mathbf{x}} + \frac{\mathbf{p}}{ma} \cdot \nabla f - ma \nabla_{\mathbf{x}} \Phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0}$$

We can integrate this equation over momenta ...

Boltzmann hierarchy

$$\left(\frac{\partial \delta}{\partial \eta}\right)_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot \left[(1+\delta)\,\mathbf{v}\right] = 0$$

$$\left(\frac{\partial v}{\partial \eta} + v_j \partial^j\right) v_i + \mathcal{H} v_i = -\partial_i \Phi - \frac{1}{\rho}\partial^j \left(\rho\, \sigma_{ij}\right)$$

$$\partial_\eta \sigma^{ij}(\eta, \mathbf{x}) + 2\mathcal{H}\sigma^{ij} + v^k \partial_k \sigma^{ij} + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = \cancel{\frac{1}{\rho}\partial_k \left(\rho \sigma^{ijk}\right)}$$

$$\dots$$

We truncate the Boltzmann hierarchy setting $\sigma^{ijk} \equiv \langle u^i u^j u^k \rangle_p = 0$

- vanishing background value
- it contains additional p/m for non-relativistic particles

vs perfect fluid approximation: only first two momenta are considered

What is this the velocity dispersion tensor σ_{ij}

Definition

$$\sigma^{ij} \equiv \left\langle \frac{dx^i}{d\eta}\frac{dx^j}{d\eta} \right\rangle_p - \left\langle \frac{dx^j}{d\eta} \right\rangle_p \left\langle \frac{dx^j}{d\eta} \right\rangle_p$$

"Physical" parametrization

$$\sigma_{ij} = P\delta_{ij} + \Sigma_{ij} = \begin{pmatrix} P & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{12} & P & \Sigma_{23} \\ \Sigma_{13} & \Sigma_{23} & P \end{pmatrix}$$

pressure of DM fluid

anisotropic stress of DM fluid

Set of equations describing CDM with velocity dispersion

$$\left(\frac{\partial \delta}{\partial \eta}\right)_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot \left[(1+\delta)\,\mathbf{v}\right] = 0$$

$$\left(\frac{\partial v}{\partial \eta} + v_j \partial^j\right) v_i + \mathcal{H} v_i = -\partial_i \Phi - \frac{1}{\rho}\partial^j \left(\rho\, \sigma_{ij}\right)$$

$$\partial_\eta \sigma^{ij}(\eta, \mathbf{x}) + 2\mathcal{H}\sigma^{ij} + v^k \partial_k \sigma^{ij} + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = 0$$

Vorticity equation (curl of Euler equation), $\mathbf{w} \equiv \nabla_x \wedge \mathbf{v}$

$$\frac{\partial \mathbf{w}}{\partial \eta} + \mathcal{H}\mathbf{w} - \nabla_{\mathbf{x}} \wedge \left[\mathbf{v} \wedge \mathbf{w}\right] = -\nabla_{\mathbf{x}} \wedge \left(\frac{1}{\rho}\nabla_{\mathbf{x}}\left(\rho\sigma\right)\right)$$

where $(\nabla_{\mathbf{x}}\sigma)^i \equiv \partial_j \sigma^{ji}$

- limit perfect fluid $\sigma = 0 \rightarrow \omega = 0$
- equation for σ_{ij} homegeneous: we need initial velocity dispersion!
- NON-perturbative results

Equation for vorticity

Vorticity equation (curl of Euler equation)

$$\frac{\partial \mathbf{w}}{\partial \eta} + \mathcal{H}\mathbf{w} - \nabla_{\mathbf{x}} \wedge [\mathbf{v} \wedge \mathbf{w}] = -\nabla_{\mathbf{x}} \wedge \left(\frac{1}{\rho} \nabla_{\mathbf{x}} (\rho\sigma) \right)$$

where $(\nabla_{\mathbf{x}}\sigma)^i \equiv \partial_j \sigma^{ji}$

Source is non-vanishing in two cases. Recalling $\sigma_{ij} = P\delta_{ij} + \Sigma_{ij}$

1. $\Sigma_{ij} = 0$, non barotropic fluid $P \neq P(\rho) \rightarrow \nabla P \wedge \nabla \rho \neq 0$
2. $\Sigma_{ij} \neq 0$ non vanishing anisotropic stress

Summary until now

We achieved our goal to go beyond the perfect fluid description for CDM

- DM described in terms δ, $\mathbf{v}$, pressure P and anisotropic stress Σ_{ij}
- new source in Euler equation proportional to $\sigma_{ij} = P\delta_{ij} + \Sigma_{ij}$
- equation for the evolution of σ_{ij}
- σ_{ij} acts as a source for vorticity

This formalism allows vorticity to be generated!

How we solve our system of equations

System of equations that we need to solve

Euler equation and evolution equation for the velocity dispersion tensor

$$\left(\frac{\partial \delta}{\partial \eta}\right)_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot \left[(1+\delta)\,\mathbf{v}\right] = 0$$

$$\left(\frac{\partial v}{\partial \eta} + v_j \partial^j\right) v_i + \mathcal{H} v_i = -\partial_i \Phi - \frac{1}{\rho}\partial^j \left(\rho\, \sigma_{ij}\right)$$

$$\partial_\eta \sigma^{ij}(\eta, \mathbf{x}) + 2\mathcal{H}\sigma^{ij} + v^k \partial_k \sigma^{ij} + \sigma^{ik}\partial_k v^j + \sigma^{jk}\partial_k v^i = 0$$

How to solve it? Eulerian picture? Lagrangian picture?

Lagrangian picture and Eulerian picture

We need to solve equations in a perturbation scheme: Eulerian? Lagrangian?

- Lagrangian picture: observer follows an individual fluid element as it moves in space

$$\mathbf{q} \to \boldsymbol{\mathcal{S}}(\eta, \mathbf{q}) \qquad \text{pathline of the volume}$$

  I sit in a boat drifting down a river

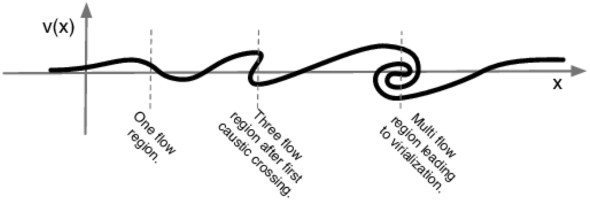
- Eulerian picture: observer focuses on specific locations in space through which the fluid flows as time passes

$$\mathbf{x} = \mathbf{q} + \boldsymbol{\mathcal{S}}(\eta, \mathbf{q})$$

  I sit on the bank of a river and I watch the water passing a fixed location

Lagrangian picture vs Eulerian picture

We use Lagrangian picture and Lagrangian perturbation theory (LPT)



Two main advantages of Lagrangian picture:

1. δ is not a dynamical field: dimensional reduction of the system
2. we do not need to linearize over δ: we can describe mildly non-linear regime $\delta \sim 1$ (where SPT breaks down)

Important! No analytic access to shell-crossing region

Lagrangian picture with velocity dispersion

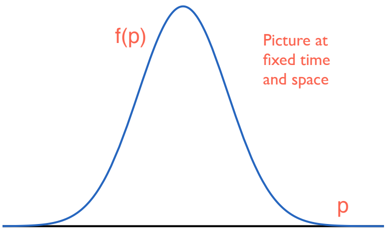
We define a Lagrangian map $\boldsymbol{\mathcal{S}}(\eta, \mathbf{q}, \mathbf{u})$

$$\mathbf{x} = \mathbf{q} + \boldsymbol{\mathcal{S}}(\eta, \mathbf{q}, \mathbf{u})$$

Peculiar velocity of a fluid element is given by the implicit equation

$$\mathbf{u}(\eta, \mathbf{x}) \equiv \frac{d\mathbf{x}}{d\eta} = \frac{d\boldsymbol{\mathcal{S}}}{d\eta}(\eta, \mathbf{q}, \mathbf{u})$$

velocity dispersion induces stochasticity in the velocity of a particle in given $\mathbf{x}$



VDT stochasticity vs shell crossing

Shell crossing

- real crossing of pathlines!
- $\delta \gg 1$
- fluid approximation breaks down

$(\eta, \mathbf{x})$: crossing 2 volume elements

Velocity dispersion

- stochastic process
- $\delta \neq 1$

$(\eta, \mathbf{x})$ associated probability having volume element with given velocity

Lagrangian picture with velocity dispersion

$$\mathbf{x} = \mathbf{q} + \boldsymbol{\mathcal{S}}(\eta, \mathbf{q}, \mathbf{u})$$

Lagrangian map has a standard part $\boldsymbol{\Psi}$ and a stochastic part $\boldsymbol{\Gamma}$

$$\boldsymbol{\mathcal{S}}(\eta, \mathbf{q}, \mathbf{u}) \equiv \boldsymbol{\Psi} + \boldsymbol{\Gamma}$$

Standard langrangian displacement field: average of $\boldsymbol{\mathcal{S}}$ over momenta

$$\mathbf{v} \equiv \left\langle \frac{\partial \mathbf{x}}{\partial \eta}(\eta, \mathbf{q}) \right\rangle_p = \left\langle \frac{\partial \boldsymbol{\mathcal{S}}}{\partial \eta}(\eta, \mathbf{q}) \right\rangle_p \equiv \frac{\partial \boldsymbol{\Psi}}{\partial \eta}(\eta, \mathbf{q})$$

We can relate the stochastic part to the velocity dispersion tensor via

$$\sigma^{ij} = \left\langle \frac{dx^i}{d\eta} \frac{dx^j}{d\eta} \right\rangle_p - \left\langle \frac{dx^i}{d\eta} \right\rangle_p \left\langle \frac{dx^j}{d\eta} \right\rangle_p = \langle \dot{\Gamma}^i \dot{\Gamma}^j \rangle_p \qquad \cdot \equiv \partial_\eta \,|_{\mathbf{q}}$$

Some technicalities

- $\mathbf{q} \to \mathbf{x} = \mathbf{q} + \boldsymbol{\mathcal{S}}(\eta, \mathbf{q}, \mathbf{u})$ invertible for a given $\mathbf{u}$
- jacobian transformation

$$J_{ij} \equiv \frac{\partial x^i}{\partial q^j} = \delta_{ij} + \frac{\partial \boldsymbol{\mathcal{S}}^i}{\partial q^j} = \delta_{ij} + \frac{\partial \boldsymbol{\Psi}^i}{\partial q^j} + \frac{\partial \Gamma^i}{\partial q^j}$$

stochastic part jacobian

- transformation spatial derivatives

$$\frac{\partial}{\partial \mathbf{x}} = \frac{\partial \mathbf{q}}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{q}} = \mathbf{J}^{-1} \frac{\partial}{\partial \mathbf{q}}$$

- we neglect stochastic contributions (consistency check a posteriori...)

Equations in Lagrangian coordinates

Euler equation (curl+divergence) and evolution equation for σ_{ij}

$$
\begin{aligned}
\left(\hat{\mathcal{T}} - 4\pi G a^2 \bar{\rho}\right) \nabla \cdot \boldsymbol{\Psi} + \epsilon_{ijk}\epsilon_{ipq}\Psi_{j,p}\left(\hat{\mathcal{T}} - 2\pi G a^2 \bar{\rho}\right)\Psi_{k,q} + & \\
+\epsilon_{ijk}\epsilon_{pqr}\Psi_{i,p}\Psi_{j,q}\left(\hat{\mathcal{T}} - \frac{4\pi G a^2}{3}\bar{\rho}\right)\Psi_{k,r} &= S_{\text{div}} \\
\hat{\mathcal{T}}\left(\nabla \wedge \boldsymbol{\Psi}\right)_j - \left(\nabla\Psi_k \wedge \hat{\mathcal{T}}\nabla\Psi_k\right)_j &= \left(S_{\text{curl}}\right)_j \\
\dot{\sigma}_{ij} + 2\mathcal{H}\sigma_{ij} &= \left(S_\sigma\right)_{ij}
\end{aligned}
$$

where $\hat{\mathcal{T}} = \partial_\eta^2 + \mathcal{H}\partial_\eta$; all time derivatives are at $\mathbf{q} =$ constant. The sources are

$$
\begin{aligned}
S_{\text{div}} &= f_{\text{div}}(\Psi, \sigma) \quad +[\text{s.t}]\,, \\
\left(S_{\text{curl}}\right)_j &= f_{\text{curl}}(\Psi, \sigma) \quad +[\text{s.t}] \\
\left(S_\sigma\right)_{ij} &= f_\sigma(\Psi, \sigma) \quad +[\text{s.t}]
\end{aligned}
$$

where [s.t.] indicates stochastic contributions

Equations for vorticity in Lagrangian picture

From the Euler equation: evolution equation for vorticity in Lagrangian picture

$$\partial_\eta \omega_\ell + \mathcal{H}\omega_\ell = \left(S^A_\omega\right)_\ell + \left(S^B_\omega\right)_\ell$$

where

$$\left(S^A_\omega\right)_\ell \equiv f(\Psi, \omega) \qquad homogeneous!$$

$$\left(S^B_\omega\right)_\ell \equiv g(\Psi, \sigma) \qquad SOURCE!$$

Lagrangian perturbation theory

- Perturbative expansion for displacement field, σ_{ij} and vorticity

$$\mathbf{\Psi} = \sum_{n=1}^{\infty} \mathbf{\Psi}^{(n)} , \qquad \sigma_{ij} = \sum_{n=0}^{\infty} \sigma_{ij}^{(n)} \qquad \boldsymbol{\omega} = \sum_{n=1}^{\infty} \boldsymbol{\omega}^{(n)}$$

- EdS universe (pure matter dominated universe)

- 'time' variable $\tau = \log a$

Background configurations

Only σ has a non-vanishing background contribution (by symmetry)

$$\mathcal{H}\left[\frac{\partial}{\partial\tau}+2\right]\sigma_{ij}^{(0)}=0$$

$$\rightsquigarrow \sigma_{ij}^{(0)}=\sigma^{(0)}\delta_{ij}=\frac{\sigma_0}{3}a^{-2}\delta_{ij} \qquad trace!$$

- $\sigma_{ij}=P\delta_{ij}+\Sigma_{ij}\rightsquigarrow P^{(0)}=a^{-2}\sigma_0/3$
- non-relativistic particles: Maxwell-Boltzmann distribution $\sigma^{(0)}\propto T/m$

$$a_0=1\rightsquigarrow \boxed{\sigma_0\equiv T_0/m}$$

Final results for vorticity

Final results for vorticity

We can solve the evolution equation for vorticity

$$\partial_\eta \boldsymbol{\omega}^{(n)} + \mathcal{H}\boldsymbol{\omega}^{(n)} \simeq \boldsymbol{\omega}^{(n-1)} a + a^{n-2}$$

$\rightsquigarrow \boldsymbol{\omega}^{(n)} \propto a^{n-3/2}$ growing modes from second order!

Vorticity is a gaussian field characterized by its power spectrum...

Final results for vorticity: power spectrum

$$\langle \omega_i^{(2)}(\mathbf{k},\eta)\omega_j^{(2)\,*}(\mathbf{k}',\eta)\rangle = (2\pi)^3\left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)\delta(\mathbf{k}-\mathbf{k}')P_\omega(k,\eta)$$

vorticity is divergence free, $\boldsymbol{\omega}\cdot\mathbf{k} = 0$

$$\boxed{P_\omega(k,\eta) = \frac{1}{9}\frac{\sigma_0^2\, a(\eta)}{\mathcal{H}_0^2\Omega_m}\int\frac{\mathrm{d}^3\mathbf{w}}{(2\pi)^3}\ (\text{kernel})\ P_\delta(w)P_\delta(|\mathbf{k}-\mathbf{w}|)}$$

For the rotational component of peculiar velocity $\mathbf{v}^R(k) = ik^{-2}\mathbf{k}\wedge\boldsymbol{\omega}(k)$

$$\langle v_i^R(k,\eta)v_j^{R*}(k',\eta)\rangle = (2\pi)^3\left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)\delta(\mathbf{k}-\mathbf{k}')P_{v_R}(k,\eta)$$

$$\boxed{P_{v_R}(k,\eta) = \frac{1}{k^2}P_\omega(k,\eta)}$$

Final results for vorticity: some numbers

Amplitude of power spectra P_ω and P_{v_R} depends quadratically on $\sigma_0 = T_0/m$

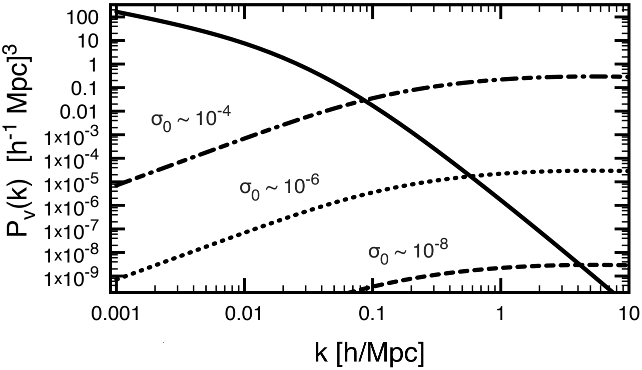
CDM: non-relativistic species at the moment of decoupling, t_*

$$\sigma_0 \propto T_0 = T_*/(1+z_*)^2 \simeq 10^{-14}$$

Piattella et al. 1507.00882

WDM: typical decoupling velocities are still relativistic

$$\sigma_0 \propto T_0 = T_*/(1+z_*)$$



100
10
1
0.1
0.01
1x10-3
1x10-4
1x10-5
1x10-6
1x10-7
1x10-8
1x10-9
$P_v(k)$ [h-1 Mpc]3
$\sigma_0 \sim 10^{-4}$
$\sigma_0 \sim 10^{-6}$
$\sigma_0 \sim 10^{-8}$
0.001
0.01
0.1
1
10
k [h/Mpc]

Summary

Velocity dispersion (i.e. pressure and anisotropic stress) included in DM description

- Boltzmann hierarchy truncated at the third momentum
- equation for vorticity is sourced $\rightarrow$ power spectrum of generated vorticity
- result depends on $\sigma_0 \propto T_0$, present dark matter temperature
- for warm dark matter at small scales $v_R \sim v_G$!

Open routes

Vorticity is measured in N-body   Plueblas et Scoccimarro [0809.4606], Paduroiu et al. [1506.03789]

- is it due to shell crossing/large scale effect induced by small scale?
- is velocity dispersion generated in the evolution?
- how is vorticity evolving with time?

Comparison with N-body simulation with our initial conditions implemented

Our description breaks down when shell crossing occurs:

- N-body domain!
- Analytic methods to access non-linear regime?

Thank you

Method used to solve perturbation equations

Equations in Lagrangian coordinates

Euler equation (curl+divergence) and evolution equation for σ_{ij}

$$\begin{aligned}
\left(\hat{\mathcal{T}} - 4\pi G a^2 \bar{\rho}\right) \nabla \cdot \mathbf{\Psi} + \epsilon_{ijk}\epsilon_{ipq}\Psi_{j,p}\left(\hat{\mathcal{T}} - 2\pi G a^2 \bar{\rho}\right)\Psi_{k,q} + & \\
+\epsilon_{ijk}\epsilon_{pqr}\Psi_{i,p}\Psi_{j,q}\left(\hat{\mathcal{T}} - \frac{4\pi G a^2}{3}\bar{\rho}\right)\Psi_{k,r} &= S_{\text{div}} \\
\hat{\mathcal{T}}\left(\nabla \wedge \mathbf{\Psi}\right)_j - \left(\nabla\Psi_k \wedge \hat{\mathcal{T}}\,\nabla\Psi_k\right)_j &= \left(S_{\text{curl}}\right)_j \\
\dot{\sigma}_{ij} + 2\mathcal{H}\sigma_{ij} &= (S_\sigma)_{ij}
\end{aligned}$$

where $\hat{\mathcal{T}} = \partial^2_\eta + \mathcal{H}\partial_\eta$; all time derivatives are at $\mathbf{q} =$ constant. The sources are

$$\begin{aligned}
S_{\text{div}} &= f_{\text{div}}(\Psi, \sigma) \quad +[\text{s.t}]\,, \\
\left(S_{\text{curl}}\right)_j &= f_{\text{curl}}(\Psi, \sigma) \quad +[\text{s.t}] \\
(S_\sigma)_{ij} &= f_\sigma(\Psi, \sigma) \quad +[\text{s.t}]
\end{aligned}$$

where [s.t.] indicates stochastic contributions

Standard LPT result ($\sigma_{ij} = 0$)

Growing leading modes

$$\tilde{\mathbf{\Psi}}^{(1)}(\mathbf{k}) = i\frac{\mathbf{k}}{k^2}\,\delta_0(\mathbf{k})a(\tau)$$

and

$$\tilde{\mathbf{\Psi}}^{(2)}(\mathbf{k}) = i\frac{3}{14}\frac{\mathbf{k}}{k^2}\,\alpha_{00}(\mathbf{k})a(\tau)^2$$

where

$$\alpha_{00}(\mathbf{k}) \equiv \int \frac{\mathrm{d}^3 w}{(2\pi)^3}\;\frac{(\mathbf{w}\wedge\mathbf{k})^2}{w^2|\mathbf{k}-\mathbf{w}|^2}\delta_0(\mathbf{w})\delta_0(\mathbf{k}-\mathbf{w})$$

Method used to solve LPT system

In the presence of velocity dispersion we can write the displacement field as

$$\Psi = \Psi_{\text{st}} + \delta\Psi_\sigma$$

Idea:

- solve eq. for σ_{ij} with standard LPT result in the source
- plug σ_{ij} found in the source of eq for $\Psi \rightsquigarrow \delta\Psi_\sigma$
- eq. for σ_{ij} with corrected $\Psi = \Psi_{\text{st}} + \delta\Psi_\sigma$ in the source
- reiterate the procedure ...

However:

- correction $\delta\Psi_\sigma$ induced by coupling to VDT is subleading wrt Ψ_{st}
- VDT solution introduces small σ_0 which further suppresses this correction

E.g. first order

$$\Psi^{(1)}_{\text{st}} \propto D_+ \quad , \quad \delta\Psi^{(1)}_\sigma \propto \sigma_0 D_+^{-2}$$

$\rightsquigarrow$ we can just use in the source for σ_{ij} the standard LPT result for Ψ

Contributions stochastic terms

Time dependence of the stochastic term $\Gamma^i \equiv \mathcal{S}^i - \Psi^i$ can be determined as

$$\sigma_{ij}^{(n)} \propto \langle \dot{\Gamma}_i \dot{\Gamma}_j \rangle_p^{(n)} \propto \sigma_0 D_+^{n-2}$$

$$\rightsquigarrow \quad \Gamma_i^{(n)} \propto \sqrt{\sigma_0} D_+^{n-\frac{1}{2}} \quad vs \quad \mathbf{\Psi}^{(n)} \propto D_+^{(n)}$$

Every time we have neglected in the sources a terms in $\Gamma_{k,j}$ we have considered an identical term in $\Psi_{k,j}$:

- which grows faster
- and it is not suppressed by a factor $\sqrt{\sigma_0}$

$\rightsquigarrow$ for sufficiently small σ_0 it is justified to neglect the stochastic contribution!

Continuity equation automatically implemented in Lagrangian picture

The continuity equation can be rewritten as

$$d^3\mathbf{x}\,\rho(\eta,\mathbf{x}) = d^3\mathbf{q}\,\rho(\mathbf{q}) \quad \text{or} \quad \rho(\eta,\mathbf{x}) = \rho(\mathbf{q})/J(\eta,\mathbf{q})$$

Neglecting stochastic contributions, it follows $\leftarrow d\left(\det A\right)/dt = \det A\,\mathrm{tr}\left(A^{-1}dA/dt\right)$

$$\frac{dJ}{d\eta} = J\,\mathrm{Tr}\left(\mathbf{J}^{-1}\frac{d\mathbf{J}}{d\eta}\right) = J\,\nabla\cdot\mathbf{v}$$

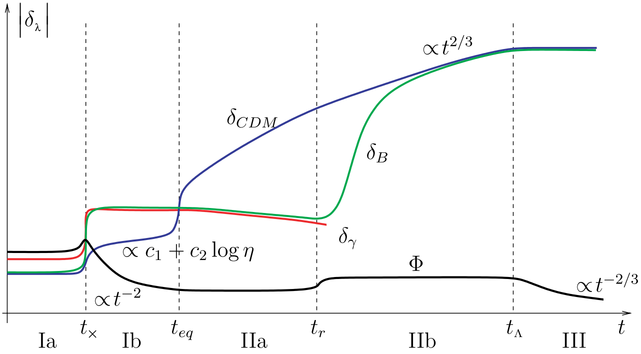
Using $\rho(\mathbf{q}) = \rho(\eta,\mathbf{x})J(\eta,\mathbf{q})$, we get

$$0 = \frac{\partial}{\partial\eta}\left(\rho\,J\right) = J\left(\frac{\partial\rho}{\partial\eta} + \rho\nabla\cdot\mathbf{v}\right)$$

in the Lagrangian picture the continuity equation is automatically implemented, independently on the specific form of the map between Lagrangian and Eulerian coordinates

Evolution CDM overdensity

Mode is entering horizon in radiation domination



from Rubakov's book