Killing Horizons and Orthogonally Transitive Groups in Space–Time

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Some concepts which have been proven to be useful in general relativity are characterized, definitions being given of a local isometry horizon, of which a special case is a Killing horizon (a null hypersurface whose null tangent vector can be normalized to coincide with a Killing vector field) and of the related concepts of invertibility and orthogonal transitivity of an isometry group in an n-dimensional pseudo-Riemannian manifold (a group is said to be orthogonally transitive if its surfaces of transitivity, being of dimension p, say, are orthogonal to a family of surfaces of conjugate dimension $n - p$). The relationships between these concepts are described and it is shown (in Theorem 1) that, if an isometry group is orthogonally transitive then a local isometry horizon occurs wherever its surfaces of transitivity are null, and that it is a Killing horizon if the group is Abelian. In the case of $(n - 2)$-parameter Abelian groups it is shown (in Theorem 2) that, under suitable conditions (e.g., when a symmetry axis is present), the invertibility of the Ricci tensor is sufficient to imply orthogonal transitivity; definitions are given of convection and of the flux vector of an isometry group, and it is shown that the group is orthogonally transitive in a neighborhood if and only if the circulation of convective flux about the neighborhood vanishes. The purpose of this work is to obtain results which have physical significance in ordinary space–time ($n = 4$), the main application being to stationary axisymmetric systems; illustrative examples are given at each stage; in particular it is shown that, when the source-free Maxwell–Einstein equations are satisfied, the Ricci tensor must be invertible, so that Theorem 2 always applies (giving a generalization of the theorem of Papapetrou which applies to the pure-vaccuum case).

1. INTRODUCTION

The purpose of this paper is to develop in a coherent way some concepts which are currently being found useful in work on general relativity in connection with isometries, and to point out some of the relationships between them and show how they may be applied. Although the motive for this study is to obtain physical applications to 4-dimensional space–time, the results are all derived in n dimensions, because, on one hand, very little extra work is required, while, on the other hand, considerably greater mathematical insight is obtained.

The main subject of discussion will be certain types of horizons which we shall now define.

A null hypersurface in a pseudo-Riemannian manifold is said to be a local isometry horizon (which we henceforth abbreviate to LIH) with respect to a group of isometries if (I) it is invariant under the group, and (II) each null-geodesic generator is a trajectory of the group.

The special case of a null surface which is an LIH with respect to a one parameter group (or subgroup) is said to be a Killing horizon. In other words, a Killing horizon is a null surface whose generating null vector can be normalized so as to coincide with a Killing vector field.

The purpose of these definitions is to isolate the characteristic features of the class of functions of which the Schwarzschild horizon[1] is the most familiar example in so far as these features can be described in terms of purely local concepts. The physical significance of an LIH is that on it a particle may at once be travelling at the speed of light (along one of the null generators) and standing still (in the sense that no change in its surroundings can be detected as its affine parameter varies because it is moving along a trajectory of a motion which leaves invariant both the intrinsic structure of space–time and the position of the null surface itself). As a general consequence, infinite red or blue shifts will be observable in relation to the frames of reference naturally determined by the isometry group.

LIH's are worth studying because, in addition to their local significance, they may have considerable importance in the global structure of space–time, for example as event horizons[2] or Cauchy horizons,[3] etc. Killing horizons in particular are interesting in four dimensions because any spacelike 2-surface within such a horizon will be marginally locally trapped according to the definition of Penrose.[4] This is because the null vectors generating the Killing horizon must have zero expansion, rotation, and shear (i.e., $\rho = \sigma = 0$ in Newman–Penrose language).[5] The vanishing of the first of these means that one family of null normals to the 2-surface is not expanding in either direction, and so there must be a sense of time

[1] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).

[2] W. Rindler, Monthly Notices Roy. Astron. Soc. **116**, 662 (1956).

[3] S. W. Hawking, Proc. Roy. Soc., London **A294**, 511 (1966).

[4] R. Penrose, Phys. Rev. Letters **14**, 57 (1965).

[5] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).

direction in which both families of null normals are nonexpanding. If the required sense happened to be the same over the whole 2-surface, then it would follow, in the case of a compact 2-surface, that it would be a closed marginally trapped surface. (A marginally trapped surface is one which satisfies the condition that neither family of null normals diverges, but not the strict condition that both converge.) However, although such global properties as these provide much of the motivation for studying LIH's, we restrict attention to purely local concepts in this paper.

It is worth emphasizing that the conditions (I) and (II) in the definition of an LIH are independent of each other, and that they are both essential if the condition is to be sufficiently restrictive to be useful. In physical terms, they are both necessary if a particle moving along a null generator is to be able to be thought of as also standing still, since if (II) were not satisfied it would have motion with respect to the intrinsic structure of space-time, while if (I) were not satisfied the null surface itself would define a structure with respect to which motion could be defined. These points may be made clearer by consideration of a few simple examples.

A trivial example is provided by the null cone of a point in Minkowski space, which is an LIH with respect to the Lorentz group at that point, but not with respect to the full Poincare group [since (I) would not be satisfied] nor with respect to the rotation group at the point [since (I) would not be satisfied]. It is not a Killing horizon.

The classic example is the Schwarzschild horizon,[1] which is an LIH with respect to the one-parameter group of static displacements, and is therefore a Killing horizon. It is also an LIH with respect to larger groups such as the Abelian group (static displacements) $\oplus$ (rotations about an axis). Within the Schwarzschild horizon there are many null hypersurfaces which satisfy (I), but the definition excludes them from being counted as LIH's because they do not satisfy (II).

A slightly more complicated example is provided by the Kerr solution[6] when $a > m$ (in the standard notation as used in Ref. 4), in which there are LIH's with respect to the Abelian group (stationary displacements) $\oplus$ (rotation about the axis), but not with respect to any larger groups. These LIH's also are Killing horizons, but this is less obvious than in the case of the Schwarzschild solution because the Killing fields involved are not the same as the unique Killing field which is timelike at infinity. This will be further discussed in Sec. 4.

[6] R. H. Boyer and T. G. Price, Proc. Cambridge Phil. Soc. **61**, 531 (1965).

It is now natural to wonder under what conditions LIH's and Killing horizons are likely to occur. A casual glance at the Schwarzschild solution might suggest that they occur where a Killing vector field becomes null. However, a little consideration shows that this is neither sufficient nor (except for a Killing horizon) necessary. For example, in Minkowski space one may form a whole class of Killing vector fields by taking different linear combinations of a static displacement and a rotation about an axis, but the hypersurfaces on which these fields become null are not themselves even null, but are timelike.

Further investigation of this question constitutes the principal content of this paper. With this end in view we introduce, in Sec. 2, the idea of an isometry group being orthogonally transitive, meaning that the surfaces of transitivity are orthogonal to a family of surfaces of conjugate dimension. It is a convenient consequence of orthogonal transitivity that it is possible (where the surfaces of transitivity are not null) to choose coordinates in two sets, constant on the surfaces of transitivity and the orthogonal surfaces, respectively, in such a way that the resulting form of the metric tensor makes manifest, as far as possible, the isometries, and at the same time contains no cross terms between the two sets.

One of the main results of this investigation is given in Sec. 3, where it is shown that, wherever the surfaces of transitivity of an orthogonally transitive group do become null, an LIH occurs. In Sec. 4 it is shown in addition that if the group is Abelian, such an LIH is a Killing horizon.

In Secs. 5 and 6 it is shown that orthogonal transitivity is not merely a condition imposed for mathematical convenience (although it has often been assumed in past investigations without any other justification) but that it may be expected to occur naturally under fairly general conditions, provided the group is Abelian and provided also that its surfaces of transitivity have $(n - 2)$ dimensions where n is the dimension of the manifold [orthogonal transitivity being trivial in the $(n - 1)$-dimensional case]. These conditions are given a physical interpretation in general relativity in terms of the vanishing of the convective circulation of matter around a region.

As the paper progresses the class of groups under consideration has to be progressively restricted: from general groups in Sec. 3 to Abelian groups in Sec. 4 to Abelian $(n - 2)$-parameter groups in Secs. 5 and 6. However, all the results apply to 2-parameter (and trivially to 3-parameter) Abelian groups in four-dimensional space–time, and therefore to stationary axisymmetric systems in particular.

2. INVERTIBILITY AND ORTHOGONAL TRANSITIVITY

We now introduce the related concepts of orthogonal transitivity and invertibility of a group of isometries.

Consider an open region $\mathcal{U}$ on an n-dimensional manifold such that there is a continuous group of isometries whose surfaces of transitivity have dimensionality p $(1 \leq p \leq n-1)$ in $\mathcal{U}$.

Then the group is said to be orthogonally transitive in $\mathcal{U}$ if there exists a family of $(n-p)$-dimensional surfaces which are orthogonal to the surfaces of transitivity at each point in $\mathcal{U}$.

The group is said to be invertible at a point P in $\mathcal{U}$ if there is an isometry leaving P fixed which simultaneously inverts the sense of the p independent directions in the surface of transitivity at P, but leaves unaltered the sense of the directions orthogonal to the surface of transitivity at P. If such an isometry exists, it is clear that it is an involution and that it is uniquely determined.

It is important to note that a group cannot be invertible at a point P if the surface of transitivity through P is null, since in this case there is a direction in the surface of transitivity which is also orthogonal to it. This situation is not merely due to an inadequacy in the definition of invertibility, but is a result of the fact that, even when the group is invertible on the other surfaces of transitivity in the immediate neighborhood of P, there is generally a real distinction between the two opposed arrangements of the direction senses in the null surface of transitivity. This somewhat paradoxical state of affairs may be made intelligible by means of an illustration. Consider the 1-dimensional group generated by stationary displacements in Kruskal's completed Schwarzschild solution.[1] This group is invertible everywhere except on the horizon, where the Killing vector becomes null. It is immediately clear that there is a distinction between the two senses of direction along a line of transitivity there, since in one sense the line approaches a fixed point of the group, while in the other it continues to infinity without interruption.

It can easily be seen that orthogonal transitivity is a necessary condition for a group to be invertible in a neighborhood. For suppose we have an n-dimensional manifold with a group of isometries whose surfaces of transitivity are p-dimensional, and which is invertible in the neighborhood of a point P. Construct the set of all differentiable paths in the neighborhood which pass through P and which are everywhere orthogonal to the surfaces of transitivity. This set of paths intersects each surface of transitivity in a unique point: for consider a pair of paths PQ and PQ', where Q and Q' lie on the same surface of transitivity; since the directed compound path QPQ' could be defined without reference to any sense of direction in the surfaces of transitivity, it follows that Q and Q' must coincide, because otherwise the ordered pair Q, Q' would give rise to an intrinsically defined sense of direction in the surface of transitivity at Q. It follows that this set of paths generates an $(n-p)$-surface through P which is orthogonal to the surfaces of transitivity. By a similar construction at each point in the neighborhood of P, a complete family of orthogonal $(n-p)$-surfaces can be built up.

Thus, in order that a group should be invertible, it is necessary that it be orthogonally transitive, and also that the surfaces of transitivity be nonnull. These conditions are not in general sufficient. For consider as a counterexample the 4-dimensional space with metric given by

$$ds^2 = a(z,t)e^{-2y}\,dx^2 + 2b(z,t)e^{-y}\,dx\,dy + c(z,t)\,dy^2 + dz^2 - dt^2, \quad (1)$$

with $a(z,t)c(z,t) > b^2(z,t)$. Then the Killing vectors $\partial/\partial x$ and $x(\partial/\partial x) + (\partial/\partial y)$ generate a non-Abelian group which is orthogonally (and simply) transitive over the 2-surfaces, $z = \text{const}$, $t = \text{const}$, these surfaces being orthogonal to the family of 2-surfaces, $x = \text{const}$, $y = \text{const}$. The surfaces of transitivity are nonnull. Nevertheless, it can easily be checked that, except for some specially simple choices of the functions $a(z,t)$, $b(z,t)$, $c(z,t)$, the group is not invertible.

Suppose, however, that we have an Abelian group. In this case the requirement that the group be orthogonally transitive with nonnull surfaces of transitivity is not only necessary but also sufficient for the group to be invertible.

In order to see this, consider an n-dimensional manifold with an orthogonally transitive Abelian isometry group which has nonnull p-dimensional surfaces of transitivity in some neighborhood. We construct a manifestly invertible coordinate patch as follows. Let $y^1, \cdots, y^{n-p}$ be any well-behaved coordinate system on one of the orthogonal $(n-p)$-surfaces. There will be a nondegenerate induced metric $ds^2 = g_{ij}\,dy^i\,dy^j$, $i,j = 1, \cdots, n-p$. By dragging the system along under the operations of the group, we equip all the orthogonal $(n-p)$-surfaces with coordinates in which the induced metric has an identical form, since, being nonnull, the surfaces of transitivity through the original orthogonal $(n - p)$-surface span the whole neighborhood. It is for the next stage that we need the group to be Abelian. We choose p linearly independent Killing vector fields generating the group, which we shall suggestively label $\partial/\partial\psi^1, \cdots, \partial/\partial\psi^p$. We proceed to attach a set of coordinate values $\psi^1, \cdots, \psi^p$ to each of the orthogonal $(n - p)$-surfaces in the obvious way, i.e., we first choose one of the $(n - p)$-surfaces as the origin, $\psi^1 = \cdots = \psi^p = 0$; we then drag this one along under the Killing vectors $\partial/\partial\psi^2, \cdots, \partial/\partial\psi^p$, thereby generating a hypersurface (since the Killing vectors commute) which we label $\psi^1 = 0$; from here we form the family of hypersurfaces $\psi^1 = \text{const}$ by dragging this one along under $\partial/\partial\psi^1$ by corresponding values of the parameter ψ^1; finally, we repeat the process for $\psi^2, \cdots, \psi^p$. As a result of the commutation, each hypersurface $\psi^k = \text{const}$ is invariant under the Killing vectors other than $\partial/\partial\psi^k$. It follows that in each p-surface of transitivity the induced metric is given by $ds^2 = h_{kl}\, d\psi^k\, d\psi^l$, $(k, l = 1, \cdots, p)$, where the coefficients h_{kl} are independent of $\psi^1, \cdots, \psi^p$. Due to the orthogonality, the metric on the whole space has the form $ds^2 = g_{ij}\, dy^i\, dy^j + h_{kl}\, d\psi^k\, d\psi^l$. Now consider the inversion mapping $(y^1, \cdots, \psi^p) \to (\tilde{y}^1, \cdots, \tilde{\psi}^p)$ where $\tilde{\psi}^i = y^i$ $(i = 1, \cdots, n - p)$ and

$$\tilde{\psi}^k = -\psi^k \; (k = 1, \cdots, p).$$

Since g_{ij} and h_{kl} are independent of $\psi^1, \cdots, \psi^p$, this is clearly an isometry; thus the group is invertible.

We note in passing that the concept of being static is the special case of orthogonal transitivity which refers to a 1-parameter group (applying in the stricter sense only when the Killing vector is time like). Since a 1-parameter group is automatically Abelian, orthogonal transitivity and invertibility are equivalent here when the Killing vector is nonnull.

By a rough analogy we can transfer these ideas from groups to tensors. Let ${}_{(i)}\zeta^\lambda$ $(i = 1, \cdots, p)$ be a set of independent vectors spanning a p-dimensional surface element at a point P, and let ${}^{(j)}\eta_\mu$ $(j = p + 1, \cdots, n)$ be a set of independent vectors spanning the orthogonal $(n - p)$ element at P. Then a tensor T is said to be orthogonal to the p-surface element at P with respect to a particular subset of s of its indices if, when we form the mixed components $T^{\mu_1 \cdots \mu_r}{}_{\lambda_1 \cdots \lambda_s}$, which are covariant in the indices of the subset and contravariant in the others, the contraction

$${}^{(\alpha_1)}\eta_{\mu_1}, \cdots, {}^{(\alpha_r)}\eta_{\mu_r} T^{\mu_1 \cdots \mu_r}{}_{\lambda_1 \cdots \lambda_s}\, {}_{(\beta_1)}\zeta^{\lambda_1}, \cdots, {}_{(\beta_s)}\zeta^{\lambda_s}$$

vanishes for all possible choices of $\alpha_1, \cdots, \alpha_s$ and $\beta_1, \cdots, \beta_s$. The tensor is said to be invertible in the p element at P [or invertible about the orthogonal $(n - p)$ element at P] if each of the scalars obtained by contracting any combinations of its indices with any choice of the ${}_{(i)}\zeta^\lambda$ and the ${}^{(j)}\eta_\mu$ is invariant when ${}_{(i)}\zeta^\lambda \to -{}_{(i)}\zeta^\lambda$ and ${}^{(j)}\eta_\mu \to {}^{(j)}\eta_\mu$ for all i, j. Obviously, these definitions are independent of the choice of basis vectors in the elements. Clearly also, the statement that a tensor is invertible in an element is equivalent to the statement that it is orthogonal to the element with respect to every subset consisting of an odd number of its indices. The definition of invertibility is quite straightforward when the element is nonnull, so that ${}_{(i)}\zeta^\lambda$ and the ${}^{(j)}\eta^\lambda$ are linearly independent. It is slightly more subtle when the element is null, since there then exist directions common to these sets. The definition requires that such a direction be inverted when represented by a contravariant vector and left unaltered when represented by a covariant vector. In this way a tensor can be invertible even in a null element, although a group cannot be invertible on a null surface of transitivity.

A tensor is said to be orthogonal (with respect to a subset of indices) to a group or invertible in a group if it is orthogonal (with respect to the subset of indices) to the surfaces of transitivity or invertible in them respectively. Clearly, if a group is invertible, then any intrinsically defined tensor (such as the Ricci tensor or the Weyl tensor) must be invertible in it.

3. AN EXISTENCE THEOREM FOR LOCAL ISOMETRY HORIZONS WHERE AN ORTHOGONALLY TRANSITIVE GROUP HAS NULL SURFACES OF TRANSITIVITY

We can use the concepts of the previous section to proceed further with the question raised in Sec. 1.

Before doing so we explain our notation and state Frobenius's theorem, which is fundamental to questions of orthogonality. We use square brackets to denote antisymmetrization and round brackets for symmetrization; when two such operations are to be performed in a context where the order is important, we shall indicate the operation to be performed first by using boldface brackets as, e.g., $[\cdots \mathbf{[} \cdots] \cdots \mathbf{]}$. We define the p vector generated by a set of vectors ${}_{(1)}\zeta^\mu, \cdots, {}_{(p)}\zeta^\mu$ as

$$w^{\kappa_1 \cdots \kappa_p} = {}_{(1)}\zeta^{[\kappa_1} \cdots {}_{(p)}\zeta^{\kappa_p]} \tag{2}$$

and define the orthogonal conjugate in n dimensions as

$${}^*w_{\mu_{p+1} \cdots \mu_n} = \frac{1}{p!} \epsilon_{\kappa_1 \cdots \kappa_p \mu_{p+1} \cdots \mu_n} w^{\kappa_1 \cdots \kappa_p}, \tag{3}$$

where $\epsilon_{\mu_1 \cdots \mu_n}$ is the alternating tensor. Frobenius's Theorem (see, e.g., Schouten[7]) states that a necessary

[7] J. A. Schouten, *Ricci Calculus* (Springer-Verlag, Berlin, 1954), p. 81.

and sufficient condition for a field of such p vectors to be orthogonal (locally) to a family of $(n-p)$-surfaces in n dimensions is

$$w^{[\kappa_1 \cdots \kappa_p;\mu}w^{\nu]\lambda_2 \cdots \lambda_p} = 0. \quad (4)$$

It is convenient for future reference to have the expansion

$$w^{[\kappa_1 \cdots \kappa_p;\mu}w^{\nu]\lambda_2 \cdots \lambda_p} = \frac{1}{p}\sum_{i=1}^{p}(-1)^{i-1}w^{[\kappa_1 \cdots \kappa_p}{}_{(i)}\zeta^{\mu;\nu]} \times {}_{(1)}\zeta^{[\lambda_1} \cdots {}_{(i-1)}\zeta^{\lambda_i}{}_{(i+1)}\zeta^{\lambda_{i+1}} \cdots {}_{(p)}\zeta^{\lambda_p]}. \quad (5)$$

The following theorem (which covers the cases of the Schwarzschild and Kerr solutions) shows that LIH's may be expected to occur in a fairly wide class of circumstances.

Theorem 1: Let $\mathcal{U}$ be an open subregion of an n-dimensional C^2 manifold with a C^1 pseudo-Riemannian metric, such that there is a continuous group of isometries whose surfaces of transitivity have constant dimension p $(1 \le p \le n-1)$ in $\mathcal{U}$.

Let $\mathcal{N}$ be the subset (which must obviously be closed in $\mathcal{U}$) where the surfaces of transitivity become null, and suppose that they are never more than singly null (i.e., the rank of the induced metric on the surfaces of transitivity drops from p to $p-1$ on $\mathcal{N}$, but is never lower).

Then, if the group is orthogonally transitive in $\mathcal{U}$, it follows that $\mathcal{N}$ is the union of a family of non-intersecting hypersurfaces which are LIH's with respect to the group, and consequently (since $\mathcal{N}$ is closed) that the boundaries of $\mathcal{N}$ are members of the family.

Proof: In the neighborhood of any point in $\mathcal{U}$ we choose a linearly independent set ${}_{(i)}\zeta^{\mu}$ $(i = 1, \cdots, p)$ of the Killing vectors generating the group, and form the Killing p vector tangent to the surfaces of transitivity

$$w^{\kappa_1 \cdots \kappa_p} = {}_{(1)}\xi^{[\kappa_1} \cdots {}_{(p)}\xi^{\kappa_p]}. \quad (6)$$

We now substitute this in the identity (5), make a further expansion of the right-hand side, and finally antisymmetrize the whole over the indices μ, ν, $\lambda_2, \cdots, \lambda_p$. Most of the terms then drop out, leaving the reduced identity

$$w^{[\kappa_1 \cdots \kappa_p;[\mu}w^{\nu]\lambda_2 \cdots \lambda_p]} = \frac{2}{p(p+1)(p+2)}\Big\{2w^{\kappa_1 \cdots \kappa_p;[\mu}w^{\nu\lambda_2 \cdots \lambda_p]} - w^{\kappa_1 \cdots \kappa_p}w^{[\nu\lambda_2 \cdots \lambda_p;\mu]} - 2\sum_{i=1}^{p}{}_{(1)}\xi^{[\kappa_1} \cdots {}_{(i-1)}\xi^{\kappa_{i-1}}{}_{(i+1)}\xi^{\kappa_{i+1}} \cdots {}_{(p)}\xi^{\kappa_p} \times {}_{(i)}\xi^{(\kappa_i]);[\mu)}w^{\nu\lambda_2 \cdots \lambda_p]}\Big\}. \quad (7)$$

Since the surfaces of transitivity are $(n-p)$-surface orthogonal, Eq. (4) holds, and so the left-hand side of Eq. (7) vanishes. Since the ${}_{(i)}\xi^{\mu}$ satisfy Killing's equation ${}_{(i)}\xi^{(\mu;\nu)} = 0$, it follows that the last term vanishes also. This leaves the relation

$$2w^{\kappa_1 \cdots \kappa_p;[\mu}w^{\lambda_1 \cdots \lambda_p]} = w^{\kappa_1 \cdots \kappa_p}w^{[\lambda_1 \cdots \lambda_p;\mu]}. \quad (8)$$

Contracting with $w_{\kappa_1 \cdots \kappa_p}$ and setting

$$W = \frac{1}{p!}w^{\kappa_1 \cdots \kappa_p}w_{\kappa_1 \cdots \kappa_p}, \quad (9)$$

we obtain the result

$$W^{;[\mu}w^{\lambda_1 \cdots \lambda_p]} = Ww^{[\lambda_1 \cdots \lambda_p;\mu]}. \quad (10)$$

We shall use the orthogonal conjugate form of this equation, i.e.,

$$W_{,\rho}\,{}^*w^{\rho\kappa_{p+2} \cdots \kappa_n} = W\,{}^*w^{\rho\kappa_{p+2} \cdots \kappa_n}{}_{;\rho}. \quad (11)$$

Now the vanishing of W at a point is a necessary and sufficient condition for the p surface of transitivity to be null there, or, in other words, $W = 0$ is the equation of the set $\mathcal{N}$.

Hence in the open region $\mathcal{U} - \mathcal{N}$ we may divide by W to obtain

$$(\ln W)_{,\rho}\,{}^*w^{\rho\kappa_{p+2} \cdots \kappa_n} = {}^*w^{\rho\kappa_{p+2} \cdots \kappa_n}{}_{;\rho}. \quad (12)$$

Since the right-hand side is continuous in $\mathcal{U}$, this equation may be interpreted as implying that the left-hand side is locally bounded in $\mathcal{U} - \mathcal{N}$.

Now let us restrict attention to a particular one of the orthogonal $(n-p)$-surfaces. Suppose that this surface lies partly in $\mathcal{N}$ and partly outside. Then in the neighborhood of any point on the boundary ln W must be unbounded, and consequently the restriction of its gradient to the $(n-p)$-surface must be unbounded. But ${}^*w^{\kappa_{p+1} \cdots \kappa_n}$ is the tangent element to the orthogonal $(n-p)$-surface and is locally nonvanishing. Thus (12) implies that the restriction of the gradient of ln W to the $(n-p)$-surface is bounded, contrary to the deduction we have just made. It follows that if any part of one of the orthogonal $(n-p)$-surfaces lies in $\mathcal{N}$, then the whole of it must lie in $\mathcal{N}$.

Consider one such $(n-p)$-surface through a point P in $\mathcal{N}$. At each point on this surface $w^{\kappa_1 \cdots \kappa_p}$ and ${}^*w^{\kappa_{p+1} \cdots \kappa_n}$ together generate an $(n-1)$ element, since, being singly null, they have a unique (null) direction in common. Therefore, by dragging along the $(n-p)$-surface under the operations of the group, we obtain a uniquely defined null hypersurface through P which is contained in $\mathcal{N}$. Its null geodesic generators lie everywhere in $w^{\kappa_1 \cdots \kappa_p}$ and consequently are trajectories of the group. They cannot intersect since otherwise the member of the family passing through a point of intersection would not be unique.

We remark on a few points arising from this theorem.

(1) In Lorentz spaces (i.e., those with signature $n - 2$), and in general relativity in particular, the restriction that the p-surfaces should be at most singly null is unnecessary, since higher nullity is not possible in these spaces anyway.

(2) When $p = n - 1$, the orthogonality condition is automatically satisfied, and the conclusion of the theorem is also a trivial result.

(3) When $p = n - 1$, and also when $p = 1$, a converse theorem holds as a trivial result. The converse theorem may be stated as follows: If $\mathcal{K}$ is an LIH with respect to a group which is transitive over p-surfaces in n dimensions, then these p-surfaces are $(n - p)$-surface orthogonal on $\mathcal{K}$. However, this converse does not hold for the intermediate values $p = 2, \cdots, n - 2$.

In the case $n = 4, p = 2$, a simple counterexample is given by the space (which has Lorentz signature when $x < 1$) with metric

$$ds^2 = dx^2 + dy^2 + dz^2 - 2\,dx\,dt + 2y\,dz\,dt - (x - y^2)\,dt^2. \quad (13)$$

Here $x = 0$ is an LIH with respect to the group generated by $\partial/\partial t$ and $\partial/\partial z$. Nevertheless, the Killing bivector $\partial/\partial t \wedge \partial/\partial z$ is not 2-surface orthogonal at $x = 0$.

(4) Most commonly, $\mathcal{N}$ will consist of discrete hypersurfaces separating regions of positive and negative W, i.e., regions where the Killing p-vector is nonnull and contains, respectively, an even and an odd number of independent orthogonal timelike directions (or more simply, in a Lorentz space, where the Killing p-vector is, respectively, spacelike and timelike).

A special case, which also arises commonly, is the situation where two such hypersurfaces have coalesced to give one, so that W has the same sign on both sides and has vanishing gradient on the hypersurface.

These possibilities are very well displayed in the hybrid Kerr-Reissner-Nordstrom solution.[8] The metric form in which it was discovered is

$$\begin{aligned} ds^2 = {} & \rho^2\,d\theta^2 + 2a\sin^2\theta\,dr\,d\varphi - 2\,dr\,du \\ & + \{r^2 + a^2 + (2mr - e^2)\rho^{-2}a^2\sin^2\theta\}\sin^2\theta\,d\varphi^2 \\ & - 2a(2mr - e^2)\rho^{-2}\sin^2\theta\,d\varphi\,du \\ & - \{1 - (2mr - e^2)\rho^{-2}\}\,du^2, \end{aligned} \quad (14)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, and the parameters m, e, ma, and ea are to be interpreted as the mass, charge, angular momentum, and magnetic dipole moment, respectively. Here, and in applications to general relativity throughout this paper, the units are understood to be such that the speed of light c and Newton's gravitational constant γ are both unity.

The Killing bivector $\partial/\partial\varphi \wedge \partial/\partial u$ is 2-surface orthogonal. It becomes null on the hypersurface where $\Delta \equiv r^2 - 2mr + e^2 + a^2 = 0$. Consequently, Theorem 1 implies that the hypersurfaces $\Delta = 0$ are LIH's.

The orthogonality is not immediately apparent in the above coordinate system, but, according to the result demonstrated in Sec. 1, a manifestly invertible coordinate system must exist. It may be obtained explicitly by using the generalized Boyer-Lindquist[9] transformation:

$$dt = -du - (r^2 + a^2)\Delta^{-1}\,dr, \quad d\phi = d\varphi + a\Delta^{-1}\,dr,$$

giving the invertible form

$$\begin{aligned} ds^2 = {} & \rho^2\Delta^{-1}\,dr^2 + \rho^2\,d\theta^2 \\ & + \{r^2 + a^2 + (2mr - e^2)\rho^{-2}a^2\sin^2\theta\}\sin^2\theta\,d\phi^2 \\ & + 2a(2mr - e^2)\rho^{-2}\sin^2\theta\,d\phi\,dt \\ & - \{1 - (2mr - e^2)\rho^{-2}\}\,dt^2. \end{aligned} \quad (15)$$

This form necessarily fails when the Killing bivector becomes null, but the orthogonality is patent elsewhere and it can be deduced by continuity that it holds where $\Delta = 0$ also.

The general and special cases mentioned above correspond to distinct and continuous roots of Δ. When there are no real roots, there are no LIH's. These different cases give rise to significant differences in the global topology (see the diagrams in Carter[10]), which can be applied qualitatively to the charged case provided it is noted that the discriminant of Δ is changed from $m^2 - a^2$ to $m^2 - a^2 - e^2$, and provided $a^2 \neq 0$; when $a^2 = 0$, the appropriate topological diagrams are also given by Carter[11]). In this paper we are not concerned with global matters, but it is the intimate connection between large-scale topology and LIH's which provides one of the motives for studying the latter.

4. EXISTENCE OF A KILLING HORIZON WHERE AN ORTHOGONALLY TRANSITIVE ABELIAN GROUP HAS NULL SURFACES OF TRANSITIVITY

If we are dealing with an Abelian group, the conclusion of Theorem 1 may be considerably strengthened.

Corollary to Theorem 1: Let the postulates of Theorem 1 be satisfied. Then if in addition the group

[8] E. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. **6**, 918 (1965).

[9] R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967).

[10] B. Carter, Phys. Rev. **141**, 1242 (1966).

[11] B. Carter, Phys. Letters **21**, 243 (1966).

is Abelian, it follows that each of the resulting LIH's is a Killing horizon.

Proof: Consider one of the resulting LIH's and let its null generator be l^μ. Since l^μ lies in the surface of transitivity of the group, we have

$$l^\mu = {}^{(i)}\alpha\, {}_{(i)}\xi^\mu, \tag{16}$$

where the set of scalars ${}^{(i)}\alpha$ is determined up to a constant of proportionality. In order to show that this LIH has a Killing vector field coinciding with its null generator, we need to show that the factor of proportionality may be chosen so that the ${}^{(i)}\alpha$ are constant in the LIH.

Since the surface of transitivity is only singly null, the direction of l^μ is fully determined by the condition that it be orthogonal to the surface of transitivity, i.e., $l_\mu w^{\mu\nu_2\cdots\nu_p} = 0$; thus substituting from (14) we find that the ${}^{(i)}\alpha$ are determined by

$$a_{(i)(j)}{}^{(j)}\alpha = 0; \quad a_{(i)(j)} = {}_{(i)}\xi^\mu{}_{(j)}\xi_\mu. \tag{17}$$

The solution of these equations is given by

$${}^{(i)}\alpha = kA^{(i)(j)}, \quad \text{for fixed } j, \tag{18}$$

where k is an arbitrary constant of proportionality and $A^{(j)(i)}$ is the cofactor of $a_{(i)(j)}$. Since $a_{(i)(j)}$ is singly null, its adjoint has rank 1 (by a well-known theorem of Jacobi) and therefore this set of solutions is nonvanishing for some values of j and is the same for all such values. For convenience we take $j = p$, reordering the labels if necessary in order to obtain a nonvanishing result.

We need to show that each of the ratios ${}^{(i)}\alpha$ to ${}^{(k)}\alpha$ is constant in the LIH. Since the Killing vectors commute, this is true automatically in the surfaces of transitivity and so we need only show that the ratios do not vary in orthogonal directions, i.e., that

$${}^{[(k)}\alpha^{(i)]}\alpha_{,[\rho}w_{\nu_1\cdots\nu_p]} = 0. \tag{19}$$

Hence, by (18), we have established the required result if we can prove

$$A^{[(\kappa)|(p)|}A^{(i)](p)}{}_{,[\rho}w_{\nu_1\cdots\nu_p]} = 0. \tag{20}$$

The cofactors are given explicitly by

$$A^{(i)(p)} = (-1)^{i+p}(p-1)!\, {}_{(1)}\xi_{\kappa_1}\cdots{}_{(i-1)}\xi_{\kappa_{i-1}} \times {}_{(i+1)}\xi_{\kappa_1}\cdots{}_{(p)}\xi_{\kappa_{p-1}}\, {}_{(1)}\xi^{[\kappa_1}\cdots{}_{(p-1)}\xi^{\kappa_{p-1}]}. \tag{21}$$

Therefore, using Killings equations ${}_{(i)}\xi_{(\mu;\nu)} = 0$ and the commutation conditions

$${}_{(i)}\xi_{\mu;\nu}\,{}_{(j)}\xi^\nu = {}_{(j)}\xi_{\mu;\nu}\,{}_{(i)}\xi^\nu,$$

we obtain

$$A^{(i)(p)}{}_{,\rho} = 2(-1)^{i+p}(p-1)! \times {}_{(1)}\xi_{\kappa_1}\cdots{}_{(i-1)}\xi_{\kappa_{i-1}}\,{}_{(i+1)}\xi_{\kappa_i}\cdots{}_{(p)}\xi_{\kappa_{p-1}} \times \sum_{j=1}^{p-1}{}_{(i)}\xi^{[\kappa_1}\cdots{}_{(j)}\xi^{\kappa_j}{}_{;\rho}\cdots{}_{(p-1)}\xi^{\kappa_{p-1}]}. \tag{22}$$

Again substituting (6) into the orthogonality conditions (4) and using the expansion (5), we see that the orthogonality conditions are equivalent to

$${}_{(j)}\xi^{[\sigma;\rho}w^{\nu_1\cdots\nu_p]} = 0 \quad (\text{each } j). \tag{23}$$

On expansion this gives

$$2\,{}_{(j)}\xi^\sigma{}_{;[\rho}w_{\nu_1\cdots\nu_p]} = \sum_{l=1}^{p}(-1)^l\,{}_{(l)}\xi^\sigma \times {}_{(j)}\xi_{[\rho;\nu_1}\,{}_{(i)}\xi_{\nu_2}\cdots{}_{(l-1)}\xi_{\nu_l}\,{}_{(l+1)}\xi_{\nu_{l+1}}\cdots{}_{(p)}\xi_{\nu_p]}. \tag{24}$$

Consequently we deduce that

$$\begin{aligned}2\sum_{j=1}^{p}{}_{(1)}\xi^{[\kappa_1}\cdots{}_{(j)}\xi^{\kappa_j}{}_{;[\rho}\cdots{}_{(p-1)}\xi^{\kappa_{p-1}]}w_{\nu_1\cdots\nu_p]} &= {}_{(1)}\xi^{[\kappa_1}\cdots{}_{(p-1)}\xi^{\kappa_{p-1}]} \\ &\times\sum_{j=1}^{p-1}(-1)^j\,{}_{(j)}\xi_{[\rho;\nu_1}\,{}_{(1)}\xi_{\nu_2}\cdots{}_{(j-1)}\xi_{\nu_j}\,{}_{(j+1)}\xi_{\nu_{j+1}}\cdots{}_{(p)}\xi_{\nu_p]} \\ &-\sum_{j=1}^{p-1}(-1)^j\,{}_{(1)}\xi^{[\kappa_1}\cdots{}_{(j-1)}\xi^{\kappa_{j-1}}\,{}_{(j+1)}\xi^{\kappa_j}\cdots{}_{(p)}\xi^{\kappa_{p-1}]} \\ &\times{}_{(j)}\xi_{[\rho;\nu_1}\,{}_{(1)}\xi_{\nu_2}\cdots{}_{(p-1)}\xi_{\nu_p]}.\end{aligned} \tag{25}$$

Substituting into (22) and using (21), we obtain

$$\begin{aligned}A^{(i)(p)}{}_{,[\rho}w_{\nu_1\cdots\nu_p]} &= A^{(i)(p)}\sum_{j=1}^{p-1}(-1)^j\,{}_{(j)}\xi_{[\rho;\nu_1} \times {}_{(1)}\xi_{\nu_2}\cdots{}_{(j-1)}\xi_{\nu_j}\,{}_{(j+1)}\xi_{\nu_{j+1}}\cdots\xi_{\nu_p]} \\ &-(-1)^p\sum_{j=1}^{p}A^{(i)(j)}\xi_{[\rho;\nu_1}\xi_{\nu_2}\cdots\xi_{\nu_{p-1}]}.\end{aligned} \tag{26}$$

When this is substituted into the left-hand side of (20), it can be seen that each of the terms has as a factor a 2×2 minor of the adjoint matrix $A^{(i)(j)}$. The terms must therefore vanish since, as has been already remarked, this matrix has rank 1. Thus (20) is true and the result is established.

We can apply this result to the charged Kerr solution. In terms of the metric form (14) with coordinates numbered from 1 to 4 in the order r, θ, φ, u, we find that the normal to the hypersurface $\Delta = 0$ has covariant components $l_\mu = \delta^1_\mu$. We can use the inverse metric given in Ref. 8 to obtain the contravariant components of the null generator:

$$l^\mu = \rho^{-2}\{\Delta\delta^\mu_1 - a\delta^\mu_3 - (r^2+a^2)\delta^\mu_4\}.$$

On the surface $\Delta = 0$, r takes constant values $r_\pm$.

Therefore we see that the null generator can be normalized so as to coincide with the Killing vector $\alpha(\partial/\partial\varphi) + (r_\pm^2 + a^2)\partial/\partial u$.

It should not be concluded from the result of this section that any null hypersurface which is an LIH with respect to an Abelian group is also a Killing horizon. A counterexample is provided by the metric described in note (3) after Theorem 1. It contains an LIH at $r = 0$ with respect to the Abelian group generated by $\partial/\partial z$ and $\partial/\partial t$. However, since the orthogonal transitivity condition does not hold, there is no reason why it should be also a Killing horizon, and indeed it is not. The null generator is $\partial/\partial t - y\partial/\partial z$. Therefore it cannot be normalized so as to coincide with any Killing vector field.

5. ORTHOGONAL TRANSITIVITY AND INVERTIBILITY OF AN $(n - 2)$-PARAMETER ABELIAN ISOMETRY GROUP WITH INVERTIBLE RICCI TENSOR

It is worthwhile to enquire when orthogonal transitivity and invertibility are likely to occur, not only because of their connection with LIH's, but also because they give rise to useful simplifications generally. Since in fact a large proportion of the known solutions of the general relativity equations have been obtained with the aid of various preassumed invertibility conditions (usually introduced with no other justification than algebraic convenience), it would probably be helpful in the future to know when such assumptions are reasonable and when they involve undesired restrictions.

One might also ask the specific question whether the orthogonal transitivity of the Kerr–Reissner–Nordstrom solution is merely a convenient algebraic coincidence, or whether there is a deeper reason for it.

Papapetrou has pointed out[12] that in the uncharged case there is a deeper reason, since he has shown that any stationary axisymmetric space–time satisfying the empty-space equations (i.e., vanishing Ricci tensor) in a region including the axis of symmetry must be orthogonally transitive in that region.

The objective of this section is to show that this rather striking result is a special case of a theorem with considerably wider significance. Thus Papapetrou's result can be extended in several directions: to a wider class of groups, to cases where the condition that the Ricci tensor vanishes is replaced by the very much weaker condition that it be invertible with respect to the group (which covers the charged case above), and to cases where the region under consideration does not include a symmetry axis but satisfies certain alternative conditions (an aspect which is further developed in Sec. 6).

When the general question of orthogonal transitivity of a p-transitive group in n dimensions is examined, it turns out that, for $p = n - 1$, the problem is trivial as has already been remarked, while for $p < n - 3$ the problem becomes very complicated, as it does even for $p = n - 2$ in the non-Abelian case. Therefore in the remainder of this paper we only attempt to deal with Abelian groups, and we are soon obliged to make the restriction $p = n - 2$.

Our results depend on the following lemma which gives a connection between the orthogonality condition and the Ricci tensor.

Lemma: Let $w^{\lambda_1\cdots\lambda_p} = {}_{(1)}\xi^{[\lambda_1}\cdots{}_{(p)}\xi^{\lambda_p]}$, where ${}_{(1)}\xi^\lambda, \cdots, {}_{(p)}\xi^\lambda$ are a set of generators of a p-parameter Abelian isometry group on an n-dimensional C^3 manifold with C^2 metric. Then

$$\{w^{[\lambda_1\cdots\lambda_p}\xi^{\mu;\rho]}\}_{;\rho} = \frac{2}{p+2}\, w^{[\lambda_1\cdots\lambda_p}R^{\mu]}{}_{\rho}\,{}_{(i)}\xi^\rho, \qquad i = 1, \cdots, p, \tag{27}$$

where R^μ_ν is the Ricci tensor.

Proof: For any set of C^2 vector fields,

$$\begin{aligned}
&{}_{(1)}\xi^\mu, \cdots, {}_{(p)}\xi^\mu \binom{p+2}{3}\, {}_{(1)}\xi^{[\lambda_1}\cdots{}_{(p-1)}\xi^{\lambda_{p-1}}\,{}_{(p)}\xi^{\lambda_p}\,{}_{(p)}\xi^{\lambda_{p+1};\rho]} \\
&= \binom{p+1}{2}\, {}_{(1)}\xi^{[\lambda_1}\cdots{}_{(p-1)}\xi^{\lambda_{p-1}}\,{}_{(p)}\xi^{[\lambda_p}\,{}_{(p)}\xi^{\lambda_{p+1}];\rho]} \\
&\quad + \frac{1}{(p-1)}\binom{p+1}{3}\sum_{i=1}^{p-1}(-1)^{p-i} \\
&\quad \times {}_{(1)}\xi^{[\lambda_1}\cdots{}_{(i)}\xi^{|\rho|}\cdots{}_{(p-1)}\xi^{\lambda_{p-2}}\,{}_{(p)}\xi^{\lambda_{p-1}}\,{}_{(p)}\xi^{\lambda_p;\lambda_{p+1}]}.
\end{aligned} \tag{28}$$

When we take the contracted derivative and make suitable rearrangements, we obtain the identity

$$\begin{aligned}
&(p+2)\{{}_{(1)}\xi^{[\lambda_1}\cdots{}_{(p-1)}\xi^{\lambda_{p-1}}\,{}_{(p)}\xi^{\lambda_p}\,{}_{(p)}\xi^{\lambda_{p+1};\rho]}\}_{;\rho} \\
&= 3\,{}_{(1)}\xi^{[\lambda_1}\cdots{}_{(p-1)}\xi^{\lambda_{p-1}}\{{}_{(p)}\xi^{[\lambda_p}\,{}_{(p)}\xi^{\lambda_{p+1}];\rho]}\}_{;\rho} \\
&\quad + 3\sum_{i=1}^{p-1}{}_{(1)}\xi^{[\lambda_1}\cdots{}_{(i)}\xi^{\lambda_i}{}_{;\rho}\cdots{}_{(p-1)}\xi^{\lambda_{p-1}}\xi^{[\lambda_p}\xi^{\lambda_{p+1}];\rho]} \\
&\quad - \sum_{i=1}^{p-1}{}_{(1)}\xi^{[\lambda_1}\cdots{}_{(i)}\xi^{|\rho|}\cdots{}_{(p-1)}\xi^{\lambda_{p-1}}\{{}_{(p)}\xi^{\lambda_i}\,{}_{(p)}\xi^{\lambda_p;\lambda_{p+1}]}\}_{;\rho} \\
&\quad - \sum_{i=1}^{p-1}{}_{(1)}\xi^{[\lambda_1}\cdots{}_{(i)}\xi^{|\rho|}{}_{;\rho}\cdots{}_{(p-1)}\xi^{\lambda_{p-1}}\,{}_{(p)}\xi^{\lambda_i}\,{}_{(p)}\xi^{\lambda_p;\lambda_{p+1}]} \\
&\quad + 2\sum_{i=2}^{p-1}\sum_{j=1}^{i-1}(-1)^{i-j-1} \\
&\quad \times {}_{(1)}\xi^{[\lambda_1}\cdots{}_{(j-1)}\xi^{\lambda_{j-1}}\,{}_{(j+1)}\xi^{\lambda_{j+1}}\cdots{}_{(i-1)}\xi^{\lambda_{i-2}} \\
&\quad \times {}_{(i+1)}\xi^{\lambda_{i-1}}\cdots{}_{(p-1)}\xi^{\lambda_{p-3}}\,{}_{[(i)}\xi^{\lambda_{p-2}}{}_{;\rho}\,{}_{(j)]}\xi^{|\rho|} \\
&\quad \times {}_{(p)}\xi^{\lambda_{p-1}}\,{}_{(p)}\xi^{\lambda_p;\lambda_{p+1}]}.
\end{aligned} \tag{29}$$

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[12] A. Papapetrou, Ann. Inst. H. Poincaré A-IV, 83 (1966).

We now use the condition that the ${}_{(i)}\xi^\mu$ commute with each other,

$$\mathfrak{L}_{[{}_{(j)}\xi^\rho]} {}_{(i)}\xi^\mu \equiv 2\,{}_{[(i)}\xi^\mu{}_{;\rho}\,{}_{(j)]}\xi^\rho = 0. \tag{30}$$

This implies that the last term in (29) vanishes. Applying Killing's equation ${}_{(i)}\xi^{(\mu;\nu)} = 0$, we deduce that ${}_{(i)}\xi^\rho;\rho = 0$ and hence that the second last term in (29) vanishes. Combining Killing's equation with (30) we obtain

$$\mathfrak{L}_{[{}_{(i)}\xi^\rho]}\{{}_{(p)}\xi^{[\lambda}\,{}_{(p)}\xi^{\mu;\nu]}\}$$
$$\equiv \{{}_{(p)}\xi^{[\lambda}\,{}_{(p)}\xi^{\mu;\nu]}\}_{;\rho}\,{}_{(i)}\xi^\rho - 3\,{}_{(i)}\xi^{[\lambda}{}_{;\rho}\,{}_{(p)}\xi^{[\mu}\,{}_{(p)}\xi^{\nu];\rho]} = 0, \tag{31}$$

from which it follows that the third and fourth last terms in (29) cancel each other out.

Since we could have singled out any one of the ${}_{(i)}\xi^\mu$ $(i = 1, \cdots, p-1)$ instead of ${}_{(p)}\xi^\mu$, it follows that for each i we have

$$\{w^{[\lambda_1\cdots\lambda_p}\,{}_{(i)}\xi^{\mu;\rho]}\}_{;\rho}$$
$$= \frac{3}{p+2}\,{}_{(1)}\xi^{[\lambda_1}\cdots{}_{(i-1)}\xi^{\lambda_{i-1}}\,{}_{(i+1)}\xi^{\lambda_{i+1}}\cdots{}_{(p)}\xi^{\lambda_p}$$
$$\times \{{}_{(i)}\xi^{[\lambda_i}\,{}_{(i)}\xi^{\mu];\rho]}\}_{;\rho}. \tag{32}$$

At this stage we introduce the Riemann and Ricci tensors defined by

$$\zeta_{\mu;[\nu\rho]} = \tfrac{1}{2}R^\sigma{}_{\mu\nu\rho}\zeta_\sigma; \quad R_{\mu\nu} = R_{\mu\rho}{}^\rho{}_\nu. \tag{33}$$

If we substitute any Killing vector ${}_{(i)}\xi^\mu$ in (33) and use the full Riemann-tensor symmetries together with Killing's equation, we obtain

$${}_{(i)}\xi^{\mu;\nu\rho} = R^{\mu\nu\rho}{}_\sigma\,{}_{(i)}\xi^\sigma. \tag{34}$$

Contracting Eq. (21) gives

$${}_{(i)}\xi^{\mu;\rho}{}_{;\rho} = R^\mu{}_\rho\,{}_{(i)}\xi^\rho. \tag{35}$$

From (22), with further use of Killing's equation, we can deduce

$$\{{}_{(i)}\xi^{[\mu}\,{}_{(i)}\xi^{\nu;\rho]}\}_{;\rho} = \tfrac{2}{3}\,{}_{(i)}\xi^{[\mu}R^{\nu]}{}_\rho\,{}_{(i)}\xi^\rho. \tag{36}$$

Finally, insertion of (36) into Eq. (32) gives Eq. (27), which is the desired result.

It is convenient to work with the orthogonal conjugate form of Eq. (27), i.e.,

$$(n-p-1)\,{}_{(i)}\chi_{[\kappa_{p+3}\cdots\kappa_n;\sigma]} = 2\,{}_{(i)}\xi^\rho R_\rho{}^\mu\,{}^*w_{\mu\kappa_{p+3}\cdots\kappa_n\sigma}, \tag{37}$$

where we have introduced a set of twist tensors ${}_{(i)}\chi_{\kappa_{p+3}\cdots\kappa_n}$ $(i = 1, \cdots, p)$ by

$${}_{(i)}\chi_{\kappa_{p+3}\cdots\kappa_n} = {}_{(i)}\xi^{\mu;\rho}\,{}^*w_{\mu\rho\kappa_{p+3}\cdots\kappa_n}. \tag{38}$$

The significance of the twist tensors can be seen by taking the orthogonal conjugate of Eq. (23). Thus Frobenius's Theorem may be expressed in the following alternative form: The elements spanned by ${}_{(1)}\xi^\rho, \cdots, {}_{(p)}\xi^\mu$ are orthogonal to a family of $(n-p)$-surfaces if and only if all the corresponding twist tensors vanish.

The utility of Eqs. (37) lies in the fact that the right-hand sides vanish for all i if and only if the Ricci tensor is invertible in the p element. However, as the equations control only the rotation of the twist tensors, this restriction is not very strong except when $p \geq n-2$, so that the twist tensors reduce to scalars or vanish trivially. This is why, in order to make further progress, we consider only $p = n-2$. Thus we now reach the main result of this section.

Theorem 2: Let $\mathfrak{D}$ be a connected open subdomain of an n-dimensional C^3 manifold with a C^2 pseudo-Riemannian metric and an Abelian $(n-2)$-parameter isometry group, whose surfaces of transitivity, which in general are $(n-2)$-dimensional, become degenerate on a subset $\mathcal{F}$ where the group has fixed points.

Then the group will be orthogonally transitive everywhere in $\mathfrak{D}$, and consequently invertible in $\mathfrak{D}$, except where the surfaces of transitivity are null, provided that:

(I) The Ricci tensor is invertible in the group everywhere in $\mathfrak{D}$; and

(II) one of the following holds:

(a) $\mathcal{F}$ is nonempty;

(b) there is a discrete isometry in some neighborhood in $\mathfrak{D}$ consisting of an inversion in a direction orthogonal to the surfaces of transitivity (in other words, an inversion about a hypersurface to which the surfaces of transitivity are tangent);

(c) it is known, for any other reason, that the group is orthogonally transitive on at least one point in $\mathfrak{D}$.

Proof: Let ${}_{(i)}\xi^\mu$, $i = 1, \cdots, n-2$ be a set of independent generators of the group. Then the corresponding twist tensors ${}_{(i)}\chi$ are scalars and, by the preceding work, they satisfy

$${}_{(i)}\chi_{,\sigma} = 2\,{}_{(i)}\xi^\rho R_\rho{}^\mu\,{}^*w_{\mu\sigma}. \tag{39}$$

As has already been remarked, the invertibility of the Ricci tensor implies the vanishing of the right-hand side, and so we see that the ${}_{(i)}\chi$ are constant in $\mathfrak{D}$.

Thus the group will be orthogonally transitive everywhere in $\mathfrak{D}$, provided that these constants are all zero. This establishes the result when (c) holds.

To check condition (b) we observe that if there is an inversion isometry in some direction, then it follows, when the direction is orthogonal to the surfaces of transitivity, that the tensors ${}_{(i)}\xi^{\mu;\nu}$ are also invertible in this direction. In the case under consideration, Eq. (38) reduces to

$$_{(i)}\chi = {}_{(i)}\xi^{\mu;\rho}\, {}^*w_{\mu\rho}\,, \tag{40}$$

and invertibility of the ${}_{(i)}\xi^{\mu;\nu}$ in a direction orthogonal to the surfaces of transitivity implies that the right-hand side vanishes, leading to the required result.

To check condition (a) we need only to notice that on $\mathcal{F}$ the Killing p vector vanishes, and consequently the ${}_{(i)}\chi$ vanish there also by (40), giving the required result.

This theorem is useful for general relativity because of the physical significance of the conditions. Since the metric tensor is invertible in all circumstances, we could, if we wished, substitute the Einstein tensor for the Ricci tensor in the statement of Theorem 2 and substitute $-G_\mu{}^\rho$ for $R_\mu{}^\rho$ in Eqs. (37) and (39), where the Einstein tensor is defined by

$$-G_\mu{}^\rho = R_\mu{}^\rho - \tfrac{1}{2}Rg_\mu{}^\rho \tag{41}$$

so that in general relativity, with units as for Eqs. (14) and (15), the energy-momentum tensor satisfies

$$T_\mu{}^\rho = \frac{1}{8\pi}\, G_\mu{}^\rho\,. \tag{42}$$

Since $n = 4$ in ordinary space-time, the physical applications of the theorem are to 2-parameter groups. Several 2-parameter Abelian isometry groups have been used for idealized problems in general relativity, of which cylindrical symmetry is perhaps the most popular. However, the most important case is that of stationary axial symmetry, since this applies to large classes of finite astrophysical objects as a realistic approximation.

As an example of the application of Theorem 2 to this situation, we shall show that the original result of Papapetrou, which applied to solutions of the vacuum Einstein equations, is in fact equally valid for solutions of the source-free Einstein-Maxwell equations.

Let $F_{\mu\nu}$ be the electromagnetic-field tensor and let ${}_{(i)}\xi^\mu$, $i = 1, 2$, be the two commuting Killing vectors in the space. The Lie derivative of the electromagnetic-field tensor with respect to each of these must vanish, i.e.,

$$\underset{[{}_{(i)}\xi^\rho]}{\pounds} F_{\mu\nu} \equiv F_{\mu\nu;\rho}\, {}_{(i)}\xi^\rho + 2F_{[\epsilon|\rho|}\, {}_{(i)}\xi^\rho{}_{\nu]} = 0, \tag{43}$$

from which, using the condition (17) that the Killing vectors commute, we obtain

$$\{F_{\mu\nu}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu\}_{;\sigma} = 3F_{[\mu\nu;\sigma]}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu. \tag{44}$$

Similarly, we can obtain two equations identical to (30) and (31), except that $F_{\mu\nu}$ is replaced by its orthogonal conjugate ${}^*F_{\mu\nu}$. Maxwell's equations take the form

$$F_{[\mu\nu;\sigma]} = 0, \tag{45}$$

$${}^*F_{[\mu\nu;\sigma]} = (4\pi/3)\, {}^*j_{\mu\nu\sigma}\,, \tag{46}$$

where j^μ is the current vector, and so we obtain

$$\{F_{\mu\nu}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu\}_{;\sigma} = 0, \tag{47}$$

$$\{{}^*F_{\mu\nu}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu\}_{;\sigma} = 4\pi\, {}^*j_{\mu\nu\sigma}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu. \tag{48}$$

Equation (34) implies that $F_{\mu\nu}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu$ is always constant, while (48) implies that ${}^*F_{\mu\nu}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu$ is also constant when the right-hand side vanishes, which occurs if and only if the current vector lies in the 2-surface of transitivity. Therefore, if these two quantities vanish at any point in a connected region satisfying this condition, and in particular if there is a symmetry axis within the region where one of the Killing vectors vanishes, then they vanish everywhere in the region, i.e.,

$$F_{\mu\nu}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu = {}^*F_{\mu\nu}\, {}_{(1)}\xi^\mu\, {}_{(2)}\xi^\nu = 0. \tag{49}$$

This is the condition that the tensor $F_{\mu\nu}$ be skew invertible, i.e., that it be affected only by an overall change of sign when the senses of the Killing vectors are simultaneously inverted. Since the energy momentum tensor of the electromagnetic field is homogeneous quadratic in the electromagnetic field tensor $F_{\mu\nu}$, it follows that condition (49) implies that the energy-momentum tensor is invertible, and consequently, when no matter other than the electromagnetic field is present, that the Einstein tensor is invertible so that the conditions of Theorem 2 are satisfied.

Thus from Theorem 2 we deduce the following result:

If the vacuum Einstein-Maxwell equations are satisfied in a connected region of a 4-dimensional space-time with a 2-parameter Abelian group, if a symmetry axis is present in the region, and if the source current is parallel to the 2-surfaces of transitivity (and, in particular, if there is no source current, as is usually assumed to be the case when no ponderable matter is present), then the group is orthogonally transitive.

When no electromagnetic field is present, this reduces to Papapetrou's result. The more general result shows that the orthogonal transitivity of the Kerr–Reissner–Nordstrom solution could have been predicted at once, even though it was not immediately apparent in the original form (12) of the solution.

6. CONVECTIVE CIRCULATION

We have not yet fully exploited the information in Eq. (39). In order to do so, we make some further definitions with ultimate astrophysical applications in mind.

A vector is said to be nonconvective with respect to an isometry group if it is invertible in the element orthogonal to the surface of transitivity at a point; otherwise it is said to be convective; i.e., it is nonconvective if and only if it is tangent to the surface of transitivity.

We define the flux vector of a group as the two-index quantity

$$ {}_{(i)}F^{\mu} = \frac{1}{8\pi}\, {}_{(i)}\xi^{\rho} G_{\rho}^{\ \mu}, \tag{50} $$

where the ${}_{(i)}\xi^{\rho}$ are a set of generators of the group. This quantity transforms as an ordinary vector with respect to μ in the manifold, and as a covariant vector with respect to (i) under a change of basis of the Lie algebra of the group. We say that the group is nonconvective if ${}_{(i)}F^{\mu}$ is nonconvective with respect to the group for each (i). We note that the statement that the group is nonconvective is invariant in the Lie algebra, and that it is equivalent to the statement that the Ricci tensor is invertible in the group.

Suppose that in an n-dimensional manifold with an Abelian isometry group transitive over p-surfaces, we have a finite segment $\mathcal{H}$ of an invariant hypersurface generated as follows. We take a finite segment $\mathcal{S}$ of an $(n-p-1)$-surface which cuts across the surfaces of transitivity, and drag it along under a set

$$ {}_{(1)}\xi^{\mu}, \cdots, {}_{(p)}\xi^{\mu} $$

of independent generators of the group by finite values, $\Delta\psi^{(1)}, \cdots, \Delta\psi^{(p)}$ of the group parameters where the group parameters $\psi^{(1)}, \cdots, \psi^{(p)}$ may be taken to be a set of functions defined on the space in such a manner that

$$ {}_{(i)}\xi^{\mu} = \partial x^{\mu}/\partial\psi^{(i)} \tag{51} $$

($x^1, \cdots, x^n$ being the coordinate patch in the manifold to which the tensor indices refer). Then we define the convective circulation through $\mathcal{H}$ as the surface integral of the normal component of the flux vector over $\mathcal{H}$. The circulation transforms as a covariant vector in the Lie algebra.

If $\mathcal{H}$ is generated by unit parameter changes $\Delta\psi^{(1)} = \cdots = \Delta\psi^{(p)} = 1$, we say that it is the unit hypersurface $\mathcal{J}(\mathcal{S})$ through $\mathcal{S}$ and that the circulation over it is the unit convective circulation over $\mathcal{S}$, which we denote by ${}_{(i)}C(\mathcal{S})$. We see that ${}_{(i)}C(\mathcal{S})$ transforms as the product of a covariant vector and a density in the Lie algebra.

When $p = n - 2$, $\mathcal{S}$ will be a line. We can now state the following result.

Corollary to Theorem 2: Let the postulates of Theorem 2 be satisfied except for the conditions (I) and (II). Then the unit convective circulation between two points in $\mathcal{D}$ is independent of the path over which it is taken; and if the group is orthogonally transitive at a point P in $\mathcal{D}$, then it is orthogonally transitive at a point Q in $\mathcal{D}$ if and only if the unit convective circulation over a path PQ between then vanishes.

Proof: By Eqs. (39) and the definitions (41) and (50) we have

$$ {}_{(i)}\chi_{,\sigma} = 16\pi\, {}_{(i)}F^{\mu}\, {}^{*}w_{\sigma\mu}. \tag{52} $$

Expanding this and expressing it in terms of differential forms, we obtain

$$ d\, {}_{(i)}\chi = \frac{16\pi}{(n-2)!}\, {}_{(i)}F^{\mu} \epsilon_{\kappa_1 \cdots \kappa_{n-2}\sigma\mu}\, {}_{(1)}\xi^{[\kappa_1} \cdots {}_{(n-2)}\xi^{\kappa_{n-2}]}\, dx^{\sigma}. \tag{53} $$

Now by (51) we have

$$ dx^{\kappa_1} \wedge \cdots \wedge dx^{\kappa_{n-2}} = (n-2)!\, {}_{(1)}\xi^{[\kappa_1} \cdots {}_{(n-2)}\xi^{\kappa_{n-2}]}\, d\psi^{(1)} \wedge \cdots \wedge d\psi^{(n-2)}. \tag{54} $$

Therefore,

$$ {}_{(i)}F^{\mu}\, d\Sigma_{\mu} = \frac{(n-2)!}{16\pi(n-1)}\, d\psi^{(1)} \wedge \cdots \wedge d\psi^{(n-2)} \wedge d\, {}_{(i)}\chi, \tag{55} $$

where we have defined the $(n-1)$-form

$$ d\Sigma_{\mu} = \frac{1}{(n-1)!}\, \epsilon_{\kappa_1 \cdots \kappa_{n-2}\sigma\mu}\, dx^{\kappa_1} \wedge \cdots \wedge dx^{\kappa_{n-2}} \wedge dx^{\sigma}. \tag{56} $$

To obtain the unit convective circulation between P and Q we integrate (35) over the unit hypersurface $\mathcal{J}(PQ)$ which gives

$$ {}_{(i)}C(PQ) \equiv \int_{\mathcal{J}(PQ)} {}_{(i)}F^{\mu}\, d\Sigma_{\mu} = \frac{(n-2)!}{16\pi(n-1)} \int_P^Q d\, {}_{(i)}\chi = \frac{(n-2)!}{16\pi(n-1)} \{ {}_{(i)}\chi(Q) - {}_{(i)}\chi(P) \}. \tag{57} $$

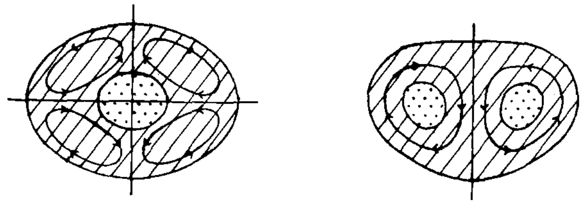
We can see at once that the result is independent of the path PQ (even if $\mathfrak{D}$ is not simply connected), and that if ${}_{(i)}\chi$ vanishes at P, it will vanish at Q if and only if ${}_{(i)}C(P, Q)$ vanishes. This establishes the required result.

The mathematical significance of Theorem 2 and its corollary seems to be that, in the circumstances to which the results apply, the existence of orthogonal transitivity is controlled almost entirely by the Ricci tensor. One might have expected *a priori* that the Weyl tensor would be able to transmit the effects of noninvertibility of the Ricci tensor in a nearby region and thereby prevent orthogonal transitivity from obtaining in a region where locally the Ricci tensor is invertible. Our results show that this can in fact happen, but only in a very restricted way, governed by the total circulation.

As we have remarked, the most suitable application for these results in general relativity is to stationary axisymmetric rotating bodies. Let us consider, in such a case, a region where the Killing bivector is timelike. (For a simple situation, such a region would have to include the whole space, or else by Theorem 1 there would exist an LIH, with, in general, dramatic consequences.) Then locally it is possible to choose a pair of Killing fields generating the group such that one of them ${}_{(1)}\xi^\mu$ is timelike, and the other ${}_{(2)}\xi^\mu$ is spacelike. We can define momentum and stress flux vectors P^μ and Γ^μ by

$$P^\mu = {}_{(1)}F^\mu = {}_{(1)}\xi^\rho T_\rho{}^\mu; \quad \Gamma^\mu = {}_{(2)}F^\mu = {}_{(2)}\xi^\rho T_\rho{}^\mu .$$

The convective components of P^μ and Γ^μ correspond to momentum across the surfaces of transitivity and shearing stress between the surfaces of transitivity, respectively. The corollary to Theorem 2 gives conservation equations for the convective components of P^μ and Γ^μ. They can be regarded as equations of conservation of momentum and balance of torque forces in the body. (Conservation of nonconvective components is trivial in consequence of the group.) The effects of gravitational potential energy and the adjustment of the correct radial factor in the torque are automatically taken care of by the varying magnitude of the Killing vector with which the energy momentum tensor is contracted.



(a) (b)

FIG. 1. Cross sections of two examples of stationary axisymmetric bodies are represented. The convective regions are shaded, with convective flow lines marked. The nonconvective regions are dotted, the only flow lines being directly into or out of the paper.

Figure 1 shows two simple examples of rotating bodies to which Theorem 2 and its corollary may be applied. We know at once in such cases that the group is orthogonally transitive in empty space outside the body, since the exterior must always contain part of the symmetry axis. (This is Papapetrou's result.) Now let us consider the interior. The first example is an object which has a nonconvective core, but which has a convective envelope containing two large convection cells, one on each side of a plane of equatorial symmetry. We can deduce that the group will be orthogonally transitive in the core either by applying condition (IIa), since the symmetry axis passes through the core, or by applying condition (IIb), since the equatorial plane also passes through the core. Hence, by the Corollary, the unit convective circulation over any line passing from the core to the outside must be zero. The second example is a smokeringlike object containing an annular nonconvective core about which the matter outside circulates; we conclude, by the corollary, that the group is certainly not orthogonally transitive in the annulus.

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