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Fields in nonaffine bundles. II. Gauge-coupled generalization of harmonic mappings and their Bunting identities

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The general-purpose bitensorially gauge-covariant differentiation procedure set up in the preceding article is specialized to the particular case of bundles with nonlinear fibers that are endowed with a (torsion-free) Riemannian or pseudo-Riemannian metric structure. This formalism is used to generalize the class of harmonic mappings between Riemannian or pseudo-Riemannian spaces to a natural gauge-coupled extension in the form of a class of field sections of a bundle having the original image space as fiber, with a nonintegrable gauge connection A belonging to the algebra of the isometry group of the fiber space. The Bunting identity that can be used for establishing uniqueness in the strictly positive metric-Riemannian case with negative image-space curvature is shown to be generalizable to this gauge-coupled extension.

I. INTRODUCTION

The purpose of this article is to construct and investigate the natural gauge-coupled generalization of the extensive class of nonlinear field models known as harmonic mappings. The work will be based on the use of the general-purpose bitensorially gauge-covariant differentiation formalism set up by the author in the preceding article,[1] of which the relevant essentials (as specialized to the torsion-free Riemannian or pseudo-Riemannian case) are summarized in Sec. II of the present article.

The general class of harmonic mappings has been studied in mathematical circles for many years. A convenient introductory review, from a physically motivated point of view, has been provided by Misner.[2] One of the reasons for interest in harmonic mappings

$$\mathscr{M} \mapsto \mathscr{X} \tag{1.1}$$

is the fact that they include the subclass known to physicists as nonlinear σ models in the case when the Riemannian image space $\mathscr{X}$ has a suitable homogeneous symmetric space structure. Wherever $\mathscr{X}$ is subject to a continuous isometry-group action (not necessarily a fully effective one as in the homogeneous case) the possibility arises of generalizing the class of simple mappings of the form (1.1) to bundle sections, whereby the base space $\mathscr{M}$ is mapped vertically into fibers of the form $\mathscr{X}$ in a bundle $\mathscr{B}$ subject to the isometry-group action in question. The purpose of the present work is to describe the natural extension of the general class of harmonic mappings of the form (1.1) to a class of *gauge-harmonic* bundle sections

$$\mathscr{M} \mapsto \mathscr{B} \tag{1.2}$$

that will be automatically determined by the specification of a bundle connection A for any given Riemannian structure on the base $\mathscr{M}$ and the fiber space $\mathscr{X}$. (For the special case of the nonlinear σ models with homogeneous symmetric space structure such a generalization has already been carried by the present author[3] using the more traditional method whereby the curved fiber space is treated by an imbedding in a higher-dimensional flat space.)

After the appropriate gauge harmonic field equations have been derived in Sec. III, they will be shown in Sec. IV to be amenable to treatment by an extension of the method recently developed in the context of ordinary harmonic mappings by Bunting[4,5] for the purpose of establishing uniqueness of solutions subject to suitable boundary conditions and inequalities.

II. THE BITENSORIALLY COVARIANT DIFFERENTIATION PROCEDURE FOR FIBERS WITH (PSEUDO-) RIEMANNIAN STRUCTURE

We start by summarizing the bitensorially covariant differentiation procedure set up in the preceding article[1] insofar as it applies to the restricted special case of fields taking values in a space $\mathscr{X}$ with a Riemannian or pseudo-Riemannian *metric structure* as specified in terms of local coordinates X^A ($A=1,\dots,m$) on $\mathscr{X}$ by

$$ds^2=\hat{g}_{AB}dX^A dX^B \tag{2.1}$$

with an associated metric connection whose components are given by the standard formula

$$\hat{\Gamma}_A{}^B{}_C=\hat{g}^{BD}(\hat{g}_{D(A,C)}-\tfrac{1}{2}\hat{g}_{AC,D}) \tag{2.2}$$

(using parentheses to denote symmetrization) where $\hat{g}^{BD}$ are components of the inverse metric to $\hat{g}_{AB}$, and the comma suffixes indicate partial derivatives with respect to the corresponding coordinates.

Despite these restrictions in relation to the more general situation considered in the preceding article (where the connection $\hat{\Gamma}$ was allowed to be quite arbitrary) the present context remains nevertheless more general than that considered by Misner[2] insomuch as we do not suppose that the field configurations in $\mathscr{X}$ are defined absolutely, but allow for the possibility of an intrinsic indeterminacy modulo the action of a gauge group $\mathscr{G}$ which must, of course, be a subgroup of the isomorphism group of $\mathscr{X}$, the existence of a nontrivial gauge freedom thus requiring the existence of a nontrivial isometry group with respect to the metric (2.1). This means that the field is to be considered as a *section* $\Phi(x)$ in a bundle $\mathscr{B}$ with fiber space $\mathscr{X}$ over a base space $\mathscr{M}$ which we shall suppose to be described by local coordinates x^μ $(\mu=1,\ldots,n)$. Such a bundle will be characterized locally by a simple but nonunique direct-product structure which may be represented by expressing the elements of a neighborhood in $\mathscr{B}$ as a couple (X,x) with corresponding local coordinates $\{X^A,x^\mu\}$ for $X\in\mathscr{X}$, $x\in\mathscr{M}$. In such a coordinate system the fiber-coordinate components of the fiber metric as induced on the bundle will depend only on the X^A, i.e, we shall have

$$\hat{g}_{AB,\mu}=0 \tag{2.3}$$

while, if we suppose that the base space has its own metric, given by

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu$$

with the resulting connection

$$\Gamma_\mu{}^\nu{}_\rho=g^{\nu\sigma}(g_{\sigma(\mu,\rho)}-\tfrac{1}{2}g_{\mu\rho,\sigma})\ , \tag{2.4}$$

the corresponding form induced on the bundle will satisfy

$$g_{\mu\nu,A}=0\ . \tag{2.5}$$

The gauge indeterminancy consists in the possibility of conserving this structure when effecting a fiber-coordinate transformation of the form

$$X^A(X,x)\mapsto G^A(X,x)\ . \tag{2.6}$$

The requirement that the property (2.3) should be preserved is expressible as the condition that the gauge transformation (2.6) should be characterized by the property

$$G^A{}_{,\mu}=G^A{}_{,B}K^B{}_\mu\ , \tag{2.7}$$

where the $K^B{}_\mu$ are components of some base-space one-form-valued vertical vector field $\mathbf{K}_\mu$ satisfying the fiber-space Killing equation

$$\hat{\nabla}^{(A}K^{B)}{}_\mu=0\ , \tag{2.8}$$

where $\hat{\nabla}$ is the ordinary Riemannian operation of covariant differentiation with respect to (2.1) and (2.2) as expressed by

$$\hat{\nabla}^AK^B{}_\mu=\hat{g}^{AC}(K^B{}_{\mu,C}+\hat{\Gamma}_C{}^B{}_DK^D{}_\mu)\ . \tag{2.9}$$

In such a context, the gauge connection A can be appropriately envisaged in the manner described in the preceding article,[1] as a gauge-patch-dependent fiber-tangent-vector-field-valued one-form $\mathbf{A}$ with local coordinates $A_\mu{}^A$ which acts as the generator of the relevant fiber-space isometry transformations, and which must therefore be characterized by the same Killing-vector property as $\mathbf{K}$, namely,

$$\hat{\nabla}^{(A}A_\mu{}^{B)}=0\ . \tag{2.10}$$

The condition characterized by (2.7) and (2.8) automatically ensures that the corresponding gauge transformation

$$A_\mu{}^A\mapsto G^A{}_{,B}A_\mu{}^B-G^A{}_{,\mu} \tag{2.11}$$

of the connector field $\mathbf{A}$ will preserve the gauge algebra property (2.10). As in the more general context considered in the preceding article,[1] the nontensorial (inhomogeneous) transformation rule (2.11) gives rise to a purely tensorial transformation rule

$$F_{\mu\nu}{}^A\mapsto G^A{}_{,B}F_{\mu\nu}{}^B$$

for the corresponding *fiber-tangent-vector-field-valued gauge-curvature two-form* $\mathbf{F}$ as defined by the basic formula

$$F_{\mu\nu}{}^A=2A_{[\nu}{}^A{}_{,\mu]}+2A_{[\nu}{}^BA_{\mu]}{}^A{}_{,B} \tag{2.12}$$

(where square brackets denote antisymmetrization) so that $\mathbf{F}$ can be considered as a *globally* well-defined bitensorial field over the entire bundle $\mathscr{B}$.

The covariant derivative $D\Phi$ of a field $\Phi(x)$ (i.e., a section of the bundle $\mathscr{B}$) over $\mathscr{M}$ was shown[1] to have bitensorial components $\Phi^A{}_{|\mu}$ given simply by

$$\Phi^A{}_{|\mu}=\partial_\mu X^A+A_\mu{}^A \tag{2.13}$$

and transforming under (2.6) according to the ordinary vectorial rule

$$\Phi^A{}_{|\mu}\mapsto G^A{}_{,B}\Phi^B{}_{|\mu}\ , \tag{2.14}$$

where $\partial_\mu X^A$ denotes the base-space gradient components of the coordinate component $X^A(\Phi(x))$ of the field Φ with respect to the local gauge coordinate patch $\{X^A,x^\mu\}$ on the bundle $\mathscr{B}$.

In order to construct higher-order similarly bitensorial derivatives, it is necessary to introduce the section-dependent connector field ω which was shown[1] to be given by

$$\omega_\mu{}^A{}_B=\Phi^C{}_{|\mu}\hat{\Gamma}_B{}^A{}_C+A_\mu{}^A{}_{,B}\ , \tag{2.15}$$

where the values of $\hat{\Gamma}_B{}^A{}_C$ and $A_\mu{}^A{}_{,B}$ are evaluated on the section $\Phi(x)$. In terms of this connector field and of the ordinary base-space connection Γ the second-order covariant derivative components are expressible as

$$\Phi^A{}_{|\mu|\nu}=(\Phi^A{}_{|\mu})_{;\nu}+\omega_\nu{}^A{}_B\Phi^B{}_{|\mu}\ , \tag{2.16}$$

where the semicolon denotes ordinary (base but not fiber) covariant derivation as defined by

$$(\Phi^A{}_{|\nu})_{;\mu}=\partial_\mu\Phi^A{}_{|\nu}-\Gamma_\mu{}^\rho{}_\nu\Phi^A{}_{|\rho}\ . \tag{2.17}$$

The antisymmetric part of this bitensor will be expressible (in the torsion-free case under consideration here) purely in terms of the curvature field $\mathbf{F}$ as evaluated on the section in the form

$$\Phi^A{}_{|[\mu|\nu]}=\tfrac{1}{2}F_{\mu\nu}{}^A\ . \tag{2.18}$$

Antisymmetrized differentiation at higher orders introduces contributions arising from the curvature of the fiber and base spaces, as represented by the corresponding Rieman tensors with components $\hat{R}_{AB}{}^{C}{}_{D}$ and $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ obtained from (2.2) and (2.4) by the standard formulas

$$\hat{R}_{AB}{}^{C}{}_{D}=2\hat{\Gamma}_{[B}{}^{C}{}_{|D|,A]}+2\hat{\Gamma}_{[A}{}^{C}{}_{|E|}\hat{\Gamma}_{B]}{}^{E}{}_{D} \tag{2.19a}$$

and

$$R_{\mu\nu}{}^{\rho}{}_{\sigma}=2\Gamma_{[\nu}{}^{\rho}{}_{|\sigma|,\mu]}+2\Gamma_{[\mu}{}^{\rho}{}_{|\tau|}\Gamma_{\nu]}{}^{\tau}{}_{\rho} . \tag{2.19b}$$

Thus, starting from the basic expression

$$\Phi^{A}{}_{|\mu|\nu|\rho}=(\Phi^{A}{}_{|\mu|\nu})_{;\rho}+\omega_{\rho}{}^{A}{}_{B}\Phi^{A}{}_{|\mu|\nu} \tag{2.20}$$

for the third-order bitensorially covariant derivative of Φ, one obtains

$$\Phi^{A}{}_{|\mu|[\nu|\rho]}=\Omega_{\nu\rho}{}^{A}{}_{B}\Phi^{B}{}_{|\mu}-R_{\nu\rho}{}^{\sigma}{}_{\mu}\Phi^{A}{}_{|\sigma} , \tag{2.21}$$

where the section-dependent total curvature bitensor Ω will be given[1] (in this torsion-free case) by

$$\Omega_{\nu\rho}{}^{A}{}_{B}=\Phi^{C}{}_{|\mu}\Phi^{D}{}_{|\nu}\hat{R}_{CD}{}^{A}{}_{B}+\hat{\nabla}_{B}F_{\mu\nu}{}^{A} . \tag{2.22}$$

III. THE GAUGE-COUPLED GENERALIZATION OF A HARMONIC MAPPING

In terms of the formalism set up in the preceding section, it is obvious how one should proceed to generalize the concept of a harmonic mapping as described, e.g., by Misner,[2] so as to incorporate a minimal gauge-invariant coupling to a nonintegrable gauge-connector field. In the following analysis we shall allow, in addition to the minimal gauge coupling, the possibility that there is also a nonlinear gauge-invariant nondifferential self-coupling term.

The field equations for such a system will be obtained by the application of the usual kind of stationary-variation principle to a Lagrangian integral of the form

$$I=\int d^{n}x\,||g||^{1/2}L(\Phi,D\Phi) \tag{3.1}$$

over the base space $\mathscr{M}$, where the Lagrangian scalar function L is taken to be a quadratic function of the gradients of the field section Φ, with the gauge-invariant form

$$L=\tfrac{1}{2}\rho\hat{g}_{AB}g^{\mu\nu}\Phi^{A}{}_{|\mu}\Phi^{B}{}_{|\nu}+\pi\mathscr{V}(\Phi) , \tag{3.2}$$

where ρ and π are given scalar fields over the base space $\mathscr{M}$ (which may occur naturally as known weight functions in certain contexts) and where $\mathscr{V}$ is a self-interaction potential that is given as a scalar field over the fiber space $\mathscr{X}$ and which, to avoid breaking the symmetry, should be required to be invariant under the gauge group action, i.e.,

$$K^{A}\mathscr{V}_{,A}=0 \tag{3.3}$$

for any member $\mathbf{K}$ of the subset of solutions of the fiber-space Killing equations

$$\hat{\nabla}^{(A}K^{B)}=0 \tag{3.4}$$

that constitutes the gauge group algebra. [Evidently if one were considering the gauge coupling of the usual kind of nonlinear σ model for which the fiber space $\mathscr{X}$ is homogeneous, and if one wished to use a *maximal* gauge group which would act effectively over the whole of $\mathscr{X}$, then the requirement (3.4) would restrict $\mathscr{V}$ to be a trivial uniform field over $\mathscr{X}$ giving no contribution to the field equations for Φ.]

The variation of the section $\Phi(x)$ and thus of the corresponding coordinate components $X^{A}(\Phi)$, while keeping the background fields ρ,π as well as the base metric g and the gauge connection A constant, leads to an infinitesimal variation δL given in terms of the section component variations δX^{A} by

$$\delta L=(\rho\Phi_{A}{}^{|\mu}\delta X^{A})_{;\mu}+\frac{\delta L}{\delta\Phi^{A}}\delta X^{A} , \tag{3.5}$$

where the Eulerian derivative takes the form

$$\frac{\delta L}{\delta\Phi^{A}}=-[(\rho\Phi_{A}{}^{|\mu})_{|\mu}-\pi\mathscr{V}_{,A}] , \tag{3.6}$$

where the base and fiber metrics g and $\hat{g}$ have been used in the normal way for the definition of (bitensorially) covariant index raising and lowering, so that explicitly

$$\Phi_{A}{}^{|\mu}=g^{\mu\nu}\hat{g}_{AB}(\partial_{\mu}X^{B}+A_{\mu}{}^{B}) . \tag{3.7}$$

In expressing the first term on the right-hand side of (3.5), use has been made of the fact that because it acts on a quantity which is scalar with respect to the fiber indices, the ordinary base-coordinate-covariant differentiation operation, as denoted by a semicolon can be used interchangeably with the gauge-covariant differentiation operation indicated by a bar. Since it thus takes the form of an ordinary divergence this first term can be eliminated in the usual way, so that one obtains the required field equations

$$\frac{\delta L}{\delta\Phi^{A}}=0 \tag{3.8}$$

expressing the condition that Φ should be a critical point of the integral I, in the form

$$(\rho\Phi_{A}{}^{|\mu})_{|\mu}=\pi\mathscr{V} . \tag{3.9}$$

This system of equations may be written out in somewhat more explicit but no longer manifestly gauge-covariant form as

$$\nabla^{\mu}(\rho\Phi^{A}{}_{|\mu})+\rho\Phi^{B\,|\mu}\hat{\nabla}_{B}A^{A}{}_{\mu}=\pi\hat{\nabla}^{A}\mathscr{V} , \tag{3.10}$$

where (using the notation of the preceding article[1])

$$\nabla^{\mu}(\rho\Phi^{A}{}_{|\mu})=(\rho\Phi^{A}{}_{|\mu})^{;\mu}+\rho\Phi^{B\,|\mu}\hat{\Gamma}_{C}{}^{A}{}_{B}\partial_{\mu}X^{C} . \tag{3.11}$$

In transforming from (3.9) to (3.10) we have taken advantage of the fact that, in addition to the obvious consequence

$$g_{\mu\nu|\rho}=0 \tag{3.12}$$

of (2.4), the conditions (2.2), (2.3), and (2.10) taken together ensure that the gauge-covariant derivative of the fiber metric $\hat{g}$, as evaluated on the section Φ, will also automatically vanish, i.e.,

$$\hat{g}_{AB|\mu}=0 \tag{3.13}$$

so that all the bitensorial index-raising and -lowering operations commute with bitensorially covariant differentiation.

In the absence of the additional self-coupling term $\mathscr{V}$ and of the gauge field A (and provided ρ is uniform) the field equations (3.8) can be seen to reduce to a system of the much studied harmonic type described, e.g., by Misner.[2]

IV. GENERALIZED BUNTING IDENTITY FOR GAUGE-HARMONIC MAPPINGS

The purpose of this final section is to extend to the gauge-coupling system that has just been presented a very useful identity involving the deviation between two hypothetically different sections of the kind that was introduced by Bunting[4,5] for systems of ordinary harmonic type for the purpose of establishing uniqueness subject to appropriate inequalities and boundary conditions. Bunting's work was motivated by the problem of establishing the uniqueness of solutions of the black-hole equilibrium problem, which had been reduced by the present author[6,7] (subject to global hypotheses which still lack an entirely complete and rigourous justification) to a boundary-value problem of the right (harmonic) type. A concise presentation of the main results of Bunting's work, and an examination of the relation between the Bunting identity and a more specialized identity constructed independently for the same purpose by Mazur[8] (including as a special case the identity of Robinson[9]) has recently been given by the present author.[10] The Mazur identity applies only in the more restricted context (which, however, includes the case of the black-hole problem) in which the harmonic system is an appropriate kind of nonlinear σ model, as characterized by a requirement to the effect that the image (fiber) space should have a fully symmetric homogeneous structure. The natural gauge-coupled extension of the Mazur identity for such fully symmetric spaces has already been described by the present author elsewhere.[3] The Bunting method, which is applicable to a much less restricted class of image spaces, uses concepts related to those introduced in a general study of harmonic systems by Schoen and Yau.[11]

The context that we wish to consider is one in which we have two distinct bundle sections $\Phi_{[0]}(x)$ and $\Phi_{[1]}(x)$ which we suppose to be homotopically connectable in the sense that in the fiber over each base point $x\in\mathscr{M}$ there is some (smooth) curve $\Phi(t;x)$ parametrized by a variable t ranging from 0 to 1, with

$$\Phi(0;x)=\Phi_{[0]}(x),\quad \Phi(1;x)=\Phi_{[1]}(x) \tag{4.1}$$

and varying smoothly as a functions of x, so that $\Phi(t;x)$ represents a well-behaved section in $\mathscr{B}$ over $\mathscr{M}$ for each fixed value of t. Without loss of generality—except for the exclusion of curves that become null in fiber spaces with indefinite-metric signature—we may, following Bunting, require that the parametrization should be adjusted so as to be affine along the curve about each base point $x\in\mathscr{M}$, with parametrization chosen so that the corresponding tangent vector with local components given by

$$s^A=\frac{dX^A}{dt} \tag{4.2}$$

should everywhere satisfy

$$s^A s_A=s^2\ ,$$

where s is the total metric length of the curve.

Let us introduce the notation

$$\hat{D}=s^A\hat{\nabla}_A \tag{4.3}$$

to denote the correspondingly parametrized operation of covariant differentiation along the fiber above any *fixed base point* $x\in\mathscr{M}$. Let us similarly use the symbol D_μ to indicate gauge-covariant differentiation, as defined in Sec. II, of quantities defined in the section $\Phi(t;x)$ determined by any *fixed affine parameter value* t. When applied successively to the fields $\Phi(t;x)$ and $\mathscr{B}$ these gauge-covariant differentiation operators satisfy the identity

$$\hat{D}(\rho D_\mu\phi^A)-\rho D_\mu s^A=0\ . \tag{4.4}$$

At the next higher order, they satisfy a commutation identity that involves the fiber-space curvature, in the form

$$\hat{D}[D_\nu(\rho D_\mu\phi^A)]-D_\nu(\rho D_\mu s^A)=\rho\hat{R}^{\ A}{}_{BCD}(D_\mu\phi^B)s^C D_\nu\phi^D\ . \tag{4.5}$$

Now the squared path length s^2 appearing in (4.2) can be considered as an ordinary (evidently gauge-independent) scalar field over the base $\mathscr{M}$. As such it will have gradient components given by

$$\begin{aligned}\tfrac{1}{2}\rho(s^2)_{;\mu}&=\tfrac{1}{2}\rho D_\mu s^2\\ &=\rho s_A D_\mu s^A\ ,\end{aligned} \tag{4.6}$$

where the right-hand side is to be evaluated at any fixed value of t in the interval (0,1), the result being manifestly independent of the choice. Taking the base-space divergence of this relation gives

$$\tfrac{1}{2}[\rho(s^2)_{;\mu}]^{;\mu}=\rho(D^\mu s_A)D_\mu s^A+s_A D^\mu(\rho D_\mu s^A)\ . \tag{4.7}$$

Now since the left-hand side is manifestly independent of t, the same must be true for the apparently t-dependent right-hand side, which will therefore be unaffected by integration with respect to t over the unit interval (0,1). Applying this integral operation to the second term on the right-hand side, and using (4.5), one obtains, after an integration by parts,

$$\int_0^1 dt\,[s_A D^\mu(\rho D_\mu s^A)]=[s_A D^\mu(\rho D_\mu\Phi^A)]_0^1-\int_0^1 dt\,[(Ds_A)D^\mu(\rho D_\mu\Phi^A)+\rho s_A R^{\ A}{}_{BCD}(D^\mu\Phi^B)s^C D_\mu\Phi^D]\ . \tag{4.8}$$

It is to be noticed that the end-point contributions in the first term on the right-hand side of this relation are proportional (via a contraction with s_A) to the first term in the field equations (3.7). We can construct an analogous expression involving the other term in (3.7) by a similar integration by parts of the form

$$\int_0^1 dt[s^A s^B \mathscr{M}_{AB}]=[s^A \mathscr{V}_{,A}]_0^1-\int_0^1 dt[(\hat{D}s^A)\mathscr{V}_{,A}]\ , \tag{4.9}$$

where the symmetric mass-tensor field on the fibers is defined by

$$\mathscr{M}_{AB}=\hat{\nabla}_A\hat{\nabla}_B\mathscr{V}\ . \tag{4.10}$$

Combining (4.7), (4.8), and (4.9) we can obtain an identity of the form

$$\tfrac{1}{2}[\rho(s^2)_{;\mu}]^{;\mu}+\left[\frac{\delta L}{\delta\Phi^A}s^A\right]_0^1-\int_0^1 dt\frac{\delta L}{\delta\Phi^A}\hat{D}s^A=\int_0^1 dt[\rho(D^\mu s_A)D_\mu s^A-\rho\hat{R}_{ABCD}s^A(D^\mu\Phi^B)s^C D_\mu\Phi^D+\pi\mathscr{M}_{AB}s^A s^B] \tag{4.11}$$

with $\delta L/\delta\Phi^A$ as given by (3.6) so that the left-hand side evidently reduces to a (weighted) Laplacian of the squared fiber distance $s^2(x)$ between the sections $\Phi_{[0]}(x)$ and $\Phi_{[1]}(x)$ at each base point $x\in\mathscr{M}$ if *all* the intermediate section $\Phi(t;x)$ of the homotopy are solutions of the field equations (3.8).

For the purpose of establishing the uniqueness of solutions to the appropriate global boundary condition problems, it will *not* do to assume that the intermediate sections $\Phi(t;x)$ are solutions of the field equations: in examining the possible deviation—as measured by s^2—between the solutions $\Phi_{[1]}(x)$ and $\Phi_{[0]}(x)$ one will be justified in setting $\delta L/\delta\Phi^A$ equal to zero *only* at the end points $t=0$ and $t=1$. We can, however, get rid of the integral involving the values of $\delta L/\delta\Phi^A$ at intermediate values of t if we now *restrict* the homotopy (which up till this stage has been left arbitrary apart from the parametrization) by requiring that the fiber-space curves $\Phi(t;x)$ for each *fixed* base point x should be *geodesic,* which means (since the parametrization has already been restricted to be affine) that they should satisfy the equation

$$\hat{D}s^A=0\ . \tag{4.12}$$

In practice this is likely to be a less serious restriction than might at first appear, since one of the most convincing ways of establishing the existence of the homotopy itself in many cases will be to construct it explicitly in terms of geodesics in the first place: all that is needed is to be sure that for each base point $x\in\mathscr{M}$ the required geodesic exists and that it is unique, at least subject to conceivably relevant additional restrictions (e.g., on the maximum allowed length s) of such a nature as to guarantee the continuous variation of the geodesic as a function of the end points.

In the particular case of fiber space with a complete (negative) *positive* definite metric and a (positive) *negative* definite Riemann curvature then it is a well-known theorem (see, e.g., Kobyashi and Nomizu[12]) that the required geodesic between any two points exists and is unique, thereby establishing the existence (and uniqueness) of the required geodesic homotopy. Under these conditions, integration of (4.11) over a base domain Σ [using the same volume measure as in the variational integral (3.1)] gives

$$\oint_{S=\partial\Sigma}\rho s s^{;\mu}dS_\mu=\int_\Sigma d\Sigma\int_0^1 dt[\rho(D^\mu s_A)D_\mu s^A-\rho\hat{R}_{ABCD}s^A(D_\mu\Phi^B)s^C(D_\mu\Phi^D)+\pi\mathscr{M}_{AB}s^A s^B]\ , \tag{4.13}$$

where the right-hand side will be a manifestly positive (negative) definite function of the distance s (vanishing only if $s=0$, i.e., if the two solutions $\Phi_{[0]}$ and $\Phi_{[1]}$ coincide) provided that the density ρ and the base-space metric are positive (negative) definite and that the potential $\mathscr{V}$ has the appropriate convexity property as expressed by the condition that the mass tensor $\mathscr{M}$ (weighted by π) should be positive (negative) definite. Under such conditions, it will suffice if the boundary conditions ensure the vanishing of the surface integral on the left-hand side of (4.12) in order for one to be able to conclude that s must vanish throughout the domain and thus that the solution Φ is unique (for the given fiber metric $\hat{g}$, connection A, self-interaction potential $\mathscr{V}$, and the given base-space fields g,ρ,π).

Even if the required ngativity or positivity, as the case may be, of the fiber-curvature tensor and the mass tensor held only indefinitely—as would be the case, for example, if the potential $\mathscr{V}$ were absent—then the right-hand side of (4.13) would still be positive definite as a function at least of $D_\mu s^A$. In these rather more general circumstances, boundary conditions ensuring the vanishing of the surface integral on the right-hand side of (4.13) would be sufficient, by (4.6), to guarantee at least the vanishing of the gradient of s, i.e.,

$$s_{,\mu}=0\ . \tag{4.14}$$

After one had thus established the uniformity of the fiber-metric distance s between the two hypothetical solutions, it would suffice in addition for the field to be fully determined even at just a single limit point on the boundary, S, in order to be able to conclude that s vanishes everywhere and thus that the solution is entirely unique.

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