Separability of the Killing–Maxwell system underlying the generalized angular momentum constant in the Kerr–Newman black hole metrics

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The concept of a Killing–Maxwell system may be defined by the relation $\hat{A}_{[\mu;\nu];\rho} = (4\pi/3)\hat{j}_{[\mu}g_{\nu]}$. In such a system the one-form $\hat{A}_\mu$ is interpretable as the four-potential of an electromagnetic field $\hat{F}_{\mu\nu}$, whose source current $\hat{j}^\mu$ is an ordinary Killing vector. Such a system determines a canonically associated duality class of source-free electromagnetic fields, its own dual being a Killing–Yano tensor, such as was found by Penrose [Ann. N.Y. Acad. Sci. **224**, 125 (1973)] (with Floyd) to underlie the generalized angular momentum conservation law in the Kerr black hole metrics, the existence of the Killing–Yano tensor being also a sufficient condition for that of the Killing–Maxwell system. In the Kerr pure vacuum metric and more generally in the Kerr–Newman metrics for which a member of the associated family of source-free fields is coupled in gravitationally, it is shown that the gauge of the Killing–Maxwell one-form may be chosen so that it is expressible (in the standard Boyer–Lindquist coordinates) by $\frac{1}{2}(a^2\cos^2\theta - r^2)dt + \frac{1}{2}a(r^2-a^2)\sin^2\theta\, d\phi$, the corresponding source current being just $(4\pi/3)(\partial/\partial t)$. It is found that this one-form (like that of the standard four-potential for the associated source-free field) satisfies the special requirement for separability of the corresponding coupled charged (scalar or Dirac spinor) wave equations.

I. INTRODUCTION

Although it is well known that the charged black hole uniqueness and no hair theorems[1–7] allow only two electromagnetic degrees of freedom (or just one if a magnetic monopole moment is deemed to be physically unrealistic) for regular electromagnetic perturbations that are *source-free* and *asymptotically vanishing*, the dropping of these latter restrictions permits one to envisage many other possibilities. Among these, one particular example is specially singled out (if not for any obvious astrophysical relevance, at least for its remarkable mathematical properties), namely what we shall refer to as the *Killing–Maxwell* field. It is demonstrated in this paper that if this field is taken seriously, in the sense of being considered to act in the usual way on charged scalar or spinor fields and discrete classical particles on the black hole background, then the resulting coupled systems have the same kind of very special separability properties as have already been found, respectively,[8–10] when such charged fields and particles are coupled to the familiar *source-free* electromagnetic perturbations allowed by the no hair theorems.

The existence of a Killing–Maxwell system in the sense to be defined below is an equivalent (necessary and sufficient) condition to the existence—in four dimensions—of a *second degree Killing–Yano* tensor,

$$f_{\lambda\mu} = f_{[\lambda\mu]}, \quad f_{\lambda(\mu;\rho)} = 0 \tag{1.1}$$

(using a semicolon for covariant differentiation, with square and round brackets for symmetrization and antisymmetrization of tensor indices). It was the culmination of a systematic attempt (using two-spinor methods) by several coworkers[11–14] to obtain (from the Weyl tensor degeneracy property that was the basis of Kerr's original discovery of his metric[15]) a simple underlying reason for the remarkable integrability properties of so many kinds of systems in the Kerr (and Kerr–Newman[16]) black hole metrics[8,10,17–21] that the existence of such a tensor in these metrics was first brought to light by Penrose[22] (with Floyd). Much further work (including the use of a Debever-type bivector formalism[23] for transcription of earlier two-spinor results into equivalent but more widely readable tensorial form) has explored the general properties of such systems, essentially confirming that the remarkable properties just referred to can indeed be considered as automatic consequences of (1.1). A recent summary and guide to many relevant references, of which only a sample can be mentioned here,[24–30] has been given by Kamran and Marck.[31] This body of work together with earlier results[8] soon made it clear that the existence of a (nonzero) solution of (1.1) is by itself sufficient to characterize the Kerr (or Kerr–Newman) solution uniquely among asymptotically flat pure vacuum Einstein (or source-free Einstein–Maxwell) solutions (and likewise for the author's asymptotically de Sitter black hole solutions [2,8,32]—though it remains a teasing mystery why the solutions of the (global) black hole problem should turn out to belong to this (locally) privileged class.

We start by collecting some essential conclusions that can be drawn directly from (1.1) (without recourse to Einstein or any other equations) by straightforward tensor analysis. Among the most basic of these results is the existence of an ordinary (symmetric) *Killing tensor* (whose presence in the case of the Kerr solutions was directly implied by the original discovery[10] of a quadratic generalized angular momentum constant of the motion)

$$a_{\lambda\mu} = f_\lambda{}^\rho f_{\rho\mu}, \quad a_{(\lambda\mu;\rho)} = 0, \tag{1.2}$$

together with the existence of what we shall refer to as *the primary and the secondary killing vector* (giving rise to linear constants of motion, interpretable as linear combinations of

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energy and axial angular momentum), the first defined (using the alternating tensor) as the dual of the (necessarily antisymmetric) covariant derivative of the Killing–Yano tensor,

$$k^\lambda = (1/3!)\epsilon^{\lambda\mu\rho\sigma} f_{\mu\rho;\sigma}, \quad k_{(\lambda;\mu)} = 0, \tag{1.3}$$

and the second given in terms of the first by

$$h^\lambda = a^{\lambda\rho} k_\rho, \quad h_{(\lambda;\mu)} = 0. \tag{1.4}$$

Furthermore, as well as having the Killing vector property of generating symmetries of the metric, $g_{\lambda\mu}$, these two vector fields also generate symmetries of the Killing–Yano tensor itself (and hence of the system as a whole, which entails in particular that they must commute) in the sense that the Lie derivatives of the Killing–Yano tensor (and hence of anything constructed directly from it) with respect to each of these vectors must vanish. (The primary Killing vector has the additional special property that the corresponding covariant derivatives along it must also vanish.)

II. THE CONCEPT OF A KILLING–MAXWELL SYSTEM

The basic defining equation of what we refer to henceforth as a *Killing–Maxwell system* may be taken to be

$$\hat{A}_{[\mu;\nu];\rho} = (4\pi/3)\hat{j}_{[\mu} g_{\nu]\rho}, \tag{2.1}$$

where $\hat{A}_\mu$ is a four-potential one-form, associated with a four-current vector $\hat{j}^\mu$, and $g_{\mu\nu}$ is the metric of the background space-time (and where we have introduced a circumflex to distinguish quantities pertaining to the Killing–Maxwell field from the analogous quantities pertaining to the closely related source-free Maxwell field to be mentioned below). Such a system evidently satisfies the (much less highly restrictive) ordinary Maxwell equations for the corresponding electromagnetic field tensor

$$\hat{F}_{\mu\nu} = 2\hat{A}_{[\nu;\mu]} \tag{2.2}$$

since the contraction of (2.1) leads directly to the source equation

$$\hat{F}^{\rho\mu}{}_{;\rho} = 4\pi \hat{j}^\mu. \tag{2.3}$$

By straightforward tensor algebra and the use of the Maxwell–Faraday integrability condition for (2.2),

$$\hat{F}_{[\mu\nu;\rho]} = 0, \tag{2.4}$$

it can easily be checked that the systems (1.1) and (2.1) are equivalent (modulo gauge transformations in the latter) since one can be constructed from the other and vice versa by the simple duality relation

$$f_{\mu\nu} = {}^*\hat{F}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\hat{F}_{\rho\sigma}, \tag{2.5}$$

which evidently entails that the current is to be identified, modulo a rationalization factor, with the primary Killing vector:

$$k^\mu = (4\pi/3)\hat{j}^\mu. \tag{2.6}$$

For many purposes it is convenient to work with the corresponding complex self-dual Killing–Maxwell Yano tensor

$${}^+\hat{F}_{\lambda\mu} = \hat{F}_{\lambda\mu} + i f_{\lambda\mu}. \tag{2.7}$$

Using the fact that by (2.1) its contraction with the primary Killing vector is a pure gradient,

$$k^\rho\, {}^+\hat{F}_{\rho\mu} = -\tfrac{1}{8}({}^+\hat{F}_{\rho\sigma}\, {}^+\hat{F}^{\sigma\rho})_{,\mu}, \tag{2.8}$$

it is straightforward to check that one can construct a new (complex) proportionally related self-dual field of the form

$${}^+F_{\lambda\mu} = Cs^{-3}\, {}^+\hat{F}_{\lambda\mu}, \tag{2.9}$$

which will satisfy the full set of *source-free* Maxwell equations, whose complex form is

$${}^+F^{\mu\rho}{}_{;\rho} = 0,$$

for an arbitrary value of the complex (charge) constant C, provided that s is taken to be the scalar field given in terms of the scalar invariants of the Killing–Maxwell field by

$$4s^2 = {}^+\hat{F}_{\rho\sigma}\, {}^+\hat{F}^{\sigma\rho}. \tag{2.10}$$

The fact, mentioned above, that the Killing–Yano tensor and hence also the Killing–Maxwell system (not to mention the associated source-free field that has just been constructed) will be invariant under the action generated by the primary Killing vector can at this stage be seen directly by combining the gradient property (2.8) with the condition [obtained by contracting the Killing vector with the dual of its defining relation (1.3)], which leads to a pair of equations

$$\hat{F}_{\lambda\mu;\rho} k^\rho = 0, \quad 2\hat{F}_{\rho[\lambda}\, k_{\mu]}{}^{;\rho} = 0 \tag{2.11}$$

which add up to the condition that the Lie derivative with respect to k^μ vanishes. For the secondary Killing vector h^μ, we do not have analogs of the separate equations (2.11) but we can nevertheless obtain the combination expressing the corresponding invariance condition,

$$\hat{F}_{\lambda\mu;\rho} h^\rho + 2\hat{F}_{\rho[\lambda}\, h_{\mu]}{}^{;\rho} = 0, \tag{2.12}$$

from (2.4), using the fact that the imaginary (magnetic) part of (2.8) implies a corresponding real (electric but not magnetic) gradient property for the effective electric (but not the magnetic) field as defined with respect to the secondary Killing vector:

$$h^\rho \hat{F}_{\rho\mu} = -\tfrac{1}{32}\{(\hat{F}_{\rho\sigma} f^{\sigma\rho})^2\}_{,\mu}. \tag{2.13}$$

We can use (2.12) together with (1.3) to see that the *Killing bivector* $2k^{[\lambda} h^{\mu]}$ has a dual two-form given by

$$\epsilon_{\lambda\mu\rho\sigma} k^\rho h^\sigma = 2f_{\rho[\lambda}\, h_{\mu]}{}^{;\rho}, \tag{2.14}$$

which enables us to derive the equations

$$h_{[\lambda;\mu} k_\rho h_{\sigma]} = 0, \quad k_{[\lambda;\mu} k_\rho h_{\sigma]} = 0, \tag{2.15}$$

of which the first is an obvious consequence directly of (2.15), while the second can be obtained from (2.11), which evidently entails a formally identical pair of equations with $f_{\lambda\mu}$ in place of $\hat{F}_{\lambda\mu}$. The same considerations also, respectively, imply

$$k^\rho h^\sigma f_{\rho\sigma} = 0, \quad k^\rho h^\sigma \hat{F} f_{\rho\sigma} = 0. \tag{2.16}$$

It can be seen that (2.15) and (2.16) are the same *circularity conditions* as those deduced from the generalized Papapetrou theorem in the black hole problem[2,33] from quite a different starting point (involving Einstein curvature equations and global boundary conditions) instead of the very simple equations (1.1) or equivalently (2.1), which is all that we have assumed here. In particular (2.15) is interpretable as the Frobenius integrability condition for the two-surface elements orthogonal to the Killing bivector to be themselves two-surface forming.

III. THE KILLING–MAXWELL ONE-FORM

We have so far mainly been collecting results that (although rather dispersed about the literature quoted above, and derived by perhaps more devious routes and in more specialized notation than the ordinary tensor calculus used here) are nevertheless for the most part, in principle, "well," albeit not "widely," known by now. However, we shall now concentrate our attention on what is in a sense the most fundamental element of all in the foregoing tree of relationships, which does not yet seem to have had the attention it deserves (or even to have been considered explicitly at all), namely what we have dubbed as the Killing–Maxwell one-form, $\hat{A}_\mu$. Once it has been specified (assuming that the metric tensor is also known) all the other quantities can be constructed by successive differentiations (the Killing–Yano tensor at first order, the ordinary Killing vectors at second order, and so on.) One reason for the neglect of the zero-order element at the base of the tree may be that to make it explicit one must, of course, make some specific choice of the gauge. In practice, however, there is no real ambiguity because there turns out to be a *canonical gauge* that imposes itself naturally (just as I found long ago[8,10] to be the case for what can now be interpreted as the canonically associated source-free fields).

To pin down the gauge we start by requiring that the four-potential one-form $\hat{A}_\mu$ should have the same properties of invariance under the action of the Killing vectors as the field $\hat{F}_{\lambda\mu}$ itself, properties which are simultaneously compatible in consequence of the commutation relation

$$h^\mu{}_\rho k^\rho - k^\mu{}_\rho h^\rho = 0 \tag{3.1}$$

that follows from the fact that the secondary Killing vector is constructed from quantities known [by (2.11)] to be invariant under the action of the primary Killing vector. We can thus obtain

$$\hat{A}_{\mu;\rho} k^\rho + \hat{A}_\rho k^\rho{}_{;\mu} = 0, \tag{3.2}$$

and

$$\hat{A}_{\mu;\rho} h^\rho + \hat{A}_\rho h^\rho{}_{;\mu} = 0. \tag{3.3}$$

Using the real (electric) part of (2.8) we see from (3.2) that it is possible by a further minor adjustment to arrange to have

$$\hat{A}_\rho k^\rho = -\tfrac{1}{4}\hat{F}_{\rho\sigma}\hat{F}^{\sigma\rho}, \tag{3.4}$$

while similarly, by (2.13) [again bearing in mind the compatibility property (3.1)], we see from (3.3) that it is possible also to arrange to have

$$\hat{A}_\rho h^\rho = -\tfrac{1}{32}(\hat{F}_{\rho\sigma} f^{\sigma\rho})^2. \tag{3.5}$$

Finally, leaving aside the possibility of degenerate limit cases in which the primary and secondary Killing vectors might not be independent, it can be seen that (as in the analogous stage in the black hole problem[2]) the orthogonal transitivity and field circularity properties, (2.15) and (2.16), allow us to impose the gauge circularity condition

$$\hat{A}_{[\lambda} k_\mu h_{\rho]} = 0, \tag{3.6}$$

which now ties down the gauge completely. Although there is now no longer any freedom to impose further gauge restrictions, it is apparent that (3.2), (3.3), and (3.6) together are sufficient to ensure automatically that the standard Lorentz gauge condition

$$\hat{A}_\rho{}^{;\rho} = 0 \tag{3.7}$$

is also satisfied.

IV. ALGEBRAICALLY PREFERRED COORDINATES AND SEPARABILITY

Up to this stage we have kept to fully covariant terminology, but it is now useful (at the price of leaving aside degenerate limit cases in which the two Maxwellian scalar invariants do not vary independently) to bring in *algebraically preferred* coordinates of the kind introduced by the present author[8] and commonly used in studies of the general cases[26,29,30] (as opposed to the more particular physical black hole case, for which the slightly different geometrically preferred coordinates of the type introduced by Boyer and Lindquist[34] are usually chosen). Within the present approach the algebraically preferred system may be specified to consist of two nonignorable coordinates, r and q say, given in terms of the Killing–Maxwell invariants by

$$r^2 - q^2 = \tfrac{1}{2}F_{\rho\sigma}F^{\rho\sigma}, \quad 2rq = \tfrac{1}{2}\hat{F}_{\rho\sigma}f^{\sigma\rho}, \tag{4.1}$$

together with two ignorable coordinates, $\tilde{t}$ and $\tilde{\phi}$ say, taken to be constant on the orthogonal hypersurfaces whose existence is established by (2.15) and such that the primary and secondary Killing vectors, k^μ and h^μ, can be identified, respectively, with the operators $\partial/\partial\tilde{t}$ and $\partial/\partial\tilde{\phi}$. It can be seen that the specification (4.1) is satisfied simply by taking r and q as the real and imaginary parts of the scalar field defined by (2.9), i.e., we have

$$s = r + iq. \tag{4.2}$$

In this system the gauge conditions imposed at the end of the previous section lead unambiguously to the explicit expression

$$\hat{A}_\rho \, dx^\rho = \tfrac{1}{2}(q^2 - r^2)d\tilde{t} - \tfrac{1}{2}r^2q^2 \, d\tilde{\phi}. \tag{4.3}$$

Nothing in the preceding line of reasoning makes it obvious in advance that this field should share the already known property of the associated source-free Maxwell field of satisfying the author's condition[2,8] for separability of the Klein–Gordon wave equation (and hence *a fortiori* the corresponding classical charged orbit equations) for a charged scalar field coupled to an electromagnetic field. In the present terminology this very restrictive condition is expressible as the requirement that the four-potential one-form should have the form

$$\hat{A}_\rho \, dx^\rho = \frac{\hat{X}_+(r)(d\tilde{t} + q^2 d\tilde{\phi}) - \hat{X}_-(q)(d\tilde{t} - r^2 \, d\tilde{\phi})}{r^2 + q^2}, \tag{4.4}$$

where $\hat{X}_+(r)$ is a function of r only, and $\hat{X}_-(q)$ is a function of q only. It transpires nevertheless that in the gauge (4.3) the Killing–Maxwell one-form does indeed satisfy this condition, the two single variable functions having the simplest form imaginable on dimensional grounds, namely

$$\hat{X}_+(r) = \tfrac{1}{2}r^4, \quad \hat{X}_-(q) = \tfrac{1}{2}q^4. \tag{4.5}$$

For comparison, it may be recalled that the analogous functions for the family (2.9) of source-free associated fields (including those coupled gravitationally in the Kerr–Newman[16] solutions) are correspondingly expressible[2,8,10] in terms of the real (electric charge) part Q and the imaginary (magnetic monopole) part P of the complex charge parameter C appearing in (2.8) by

$$X_+(r) = Qr, \quad X_-(q) = Pq \quad (P + iQ = C). \tag{4.6}$$

The significance of the property of being expressible in the form (4.4) is strengthened by the recent work of Kamran and McLenaghan,[28] which shows that the condition (4.4) is sufficient to ensure (undecoupled Chandrasekhar-type[19]) separability in the case where the charged scalar is replaced by a charged Dirac spinor. Although such separability properties can be studied more easily in the algebraically preferred coordinates used here, they are, of course, preserved by the transformation to the standard geometrically preferred Boyer–Lindquist[34] coordinates according to the prescription

$$r \to r, \; q \to a \cos\theta, \; \tilde{\phi} \to a^{-1}\phi, \; \tilde{t} \to t - a\phi. \tag{4.7}$$

(I would insist, by the way, that contrary to a widespread myth that has been implicitly perpetuated by a recent major treatise on the subject[35] the transformation to Boyer–Lindquist coordinates does *not* imply any need to transform to a noncanonical—e.g. Kinnersley-type[36] —tetrad in place of the maximally symmetric one.[8,29,37–40]) It is also to be remarked that the separability condition (4.4) is preserved by the trivial gauge changes corresponding to addition of *constant* multiples of $d\tilde{t}$ and $d\tilde{\phi}$. The Boyer–Lindquist form of the Killing–Maxwell potential quoted in the abstract does in fact differ from (4.3) by such a separability-preserving adjustment.

Despite the fact that the corresponding constants of the motion could have been constructed in advance as eigenfunctions of corresponding operators in both the scalar[41] and Dirac spinor[25] cases, the fact that these constants are associated with full separability still seems somewhat miraculous. In the simplest case, that of a classical particle with charge to mass ratio e/m on an orbit whose unit tangent vector u^μ evolves according to

$$u^\mu{}_{;\rho} u^\rho = (e/m) F^\mu{}_\rho u^\rho, \tag{4.8}$$

our original postulate (1.1) implies that the generalized (specific) angular momentum vector and scalar, defined by

$$l^\mu = f^\mu{}_\rho u\rho, \; l^\rho l_\rho = a_{\rho\sigma} u^\rho u^\sigma, \tag{4.9}$$

will satisfy corresponding precessing translation and conservation laws,

$$l^\mu{}_{;\rho} u^\rho = (e/m) F^\mu{}_\rho l^\rho, \; (l^\rho l_\rho)_{,\sigma} u^\sigma = 0, \tag{4.10}$$

for *any* field $F_{\lambda\mu}$ given by an expression of the form (2.9) whatever the field s may be. Now although any field satisfying the separability condition (4.4) will have the form (2.9) for some scalar field s, the converse requirement is highly restrictive.[1] It is therefore remarkable that such a requirement (which in this case is manifestly not necessary for the conservation law to apply) should turn out to hold both for the source-free solutions [with s given by (2.10) or (4.3)] and for the Killing–Maxwell field (with s uniform so that $Cs^{-3} = 1$). Indeed even in the Minkowski space limit, for which the Killing–Maxwell field is interpretable as that within a uniform spherical charge distribution, the spherical symmetry of which is *broken* by the superposition of a uniform magnetic field, the (scalar and Dirac) separability that has been revealed was hardly obvious in advance.

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