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Cosmological expansion and thermodynamic mechanisms in cosmic string dynamics

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Some of the essential general principles governing cosmic string mechanics in a conformally expanding blackbody radiation background are described. It is shown that the effect of dissipative drag damping may be given a strictly conservative (i.e., variational) representation in which the usual Goto-Nambu action is simply multiplied by an appropriate cosmological temperature-dependent conformal factor. A simplified thermodynamic description is used to investigate approximately stationary equilibrium states such as may occasionally be produced as the long term outcome of large scale damping in the case of a cosmic string loop for which the (thermal or more general) distribution of surviving microscopic wiggles on an isolated cosmic string loop is characterized by a strong preponderance of "right movers" over "left movers" (or vice versa). For nonsuperconducting strings, such states can be represented very simply using the nondispersive "warm" cosmic string model whose dynamics is characterized by a pair of "left"- and "right"-moving characteristic surface currents that will be independently conserved so long as the effective heat loss to the environment is negligible. It is predicted that one of these currents will still remain conserved in the long run when account is taken of radiative energy loss from the approximately stationary equilibrium state, which will evolve with negative specific heat, monotonically increasing its effective temperature as it contracts.

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I. INTRODUCTION

Much of the numerical work on the earlier stages of the evolution of cosmic string networks (starting with the pioneering work of Bennet and Bouchet [1]) drew attention to the likely relevance of widely differing characteristic length scales on which qualitatively different kinds of behavior occur. Whereas larger scales, ranging up to the current Hubble length, enter the picture as the outcome of subsequent dynamical developments, it is typically observed (see, e.g., discussions by Albrecht [2], Allen and Shellard [3], and Sakellariadou and Vilenkin [4]) that the simulations tend to conserve the memory of relatively much smaller scales, which were put in at the outset, typically as the scale of a Vachaspati-Vilenkin-type [5] lattice structure from which the calculation was initiated. With a pessimism representative of the point of view of the numerical simulator's approach rather than the more intuitive line of thought followed here, one of the references cited above described the situation by saying [4] that a "distressing aspect of our results is their dependence on the energy cutoff," drawing from this the obvious (and in the present framework not at all distressing) inference that "small-scale processes . . . play an essential role." It seems safe to presume that a realistic description would indeed show a similar feature whereby the buildup of large scale features is accompanied by the survival of relatively microscopic structure on scales that would typically be even smaller than could conveniently be allowed for in the computer simulations. These small scales correspond to values that might conceivably be as low as the correlation lengths characterizing the phase transition during which the strings were supposed to have been produced. In any case, they could hardly be supposed to exceed the Hubble length at the stage at which damping became unimportant, i.e., at a stage much earlier than that involved in the simulations concerned with large scale structure formation. This is in agreement with a recent work by Austin, Copeland, and Kibble [6], that the small scale structure initially does not grow, but eventually scales at a rather low level, as a result of the effect of the gravitational back reaction.

In order to study analytically the evolution of a string network, we will start by providing a coherent overview of some of the essential general principles governing cosmic string mechanics in a conformally expanding background of ambient cosmological blackbody radiation. We will then deal more specifically with very short length scale effects by describing them using the nondispersive "warm cosmic string" model. This is a specially simple macroscopic string model, which approximates the microscopic substructure of wiggly Goto-Nambu cosmic strings [7]. To study the equilibrium states of such string loops in the late time quasistationary limit, we will follow a simplified thermodynamical description.

The plan of this work is organized as follows. We start in Sec. II by recalling the Lagrangian method for both Goto-Nambu and *elastic* strings. In Secs. III and IV we analyze both the drag force exerted by a thermal background medium on elastic strings and the second fundamental tensor in order to obtain in Sec. V the string dynamical evolution in a conformally expanding universe, particularly in a Friedmann-Robertson-Walker universe. It will also be shown in this section that in suitable cosmological circumstances the effect of drag force can also be given a conservative variational treatment in terms of a modified action. Following the general presentation of the properties of the *warm* cosmic string model in Sec. VI, we shall proceed in Sec. VII by describing the quasistationary states that may be attained by closed loops for such a string model. Finally, in Sec. VIII we shall present some simple conclusions that can be drawn about the secular evolution of such quasiequilibrium states as a result of thermal energy loss by gravitational radiation.

II. PRELIMINARIES

Before considering the effects of dissipative drag forces resulting from movement relative to a background field, it will be useful to start by recalling [8] that in the *freely evolving* limit, in which such dissipative effects are absent, the motion of a cosmic string in a background spacetime with coordinates x^μ ($\mu=0,1,2,3$) and metric $g_{\mu\nu}$ will be *conservative* in the technical sense of being governed by a variation principle based on an appropriate action $\mathcal{I}$. In all such cases, the action will be constructed as an integral over the relevant world sheet, which will be specifiable by an embedding mapping of the form $\sigma^i \mapsto x^\mu$, where σ^i are internal coordinates with the index taking the values $i=0,1$ in the two-dimensional case of a string world sheet. The action will thus take the form

$$\mathcal{I}=\int d\tilde{\mathcal{S}}\,\tilde{\mathcal{L}} \tag{2.1}$$

(using a tilde to distinguish quantities defined with respect to the world sheet from their four-dimensional analogues), where $\tilde{\mathcal{L}}$ is the relevant scalar Lagrangian and $d\tilde{\mathcal{S}}$ is the surface measure element induced on the world sheet by the background metric. It will therefore be expressible as $d\tilde{\mathcal{S}}=\sqrt{-|h|}\,d\sigma^0 d\sigma^1$, where $|h|$ is the determinant of the induced world sheet metric as given by

$$h_{ab}=g_{\mu\nu}x^\mu{}_{,a}x^\nu{}_{,b} \ , \tag{2.2}$$

using a comma to indicate partial differentiation with respect to internal coordinates.

For such an action, the corresponding surface stress momentum-energy density tensor $\tilde{T}^{\mu\nu}$ is definable by the condition that the effect of an infinitesimal localized variation $\delta g_{\mu\nu}$ of the background metric be expressible by

$$\delta\mathcal{I}=\tfrac{1}{2}\int d\tilde{\mathcal{S}}\,\tilde{T}^{\mu\nu}\delta g_{\mu\nu} \ . \tag{2.3}$$

The application of the variation principle (in the absence of external coupling effects) automatically ensures [8] that this tensor will satisfy a pseudoconservation law of the form

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu}=0 \tag{2.4}$$

as a Noether identity, where $\tilde{\nabla}_\mu$ is the surface-projected covariant derivation operator,

$$\tilde{\nabla}_\mu=\eta^\nu{}_\mu\nabla_\nu \ , \tag{2.5}$$

which is constructed by contraction of the usual Riemannian covariant derivation operator ∇_ν with the tangential projector $\eta^\nu{}_\mu$. The latter is obtained by lowering an index of the *fundamental tensor* of the world sheet, which is defined as the background spacetime pullback of the index raised (inverse) version h^{ab} of the internal world sheet metric, i.e.,

$$\eta^{\mu\nu}=h^{ab}x^\mu{}_{,a}x^\nu{}_{,b} \ . \tag{2.6}$$

The projector $\eta^\mu{}_\nu$ obtained this way will be of rank 2 for a string world sheet so that the complementary *orthogonal projector*

$$\perp^\mu{}_\nu=g^\mu{}_\nu-\eta^\mu{}_\nu \tag{2.7}$$

will have rank 2 for a string in four-dimensional spacetime.

It is to be remarked in passing that, in circumstances requiring consideration of the active effect of the string on the background, it may be useful to express the action as a distributional integral over the four-dimensional background with surface measure element $d\mathcal{S}=\sqrt{\|g\|}dx^0dx^1dx^2dx^3$ in the form

$$\mathcal{I}=\int dS\,\hat{\mathcal{L}} \ ,$$

using a circumflex to indicate that the four-dimensional Lagrangian density scalar is singular, being given in Dirac notation as

$$\hat{\mathcal{L}}=\|g\|^{-1/2}\int d\tilde{\mathcal{S}}\,\tilde{\mathcal{L}}\,\delta^4\{x-x\{\sigma\}\} \ .$$

However, if one is not concerned with the inevitably singular but usually very weak active effects of the string on the background, whose treatment in any case poses delicate problems of renormalization, the use of such elaborate distributional machinery (although popular with many authors) will be quite unnecessary. For purposes such as ours, it will be quite sufficient to work exclusively with the simple formalism exemplified by (2.1) and (2.3) as expressed directly in terms of $\tilde{\mathcal{L}}$ and $\tilde{T}^{\mu\nu}$, which have the advantage of just being ordinary *regular* differentiable functions of position, the special property that is indicated by the tilde being simply that their support is confined to the world sheet.

The simplest Lagrangian for a string is that of the Goto-Nambu action: namely,

$$\tilde{\mathcal{L}}=-\frac{m^2}{\hbar}\ , \tag{2.8}$$

where $\hbar$ is the Dirac-Planck constant and m is a constant having the dimensionality of a mass. In the case of a simple cosmic string, m can be presumed to have the order of magnitude of the Higgs boson mass scale involved in the spontaneous symmetry breaking responsible for the formation of the relevant underlying Nielsen-Olesen-type vortex [9]. The corresponding Goto-Nambu string stress momentum-energy tensor is thus obtained as

$$\tilde{T}^{\mu\nu}=-\frac{m^2}{\hbar}\eta^{\mu\nu}\ . \tag{2.9}$$

The fundamental tensor for the two-dimensional world sheet of a string will be expressible in the form

$$\eta^{\mu\nu}=-\tilde{u}^{\mu}\tilde{u}^{\nu}+\tilde{v}^{\mu}\tilde{v}^{\nu}\ , \tag{2.10}$$

where $\tilde{u}^{\mu}$ is a timelike unit tangent vector and $\tilde{v}^{\mu}$ an orthogonal spacelike unit tangent vector so as to obtain an orthonormal frame that is characterized by $\tilde{u}^{\mu}\tilde{u}_{\mu}=-1$, $\tilde{u}^{\mu}\tilde{v}_{\mu}=0$, and $\tilde{v}^{\mu}\tilde{v}_{\mu}=1$. Such a frame is unique only modulo Lorentz rotations, which of course will not affect the form of the tensorial expression (2.10).

For the not quite so simple *elastic* string models [8,10] that will be needed below, the trivial Goto-Nambu Lagrangian must be replaced by a Lagrangian scalar that is variable. Such an elastic string always has conserved currents (as exemplified by superconducting strings); we can therefore define an appropriate stream function ψ, say, on the string world sheet which will be constant along the flow lines of a conserved surface current. The Lagrangian can thus be taken to be a function of the magnitude ν, say, of the gradient of the stream function ψ:

$$\tilde{\mathcal{L}}=\tilde{\mathcal{L}}\{\nu\}\ , \tag{2.11}$$

where ν is given by

$$\nu^2=-h^{ab}\psi_{,a}\psi_{,b}=-\eta^{\mu\nu}(\tilde{\nabla}_{\mu}\psi)\tilde{\nabla}_{\nu}\psi\ . \tag{2.12}$$

By using the freedom of Lorentz rotation to align the orthonormal frame introduced in (2.10) with its eigenvectors, the corresponding string stress momentum-energy density tensor will be expressible in the standard form

$$\tilde{T}^{\mu\nu}=U\tilde{u}^{\mu}\tilde{u}^{\nu}-T\tilde{v}^{\mu}\tilde{v}^{\nu}\ , \tag{2.13}$$

where the corresponding eigenvalues U and T are what will be interpretable as the *energy density* (per unit string length) and the *tension* of the string. It can be seen that this form is still applicable in the isotropic Goto-Nambu limit case for which the eigenvalues are equal and constant, being given by

$$U=T=\frac{m^2}{\hbar}\ , \tag{2.14}$$

so that there is no preferred frame.

In the general elastic string case, as in the Goto-Nambu limit, the energy density will still be given directly by the Lagrangian scalar in the form

$$U=-\tilde{\mathcal{L}}\ , \tag{2.15}$$

the difference from the Goto-Nambu case being that $\tilde{\mathcal{L}}$ will no longer be a constant but a variable function of ν. However, in the generic case the tension T will have a lower value, whose derivation requires, as an intermediate step, the evaluation of the relevant chemical potential or *effective mass* μ per idealized particle of the flow implicitly defined by the timelike streamlines on which ψ is constant, which will be given by

$$\mu=\frac{dU}{d\nu}\ . \tag{2.16}$$

[The relativistic chemical potential defined in this standard way has the dimensionality of a mass and is not to be confused with the dimensionless quantity $Gm^2/\hbar$ characterizing the gravitational coupling of a Goto-Nambu string, for which the symbol μ is frequently used by writers dealing with the specialized topic of cosmic strings of this type, particularly in the context of strings formed during grand unified theory (GUT) symmetry breaking for which the relevant value is expected to be given roughly by $Gm^2/\hbar\approx 10^{-6}$, though not in more general contexts.] In terms of this chemical potential, the string tension T itself is finally obtained as a function of the number density ν (and hence implicitily of the energy density U) in the form

$$T=U-\nu\mu\ . \tag{2.17}$$

The original reason for introducing such general elastic string models in the context of cosmic string theory was for dealing with the mechanical effects [10,11] of the currents that occur in "superconducting" cases of the kind first considered by Witten [12], but it has more recently [7] become apparent that a particular class of "warm" string models within this general elastic category are potentially useful for the large scale averaged description of the mechanical behavior of strings whose behavior on smaller scales is of the simple Goto-Nambu kind.

III. DRAG FORCE OF A THERMAL BACKGROUND MEDIUM

The purpose of the present and following sections is to summarize the general principles of elastic string mechanics that are relevant not only to the "warm" cosmic string models, to which they will be applied in the subsequent sections, but also to the more general category of models [10,11,8] that is needed for the macroscopic representation of "superconducting" cosmic strings of the kind first discussed by Witten [12]. The treatment here goes beyond previous discussions in giving particular attention to the related effects of cosmological expansion and of the drag force F^{ν}, say, exerted by the ambient thermal background medium.

The definition of the force vector to be considered is provided by the relation

$$\tilde{\nabla}_{\mu}\tilde{T}^{\mu\nu}=F^{\nu}\ ; \tag{3.1}$$

i.e., F^{ν} measures the deviation from the conservative behavior governed by the Noetherian relation (2.4) that would have been satisfied if the behavior had been governed completely by the relevant variational principle.

Although more general possibilities can be imagined (e.g., cases involving accretion from the ambient medium), our discussion here will be limited to cases whose internal dynamics remains conservative in the strict sense. This means that we only consider the kind of force that remains *orthogonal* to the world sheet in the sense that its contraction with the fundamental tangential projection tensor $\eta^{\mu}{}_{\nu}$ vanishes, i.e.,

$$\eta^{\mu}{}_{\nu}F^{\nu}=0 \ . \tag{3.2}$$

The vector F^{ν} that has just been defined has the dimensionality of a force per unit length.

Whereas most previous accounts of cosmic string dynamics have been limited to the late-time regime in which the evolution can be treated as "free" in the sense that coupling to the ambient cosmological background medium can be neglected, our purpose here is to consider allowance for a drag force F^{μ}. Such a force can be expected to be significant at earlier times because of relative motion with respect to the ambient thermal gas as characterized by the cosmological *temperature* measured in energy units (i.e., with Boltzmann's constant set to unity), which we denote by the symbol Θ (the symbol T being reserved for the tension). According to an estimate of Kibble [13], damping by the environment would continue until the cosmological temperature had dropped below the critical temperature $\Theta\approx m$ of the symmetry-breaking transition at which the strings are supposed to have been formed by a factor of the order of magnitude $\sqrt{Gm^2/\hbar}$, so that the strings could be treated as free only after the temperature had fallen below a critical value which would be given by $\Theta\approx(G/\hbar)^{1/2}m^2$, where the factor $\sqrt{Gm^2/\hbar}$ would be of the order of 10^{-3} for the usual GUT value of the Higgs boson mass scale m on which the much discussed "heavyweight" string scenario for cosmological structure formation is based. Moreover, it has been emphasized by Vilenkin [14] that the duration of the regime in which drag forces could be effective would be even longer for relatively "lightweight" strings (of which a potentially interesting class arises in modified versions of the electroweak unification scheme [15]).

In accordance with the usual idealization that is expected to provide a good description of conditions in the early Universe, we shall suppose that the ambient cosmological medium can be treated as a perfect gas that is determined by an entropy current s^{μ}, say, and that is governed by an appropriate equation of state whereby the corresponding entropy density s, as defined by

$$s^2=-s_{\mu}s^{\mu} \ , \tag{3.3}$$

will determine the corresponding energy density ρ, say. The corresponding perfect fluid stress momentum-energy density tensor will have the form

$$T^{\mu\nu}=\rho s^{-2}s^{\mu}s^{\nu}+P\gamma^{\mu\nu} \ , \tag{3.4}$$

with

$$\gamma^{\mu\nu}=g^{\mu\nu}+s^{-2}s^{\mu}s^{\nu} \tag{3.5}$$

for a pressure given by

$$P=\Theta s-\rho \ , \tag{3.6}$$

where the relevant temperature is obtained from the blackbody equation of state as

$$\Theta=\frac{d\rho}{ds} \ , \tag{3.7}$$

in units such that Boltzmann's constant is unity. The relation (3.6) is the continuum analogue of (2.17) (the fluid pressure P being the analogue of the negative of the string tension T), its form being such that the momentum-energy pseudoconservation law

$$\nabla_{\mu}T^{\mu\nu}=0 \tag{3.8}$$

automatically entails the conservation law

$$\nabla_{\mu}s^{\mu}=0 \tag{3.9}$$

for the entropy. [It is to be remarked that the latter is always a conservation law in the strict sense, whereas (3.8) is a genuine conservation law only if the background spacetime metric $g_{\mu\nu}$ is flat.]

In the relativistic limit in which the radiation gas is dominated by effectively massless particles, the equation of state will have the form

$$\rho=\frac{3\hbar}{4\alpha}s^{4/3} \ , \tag{3.10}$$

where α is a dimensionless constant of order unity (which in the familiar case of a gas consisting only of photons is given exactly by $\alpha=(\frac{4}{5})^{1/3}(\pi/3)^{2/3}$, but which would be somewhat larger when many species are involved), which corresponds to a radiation pressure given by

$$P=\frac{\alpha^3}{4\hbar^3}\Theta^4 \ , \tag{3.11}$$

while the corresponding expression for the entropy density will be given by

$$s=\frac{\alpha^3\Theta^3}{\hbar^3} \ . \tag{3.12}$$

In the absence of detailed coupling with the internal structure of the string or particle [which, if relevant, would be likely to cause the partial conservation condition (3.2) to be violated], the only covariant possibility for the specification, consistent with (3.2), of the drag force exerted by the thermal current s^{μ} is that it should have the form

$$F^{\mu}=\beta\hbar\perp^{\mu}{}_{\nu}s^{\nu} \ , \tag{3.13}$$

in which, following Vilenkin [14], we use the notation β for the relevant scalar coefficient, whose value will depend on the particular circumstances, and where the tensor $\perp^{\mu}{}_{\nu}$ is the operator of projection orthogonal to the string world sheet, as given by (2.7).

In the case of a string, the coefficient β in (3.13) will be dimensionless, being expressible in the form $\beta=r/\lambda$, where r is the effective string thickness characterizing the interaction. This would be expected to be of the order of the geometric cross section of the string if this were large compared with the thermal wavelength λ. However, in the case of a Goto-Nambu string model for a Nielsen-

Olesen-type vortex, one would expect the geometric cross section to be of the order of the relevant Higgs particle Compton radius $\hbar/m$, whereas after the epoch of formation of the string the thermal wavelength λ would be expected to have become very much larger so that it, rather than the Higgs length scale, might be expected to determine the relevant effective width r. In this case, in accordance with more detailed analysis based on more specialized assumptions by Vilenkin [14], as a rough order of magnitude estimate one would simply get $\beta \approx 1$.

For the purpose of the general considerations developed below, the particular form of the expression for β will not be important. It will be sufficient to suppose that, as is likely to be a good approximation in all the diverse string scenarios envisaged in the preceding paragraphs, the value of β should be specified only as a function of the entropy density s or equivalently as a function of the ambient temperature Θ.

IV. DYNAMICAL EVOLUTION EQUATIONS AND THE SECOND FUNDAMENTAL TENSOR

We can always use a Lagrangian function such as those described in Sec. II as a means of specifying a string model and the associated tensor $\tilde{T}^{\mu\nu}$, but in the presence of an external force the evolution of the model thus obtained will no longer be completely governed by the corresponding variational principle. Nevertheless, provided that electromagnetic and other long range forces can be neglected (which is likely to be justifiable as a good approximation, even when they are not entirely absent [16,17], at least as long as radiation remains small) so that the drag force density given by (3.13) is the only external force contribution, its world sheet orthogonality property will, by the general principles of classical relativistic brane mechanics [8,18], be sufficient to ensure the respect of the purely *internal* pseudoconservation law that is the analogue of (3.8). The orthogonality property of the drag force density (3.13) implies that the tangentially projected part of the pseudoconservation law (2.4) will still be applicable; i.e., as in the case of a free string motion, we shall still have

$$\eta^{\mu}{}_{\nu}\tilde{\nabla}_{\rho}\tilde{T}^{\rho\nu}=0 \ . \tag{4.1}$$

In the case of a simple Goto-Nambu string model as characterized by (2.8), the tangentially projected conservation law (4.1) is trivial: All it contains is the statement that, as was in any case postulated at the outset in the formulation of the variational principle, the mass parameter in the specification of the model is *constant* over the world sheet, i.e., $\tilde{\nabla}_{\mu}m=0$. However, in the case of a generic elastic string model as characterized by (2.11) and (2.13), the tangentially projected conservation law (4.1) provides the nontrivial information that is both necessary and sufficient for determining the internal mechanical evolution of the model (whereas for more complicated models than those considered here further information from other internal field equations might be needed). Part of the information contained in (4.1) will be expressed by the conservation law for surface current defined by the relation [8]

$$\nu\tilde{u}^{\mu}=\mathcal{E}^{\mu\nu}\tilde{\nabla}_{\nu}\psi \ , \tag{4.2}$$

which specifies both the number density ν and the unit eigenvector $\tilde{u}^{\mu}$ in terms of the stream function ψ. $\mathcal{E}^{\mu\nu}$ is the surface alternating tensor, which will be expressible in terms of any orthonormal tangent frame consisting of unit timelike and spacelike tangential vectors $\tilde{u}^{\mu}$ and $\tilde{v}^{\mu}$, as characterized by $\tilde{u}^{\mu}\tilde{u}_{\mu}=-1$, $\tilde{v}^{\mu}\tilde{v}_{\mu}=1$, and $\tilde{u}^{\mu}\tilde{v}_{\mu}=0$, in the form

$$\mathcal{E}^{\mu\nu}=\tilde{u}^{\mu}\tilde{v}^{\nu}-\tilde{u}^{\nu}\tilde{v}^{\mu} \ . \tag{4.3}$$

As well as the kinematically obvious surface conservation law

$$\tilde{\nabla}_{\rho}(\nu\tilde{u}^{\rho})=0 \tag{4.4}$$

obtained for this current, (4.1) also implies the existence of another, dynamical rather than merely kinematical law of the analogous form

$$\tilde{\nabla}_{\rho}(\mu\tilde{v}^{\rho})=0 \ . \tag{4.5}$$

Unlike (4.4), which is the analogue of the perfect fluid current conservation law (3.9), the second surface conservation law (4.5) has no perfect fluid analogue and its existence is not so obvious from the point of view of the variational principle. It is, however, evident that it is immediately obtainable from its partner (4.4) by application of the dual symmetry property [8,10] that is a unique distinguishing factor of strings (as opposed to point particles and to higher-dimensional membranes, and fluids in particular).

The equations just described apply directly to "superconducting cosmic strings" (of the kind proposed by Witten [12]) in circumstances compatible with the neglect of electromagnetic and other such coupling effects, allowance for which would require the inclusion of external forces of a less simple kind than that given by (3.13), in which case the relevant "superconducting current" has either the form $\mu\tilde{v}^{\rho}$ or else the form $\nu\tilde{u}^{\rho}$ depending on whether it is spacelike (the possibility that was most commonly considered in the early discussions) or timelike.

One of the general consequences that can be drawn [8,11] from the intrinsic equations (4.4) and (4.5) of the elastic string model that has just been described is that the characteristic speed c_L, say, of longitudinal "woggle" (as opposed to "wiggle") perturbations will be given by an expression that corresponds exactly to its perfect fluid analogue ($c_L^2=dP/d\rho=\frac{1}{3}$ in the case of the thermal gas discussed above), namely,

$$c_L^2=-\frac{dT}{dU} \ . \tag{4.6}$$

To complete the set of equations governing the dynamics of the string, it remains to specify the extrinsic equations of motion governing the evolution of the supporting world sheet itself, which will differ from those for the free string case by the inclusion of the force term characterized by (3.1) and (3.2) on the right-hand side. What this means is that the *extrinsic* equation of motion [as obtained from (3.1) by orthogonal projection] will take the form

$$\tilde{T}^{\lambda\mu}K_{\lambda\mu}{}^{\nu}=F^{\nu} \tag{4.7}$$

(whose validity is universal in the sense that it applies [8,18] not just to string world sheets, but also to point particle and to membranes and higher-dimensional "p-branes"), where $K_{\lambda\mu}{}^{\nu}$ is the *second fundamental tensor* or "extrinsic curvature tensor" of the world sheet, which is defined [8,11] in terms of the (first) fundamental tensor $\eta^{\mu\nu}$ by

$$K_{\lambda\mu}{}^{\nu}=\eta_{\lambda\rho}\tilde{\nabla}_{\mu}\eta^{\rho\nu} . \tag{4.8}$$

This second fundamental tensor might equivalently be considered to be implicitly defined by the condition, which can easily be seen to follow directly from (4.8), that for *any* vectors $\tilde{u}^{\mu}$ and $\tilde{v}^{\mu}$ that are tangential to the world sheet, i.e., such that

$$\perp^{\nu}{}_{\mu}\tilde{u}^{\mu}=\perp^{\nu}{}_{\mu}\tilde{v}^{\mu}=0 , \tag{4.9}$$

and hence in particular for the eigenvectors introduced in (2.13), the orthogonally projected derivative of one with respect to the other will be given by

$$\perp^{\nu}{}_{\mu}\tilde{v}^{\rho}\nabla_{\rho}\tilde{u}^{\mu}=-\tilde{u}^{\rho}\tilde{v}^{\mu}K_{\rho\mu}{}^{\nu} . \tag{4.10}$$

In terms of the internal metric components h^{ab} and the Christoffel connection components $\Gamma_{\mu}{}^{\nu}{}_{\rho}$ of the background spacetime, this second fundamental tensor is expressible more explicitly as

$$K_{\lambda\mu}{}^{\nu}=\perp^{\nu}{}_{\rho}(\eta_{\lambda\sigma}\eta_{\mu\tau}h^{ac}h^{bd}x^{\sigma}{}_{,a}x^{\tau}{}_{,b}x^{\rho}{}_{,c,d}+\eta_{\lambda}{}^{\sigma}\eta_{\mu}{}^{\tau}\Gamma_{\sigma}{}^{\rho}{}_{\tau}) . \tag{4.11}$$

As compared with (4.8), this later version has the disadvantage that the tensorial covariance is no longer immediately apparent, but it has the advantage of manifestly displaying the symmetry and tangentiality properties of the first two indices and the world sheet orthogonality property of the third index.

In the case of a string, it can be seen that substitution of the generic expression (2.13) in (4.7) gives an extrinsic equation of motion of the standard form

$$\perp^{\rho}{}_{\mu}(U\tilde{u}^{\nu}\nabla_{\nu}\tilde{u}^{\mu}-T\tilde{v}^{\nu}\nabla_{\nu}\tilde{v}^{\mu})=F^{\rho} . \tag{4.12}$$

It is to be noted that, since the expression (3.13) for the force term on the right-hand side of this equation does not involve gradients, the presence of this external force will not affect the characteristic speed c_E of extrinsic, i.e., "wiggle," perturbations (relative to the preferred frame specified by $\tilde{u}^{\mu}$), which will therefore be given by a formula of the same form as applies [8,11] in the free string case: namely,

$$c_E^2=\frac{T}{U} . \tag{4.13}$$

The preceding Eqs. (4.12) and (4.13) are applicable, in particular, to the "usual" Goto-Nambu-type model as characterized by (2.8). In this special case, substitution in (4.7) of the relevant expression (2.9) for the world sheet stress momentum-energy density tensor leads directly to a dynamic equation of motion of the simple standard form

$$TK^{\mu}+F^{\mu}=0 , \tag{4.14}$$

where T now has the constant value given by (2.14) and where K^{μ} is the extrinsic curvature vector of the world sheet, which is defined as the *trace,*

$$K^{\mu}=K^{\nu}{}_{\nu}{}^{\mu}=\tilde{\nabla}_{\nu}\eta^{\nu\mu} , \tag{4.15}$$

of the second fundamental tensor.

The force driven Goto-Nambu equation of motion (4.14) evidently agrees with what is obtained by setting $U=T$ in (4.12), but whereas the eigenvectors $\tilde{u}^{\mu}$ and $\tilde{v}^{\mu}$ involved therein are well determined in the generic case; on the other hand, in the degenerate Goto-Nambu case they can be adjusted by an arbitrarily variable two-dimensional Lorentz frame rotation, which will not have any effect on the corresponding expression

$$K^{\rho}=\perp^{\rho}{}_{\mu}(\tilde{v}^{\nu}\nabla_{\nu}\tilde{v}^{\mu}-\tilde{u}^{\nu}\nabla_{\nu}\tilde{u}^{\mu}) \tag{4.16}$$

for the curvature vector itself.

It is to be remarked that, using (4.11), the curvature vector (4.15) can, if desired, be worked out as a sum of coordinate-dependent terms, involving the internal metric h_{ab} and its determinant h, in the explicit form

$$K^{\mu}=\frac{1}{\sqrt{-h}}(\sqrt{-h}\,h^{ab}x^{\mu}{}_{,b})_{,a}+h^{ab}x^{\sigma}{}_{,a}x^{\tau}{}_{,b}\Gamma_{\sigma}{}^{\mu}{}_{\tau} , \tag{4.17}$$

whose substitution can be used to verify that the compact standard form (4.14) of the force-driven Goto-Nambu equation does in fact agree with the more unwieldy version given by Vilenkin [14].

This last expression arises naturally when the left-hand side of the equation of extrinsic motion (4.14) is derived directly from the variation that arises in the use of the Goto-Nambu action principle (whose application is violated in the present case by the presence of the force term on the right). However, compared with the simple definition (4.15), the version (4.17) has the double disadvantage of hiding both the vectorial covariance and the world sheet orthogonality property of the curvature vector. The expression (4.17) is of course susceptible to simplification by adoption of a particular choice of gauge for the internal coordinates. For example, one might choose to use a *conformal gauge* of the widely familiar kind characterized by $g_{\mu\nu}x'^{\mu}\dot{x}^{\nu}=0$ and $g_{\mu\nu}(x'^{\mu}x'^{\nu}-\dot{x}^{\mu}\dot{x}^{\nu})=0$, using the abbreviated notation scheme exemplified by $x'^{\mu}=x^{\mu}{}_{,1}$ and $\dot{x}^{\mu}=x^{\mu}{}_{,0}$, so as to obtain $\sqrt{-h}=g_{\mu\nu}x'^{\mu}x'^{\nu}=-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$, in terms of which (4.17) would reduce to the form

$$K^{\mu}=\frac{1}{\sqrt{-h}}[x''^{\mu}-\ddot{x}^{\mu}+(x'^{\sigma}x'^{\tau}-\dot{x}^{\sigma}\dot{x}^{\tau})\Gamma_{\sigma}{}^{\mu}{}_{\tau}] .$$

But in particular applications it may be shrewder to use a different, more specifically adapted, choice (such as that employed by Vilenkin [14]). It is to be remarked that even if one uses a gauge of the standard conformal type, there will still be a considerable amount of freedom in the way it is set up, just as there is a considerable degree of freedom in the choice of the orthonormal tangent frame in the corresponding covariant formula (4.16).

V. DYNAMICAL EVOLUTION IN A CONFORMALLY EXPANDING UNIVERSE

Following up a remark made by Vilenkin [14], namely, that the effect of a drag force is similar to the effect of the expansion of the Universe, the new points that we wish to make in this section concern the way the equations of motion discussed in the previous section are affected by a conformal transformation of the form

$$g^{\mu\nu}\mapsto\breve{g}^{\mu\nu}=a^2g^{\mu\nu}\ , \tag{5.1}$$

such as is discussed in the Appendix. Before proceeding, we remark that such a transformation does not affect the tangential and orthogonal projection tensors whose respective behavior is given by

$$\eta^{\mu}{}_{\nu}\mapsto\eta^{\mu}{}_{\nu},\quad \perp^{\mu}{}_{\nu}\mapsto\perp^{\mu}{}_{\nu}\ , \tag{5.2}$$

but that it does affect the corresponding contravariant tensors, so that for the world sheet fundamental tensor itself we shall have

$$\eta^{\mu\nu}\mapsto\breve{\eta}^{\mu\nu}=a^2\eta^{\mu\nu}\ , \tag{5.3}$$

while similarly

$$\perp^{\mu\nu}\mapsto\breve{\perp}^{\mu\nu}=a^2\perp^{\mu\nu}\ . \tag{5.4}$$

The first point is that when the "true" spacetime metric $g_{\mu\nu}$ is related by such a conformal transformation to an auxiliary metric $\breve{g}_{\mu\nu}$ that is comparatively simple, it may evidently be technically advantageous to evaluate the second fundamental tensor in terms of the latter. If the conformal transformation is of "homothetic" type (i.e., if the expansion factor a is uniform), the second fundamental tensor $K_{\lambda\mu}{}^{\nu}$ will be unaffected, but whenever the generalized Hubble covector H_μ defined by (A3) is nonzero, the transformation (5.1) will induce a corresponding transformation that is given [19] by the universally applicable formula

$$K_{\lambda\mu}{}^{\nu}\mapsto\breve{K}_{\lambda\mu}{}^{\nu}=K_{\lambda\mu}{}^{\nu}+\eta_{\lambda\mu}\perp^{\nu\rho}H_\rho \tag{5.5}$$

(which holds for an embedding of arbitrary dimension in a background that may also have arbitrary dimension).

Assuming that the auxiliary metric $\breve{g}_{\mu\nu}$ is simpler than the "true" spacetime metric $g_{\mu\nu}$ (as will be the case if it is stationary), then it will follow that the direct evaluation of $\breve{K}_{\lambda\mu}{}^{\nu}$ will be simpler than that of $K_{\lambda\mu}{}^{\nu}$ itself. This implies that it may be advantageous to work directly with the correspondingly transformed version of the extrinsic equation of motion (4.7) which will be expressible as

$$\tilde{T}^{\lambda\mu}\breve{K}_{\lambda\mu}{}^{\nu}=\perp^{\nu\mu}\breve{F}_\mu\ , \tag{5.6}$$

where the correspondingly transformed force density, as seen from the point of view of the simplified, e.g., stationary, static, or even flat, metric $\breve{g}_{\mu\nu}$, is identifiable just as the net *quasiforce* covector $\breve{F}_\mu$ that is introduced in the Appendix by the basic defining relation (A10).

It is to be emphasized that formulas (5.5) and (5.6) are of quite general validity, applying not only to the case of strings with which we are concerned here, but also to higher-dimensional "branes" and to particles [8,18,19]. In the case of a Goto-Nambu string, it follows from (2.14) that the required trace will be given by $\tilde{T}^{\nu}{}_{\nu}=-2m^2/\hbar$, so that in this case the net quasiforce will be given by the almost equally simple formula

$$\breve{F}_\mu=F_\mu-\frac{2m^2}{\hbar}H_\mu\ . \tag{5.7}$$

In the standard application to the conformally static Friedmann cosmological models, the entropy current vector s^μ as defined in (3.3) and the conformal Killing vector k^μ are both aligned oppositely to the Hubble vector H^μ, which is itself directed towards the big bang, so that, taking account of the conservation law (3.9), one has

$$s^\mu=\frac{2}{a}k^\mu=-\frac{s}{H}H_\mu=\frac{1}{3H}\nabla_\mu s\ . \tag{5.8}$$

It can be seen (as the generalization of a remark first made under more restricted conditions by Vilenkin [14]) that if the true force F^ρ is given just by the simple string drag formula (3.13), then the adjustment involved in the replacement of the true force by the quasiforce in (5.6) merely gives a change in the magnitude, but not in the direction in which acts the force term $\perp^{\nu\mu}\breve{F}_\mu$.

In the simplest case, that of a "parabolic," i.e., marginally open, Robertson-Walker model, the relevant conformally adjusted metric $\breve{g}_{\mu\nu}$ is just that of a flat Minkowski space, and the Hubble rate H will be given by the corresponding Friedmann equation [20] as

$$H^2=\frac{8\pi G\rho}{3}\ . \tag{5.9}$$

For the pure radiation gas specified by (3.10), the expression (5.8) will reduce to the simple form

$$H_\mu=-\nabla_\mu(\ln\Theta)\ , \tag{5.10}$$

and so in this case it follows from the Friedmann equation (5.9) that the temperature gradient itself will be given by

$$\nabla_\mu\Theta=\left[\frac{2\pi G\hbar^3}{\alpha^3}\right]^{1/2}s_\mu\ . \tag{5.11}$$

Under these circumstances, it follows that the formula (3.13) for the ("true") drag force covector F_μ acting on a point particle or string will be expressible as

$$F_\mu=\beta\left[\frac{\alpha^3}{2\pi G\hbar}\right]^{1/2}\perp^{\nu}{}_{\mu}\nabla_\nu\Theta\ ; \tag{5.12}$$

i.e., it will be proportional to the projection of the cosmological temperature gradient. The relevant total quasiforce covector will therefore be given by

$$\breve{F}_\mu=\left[\tilde{T}^{\nu}{}_{\nu}g^{\rho}{}_{\mu}-\left[\frac{\alpha^3}{2\pi G\hbar}\right]^{1/2}\beta\Theta\perp^{\rho}{}_{\mu}\right]H_\rho\ . \tag{5.13}$$

And therefore, for the parabolic pure radiation gas case, the adjusted force term on the right-hand side of (5.6) will be expressible explicitly, for a general string model with

$$\tilde{T}^{\nu}{}_{\nu}=-(U+T)\ , \tag{5.14}$$

by

$$\check{\perp}^{\mu\nu}\check{F}_\nu = \left[1+\left\{\frac{2\pi G\hbar}{\alpha^3}\right\}^{1/2}\frac{U+T}{\beta\Theta}\right]F^\mu . \quad (5.15)$$

it is to be remarked that when the extrinsic equation of motion (4.7) is transformed to the equivalent version (5.6) it can be considered as implicit that since the eigenvectors transform according to the rule

$$\tilde{u}^\mu \mapsto a\tilde{u}^\mu, \quad \tilde{v}^\mu \mapsto a\tilde{v}^\mu , \quad (5.16)$$

so as to preserve their unit normalization, the eigenvalues U and T themselves will remain invariant:

$$U \mapsto U, \quad T \mapsto T . \quad (5.17)$$

The particular case considered by Vilenkin [14] was that of the "usual" Goto-Nambu-type model with equation of motion given by (4.14). It can be seen that the transformed version (5.6) of the extrinsic equation will reduce in this case to the form

$$T\check{K}^\mu + \check{\perp}^{\mu\nu}\check{F}_\nu = 0 , \quad (5.18)$$

the advantage of the latter being that if $\check{g}_{\mu\nu}$ is simpler than $g_{\mu\nu}$ then the evaluation of $\check{K}^\mu$ will be correspondingly simpler than that of its analogue K^μ: For instance, there will be no need for the analogue of the second term involving the spacetime connection in the expression (4.17) when the conformally adjusted metric is flat.

Whereas Vilenkin's observation was that the effect of expansion in a Friedmann universe can be taken into account as a force renormalization with respect to the conformally related static background, it may now be remarked that the trick can be played the other way round in the sense that one can use an artificially accelerated rate of expansion to represent the effect of the drag force as given by (5.12). Instead of the usual conformal transformation to the stationary system determined by

$$\check{g}_{\mu\nu} = a^{-2} g_{\mu\nu} , \quad (5.19)$$

let us proceed the other way by introducing a new conformally transformed metric

$$\hat{g}_{\mu\nu} = e^{2\tau} g_{\mu\nu} , \quad (5.20)$$

in which, instead of being slowed down, the acceleration is speeded up. If we choose τ to be the function of the cosmological entropy density s that is specified, modulo a constant of integration that will merely fix the overall scale in (5.20), by the equation of state for the mass density ρ and the formula for the drag coefficient β in terms of s as the solution of the differential equation

$$\frac{d\tau}{ds} = -\frac{\hbar^2\beta}{2m^2\sqrt{6\pi G\rho}} , \quad (5.21)$$

then, by calculations analogous to those we have already carried out for the "usual" conformal transformation (5.19), it can be seen that for a Goto-Nambu string model as characterized by (2.8) subject to a force law of the form (3.13) in a parabolic Friedmann universe as characterized by (5.9), the net force (that in the analogue of our previous notation scheme would be designated by $\hat{F}_\mu$) perceived from the point of view of the new metric (5.20) will cancel out altogether (i.e., we shall get $\hat{F}_\mu = 0$). This means that the equation of motion (4.14) or equivalently (5.18) for the thermally damped string motion is equivalent to that of a *free* Goto-Nambu string,

$$\hat{K}^\mu = 0 , \quad (5.22)$$

as evaluated with respect to the artificially transformed metric (5.20).

In the case of a simple ultrarelativistic gas with equation of state given by (3.10), the usual conformal factor a will be given directly in terms of the cosmological temperature Θ by

$$a = \frac{\Theta_0}{\Theta} , \quad (5.23)$$

where Θ_0 is a constant of integration representing the temperature at some conveniently chosen time origin (such as now). If (in accordance with the considerations discussed at the end of Sec. III) we suppose that β is simply a constant, independent of s and hence of Θ, then in this same case as governed by (3.10) the relation (5.21) is integrable to give a corresponding simple formula of the form

$$\tau = \frac{\beta}{2m^2}\left\{\frac{\hbar\alpha^3}{2\pi G}\right\}^{1/2}(\Theta_0 - \Theta) . \quad (5.24)$$

The fact that the string behaves with respect to the transformed metric as if it were free means that although from a physical point of view the drag forced motion is dissipative, nevertheless from a mathematical point of view it will be conservative in the sense of being governed by a variational principle expressible in terms of an action integral of the form (2.1). The effect of the drag is simply that instead of using a Lagrangian scalar of the trivial Goto-Nambu form (2.8) we must use a less trivial, but nevertheless still extremely simple, Lagrangian that will be expressible in terms of the known (cosmological temperature dependent) scalar background field τ in the form

$$\tilde{\mathcal{L}} = -\frac{m^2}{\hbar} e^\tau . \quad (5.25)$$

The effect of the drag force on a cosmic string in a conformally expanding background is equivalent to a renormalization of the effect of the conformal expansion of the metric. In particular, the result found by Vilenkin [14] is recovered in the case of a Friedmann universe. Furthermore, it is shown that the effect of the drag force can be simulated by an artificially accelerated expansion. As a consequence, a rather simple Lagrangian exists to describe the interaction between a Goto-Nambu string and the surrounding medium.

VI. WARM STRING MODEL

The underlying intuition that was used as a basis for setting up the "warm" elastic string model [7] for the macroscopic averaged description of (thermal or other) microscopic substructure is that although (compared with the speed of light propagation that occurs in the underlying "bare" Gotu-Nambu model) the effective propagation speed should be slowed down because of the extra microscopic distance that needs to be traveled when small wiggles are present, there is no way in which dispersion can arise between distinct longitudinal and transverse modes. What appear as "woggles," i.e., longitudinal perturbations from the point of view of the macroscopic treatment, would be perceived merely as bunched packets of "wiggles," i.e., extrinsic (transverse) perturbations, when analyzed in microscopic detail. The requirement of consistency between the pictures provided by the effective macroscopic model and the underlying microscopic dynamics therefore imposes the condition that the "woggle" and "wiggle" speeds should agree, i.e.,

$$c_L = c_E \ . \tag{6.1}$$

Equating the explicit expressions (4.6) and (4.13) immediately gives a differential equation of state which can be integrated to give a relation of the constant product form

$$UT = \frac{m^4}{\hbar^2} \ , \tag{6.2}$$

where m is a constant of integration which must evidently be identified with the mass scale m characterizing the underlying "bare" Goto-Nambu model, with which the "warm" string model must agree in the limit $U = T = m^2/\hbar$.

Like any other elastic string model of the category specified by (2.11) and described in Sec. IV, this warm string model will be governed by a Lagrangian scalar which can be taken to have the explicit form

$$\tilde{\mathcal{L}} = -\frac{m^2}{\hbar}\left\{1 - \frac{3\hbar^2}{2\pi m^2} h^{ab} S_{,a} S_{,b}\right\}^{1/2} \ , \tag{6.3}$$

where S is an appropriate stream function which in this case will be interpretable as a measure of effective entropy. This Lagrangian corresponds, by (2.15), to an energy density U that is given [7] by the formula

$$U = \frac{m^2}{\hbar}\left\{1 + \frac{3\hbar^2 \tilde{s}^2}{2\pi m^2}\right\}^{1/2} \ , \tag{6.4}$$

as a function of the relevant number density $\tilde{s}$, which is specified as the magnitude of the corresponding effective entropy current vector $\tilde{s}^{\mu}$ by

$$\tilde{s}^2 = \tilde{s}^{\mu} \tilde{s}_{\mu}, \quad \tilde{s}^{\mu} = \mathcal{E}^{\mu\nu} \tilde{\nabla}_{\mu} S \ . \tag{6.5}$$

In order to obtain an agreement with the general elastic string formalism presented in Secs. II and IV, the conserved number density ν that was used could be taken to be any constant multiple of this particular choice $\tilde{s}$, whose precise normalization is motivated by thermodynamic considerations that will be explained below.

According to the principles recapitulated in the previous section, the way to calculate the tension T from such an expression is first to work out the corresponding chemical potential, which we shall in this case denote by $\tilde{\Theta}$, as given by the derivative of the energy with respect to the number density under consideration, i.e.,

$$\tilde{\Theta} = \frac{dU}{d\tilde{s}} \ , \tag{6.6}$$

and then to use a relation of the general form (2.17), which in this case will be expressible as

$$U - T = \tilde{s} \tilde{\Theta} \ . \tag{6.7}$$

Since the application of (6.6) to (6.4) gives a result expressible as

$$\tilde{\Theta} = \frac{3m^2 \tilde{s}}{2\pi U} \ , \tag{6.8}$$

it can easily be seen that (6.7) leads back to the original constant product relation (6.2).

It is evident that, by working with a rescaled number density ν taken proportional to $\tilde{s}$ with a suitably adjusted numerical factor, it would be possible to simplify the numerical coefficient of the $\tilde{s}^2$ term within the square root in (6.4). The motivation for the particular choice [7] of the numerical coefficient in the "warm string" equation of state (6.4) is that if the spectrum of the microstructure actually is that of local thermal equilibrium (so that it is given by the one-dimensional analogue of the three-dimensional bosonic photon gas spectrum that is familiar in the context of Planckian blackbody radiation), then the variable $\tilde{s}$ will be directly identifiable with the *entropy density* (per unit length of string) in units (of the kind used for describing the blackbody radiation gas discussed in Sec. III) such that Boltzmann's constant is unity. It follows that, under the same conditions of local thermal equilibrium, the associated chemical potential $\tilde{\Theta}$ will be directly identifiable with the relevant *temperature* of the microstructure on the string. As already remarked in Sec. III, our reason for not using the letter T but employing the symbol $\tilde{\Theta}$ (with a tilde to distinguish it from the background radiation temperature Θ) is that in the present context the former symbol has already been preempted to denote the *tension*, whose functional dependence on the quantity $\tilde{\Theta}$ can be seen to be expressible as

$$\left\{\frac{\hbar T}{m}\right\}^2 = m^2 - \frac{2\pi}{3}\tilde{\Theta}^2 \ . \tag{6.9}$$

The thermal interpretation of $\tilde{s}$ and $\tilde{\Theta}$ is potentially important in view of the likelihood that the microscopic excitation spectrum really will be approximately thermal in many physical circumstances, in which case use of the appropriate normalization will be helpful as a guide in making quantitative estimates. Nevertheless, it is to be emphasized that the validity of the formulas of the preceding paragraph is not limited to the case of exact or even approximate equilibrium. The formulas (6.4)–(6.9) will still characterize a well-defined "effective entropy" density $\tilde{s}$ and a well-defined "effective temperature" $\tilde{\Theta}$ regardless of whether or not the spectrum actually is in a state of exact or even approximate thermal equilibrium. Soon after the original, heuristic, derivation [7] of the relation (6.4) (on which these formulas depend), the validity of this way of representing the cumulative effect of small scale moderate amplitude "wiggles" was confirmed, without any restriction on the qualitative (thermal or other) nature of their spectrum, by a more explicit calculation by Vilenkin [21]. The convenient feature that the effective equation of state is not affected by deviations from true thermal equilibrium is not a mere simplifying approximation, but occurs as a special consequence of the one-dimensional spatial geometry of the string (which means that no such convenient simplification could be expected to arise in an analogous description of wiggles on a membrane).

Having completed this brief summary of the basic assumptions and underlying motivation of the "warm string model," we can now apply the formalism described in Sec. IV. In particular, the (in this case unique) characteristic velocity as given by (4.13) is expressed as

$$c_E = \frac{\hbar T}{m^2} . \tag{6.10}$$

The relevant effective temperature $\tilde{\Theta}$ will be given in terms of this dimensionless factor by

$$2\pi\tilde{\Theta}^2 = 3m^2(1-c_E^2) . \tag{6.11}$$

While it is to be anticipated that the description presented in this section should work very well as long as the amplitudes of the wiggles are small compared with their wavelengths (and as long as the latter are small compared with the large scale structure under consideration), it cannot plausibly be expected to remain satisfactory when the effective temperature $\tilde{\Theta}$ approaches the Hagedorn-type saturation limit that can be read off from (6.11) as

$$\tilde{\Theta}^2_{\max} = \frac{3m^2}{2\pi} , \tag{6.12}$$

at which the effective tension T and the characteristic speed c_E tend to zero. In this singular "hot" limit, the "wiggles" spill out to such an extent that a "string" (i.e., spatially one-dimensional) description ceases to make sense, so that some sort of fractal description [3,22], and ultimately just a simple low-pressure gas description, would become more appropriate. The nature and extent of deviations from the simple model characterized by (6.2) as this "hot" limit is approached remains a potentially interesting subject for further investigation [23]. It is also of interest [24] to study the generalization of this simple model (going beyond the framework of the single current elastic string formalism to which the present analysis is confined) that would be needed to allow for the effect of thermal excitations in a cosmic string of superconducting type.

Although dimensional considerations suggest that at the epoch of the symmetry-breaking phase transition by which it is formed the string network might well be characterized by an effective temperature $\tilde{\Theta}$ of the order of the corresponding Higgs boson mass m and hence comparable with the critical limit value given by (6.12), the network would at first be strongly coupled to its environment by drag forces of the kind described in the previous section, which might be expected to damp it down to an effective temperature $\tilde{\Theta}$ not greatly in excess of the ambient cosmological temperature Θ. Such drag forces might also be expected at the same time to tend to smooth out most larger scale structure on scales large compared with the thermal wavelength, but small compared with the cosmological relevant horizon radius. Under such circumstances the short run dynamical behavior of the string might be expected to be well described on a macroscopic scale by "warm string model" with effective temperatures adequately small compared with the "hot" limit value (6.12), i.e.,

$$\Theta^2 \ll \Theta^2_{\max} . \tag{6.13}$$

Insofar as longer term behavior is concerned, deviations from the strictly elastic behavior characterizing the model presented above can be expected to arise as a consequence of "heat" losses. Such losses will be caused by the action on microscopic scales of the ambient drag mechanism that has just been referred to and also by back reaction from gravitational radiation whose effect will be discussed in the following section. However, over time scales short enough for such effects to be neglected, so that the use of the strictly elastic "warm string" model is justified, the internal evolution of the string will be characterized by a pair of conserved surface currents of the kind described in the previous section. In the particular case of the model specified by (6.2), the currents in question are specifiable as the mutually orthogonal pair

$$\mathfrak{s}^\mu = \mathfrak{s} u^\mu, \quad *\tilde{\Theta}^\mu = \tilde{\Theta} v^\mu . \tag{6.14}$$

The first of these currents, namely, $\mathfrak{s}^\mu$, is of a physically familiar type, being interpretable in the local thermal equilibrium case as that of the entropy of the microstructure. The physical nature of its conjugate partner, the other conserved current $*\tilde{\Theta}^\mu$, is less immediately obvious: It can be interpreted as the surface dual, as constructed by contraction with the surface alternating tensor (4.3), of the effective *thermal momentum-energy* covector

$$\tilde{\Theta}_\mu = \tilde{\Theta} u_\mu , \tag{6.15}$$

the implication of the relevant conservation law (which will remain valid only as long as microscopic thermal emission can be neglected) being that the latter should (in this conservative case) be expressible locally as the surface gradient of a certain scalar field ϑ, say, that may be considered as a "thermal winding angle" on the world sheet.

One of the purposes of the present paper is to show that it is useful to consider the effective entropy current $\mathfrak{s}^\mu$ in the "warm" string model as being constructed from a pair of "right"- and "left"-moving *characteristic current* contributions $\mathfrak{s}^\mu_+$ and $\mathfrak{s}^\mu_-$, whose difference is proportional to the dual current:

$$\mathfrak{s}^\mu = \mathfrak{s}^\mu_+ + \mathfrak{s}^\mu_- , \quad *\tilde{\Theta}^\mu = \frac{3\hbar}{2\pi}(\mathfrak{s}^\mu_+ - \mathfrak{s}^\mu_-) , \tag{6.16}$$

where

$$\mathfrak{s}^\mu_\pm = \frac{1}{2}\mathfrak{s}^\mu \pm \frac{\pi}{3\hbar} *\tilde{\Theta}^\mu . \tag{6.17}$$

These currents share a common magnitude:

$$\tilde{s}^{\mu}_{+}\tilde{s}_{+\mu}=\tilde{s}^{\mu}_{-}\tilde{s}_{-\mu}=-\frac{\tilde{s}^2}{4\gamma^2}\ , \tag{6.18}$$

where γ is the Lorentz factor associated with the (unique) characteristic velocity (6.10), which will be given by

$$\gamma^2=\frac{U}{U-T}\ . \tag{6.19}$$

In terms of these currents, the stress momentum-energy density tensor can be rewritten as

$$\tilde{T}^{\mu\nu}=\frac{6m^2}{\pi s\Theta}\tilde{s}^{(\mu}_{+}\tilde{s}^{\nu)}_{-}\ , \tag{6.20}$$

where the parentheses denote index symmetrization. The existence of such characteristic currents is a special feature distinguishing the "warm string model" from more general elastic string models (such as are appropriate for describing the "superconductivity" phenomenon). The justification for describing them as "characteristic" is that, as well as being separately conserved,

$$\eta^{\mu}{}_{\nu}\nabla_{\mu}\tilde{s}^{\nu}_{\pm}=0\ , \tag{6.21}$$

they have the property of being everywhere tangential to the corresponding "right"- and "left"-moving characteristic directions that move relative to the locally preferred rest frame specified by $\tilde{u}^{\mu}$ with the relative velocity given by (4.6). When the external force F_{μ} is absent, the corresponding unit characteristic vectors

$$\tilde{L}^{\mu}_{\pm}=2(\gamma/\tilde{s})\tilde{s}^{\mu}_{\pm} \tag{6.22}$$

have the very convenient property [7] of being parallelly propagated along each other; i.e., one gets

$$\tilde{L}^{\nu}_{\pm}\nabla_{\nu}\tilde{L}^{\mu}_{\mp}=0\ . \tag{6.23}$$

What makes the pair of independent current conservation laws so important in applications to closed string loops is their implication that, whatever its detailed behavior may be, each such loop will be characterized by a corresponding pair of independent contour integrals, which are expressible, using the abbreviation

$$dl_{\mu}=\mathcal{E}_{\rho\mu}dx^{\rho}\ , \tag{6.24}$$

as

$$[S]=\oint dS=\oint\tilde{s}^{\mu}dl_{\mu}\ , \tag{6.25}$$

which is evidently identifiable as the total effective entropy, and as the dually related quantity

$$[\vartheta]=\oint *\tilde{\Theta}^{\mu}dl_{\mu}=\oint\tilde{\Theta}u_{\mu}dx^{\mu}\ , \tag{6.26}$$

which may be described as the thermal action. Equivalently, from the characteristic current conservation laws (6.21), one obtains a corresponding pair of conserved characteristic charges,

$$[S_{\pm}]=\oint\tilde{s}^{\mu}_{\pm}dl_{\mu}\ , \tag{6.27}$$

in terms of which the conserved circuit integrals (6.25) and (6.26) will evidently be given by

$$[S]=[S_{+}]+[S_{-}],\quad [\vartheta]=\frac{3\hbar}{2\pi}([S_{+}]-[S_{-}])\ . \tag{6.28}$$

It is to be noted that $[S]$ will automatically be positive and that, whereas $[\vartheta]$ could in principle be negative, we can in practice eliminate this eventuality by a choice of the orientation in the direction of the loop integral. Subject to this convention, it follows from the useful alternative expression

$$c_E=\frac{2\pi\tilde{\Theta}}{3\hbar\tilde{s}} \tag{6.29}$$

(which is easily derivable using the formulas in Sec. IV) for the characteristic speed $c_E=c_L$, which must of course be less than unity, that the integral (6.27) will be restricted by inequalities of the form

$$[S_{+}]\geq[S_{-}]\geq 0\ . \tag{6.30}$$

We have presented a thermodynamical description of the warm cosmic string model, defining the notion of temperature and entropy for the string microstructure and the associated conserved integral quantities, which will now be used to study the equilibrium properties of a circular rotating loop and its slow secular evolution.

VII. WARM QUASISTATIONARY EQUILIBRIUM STATES

In a wide range of physical contexts the long term effect of resistive drag forces such as were described in Sec. II, as well as of other loss mechanisms such as radiation reaction, may typically be expected to be describable in terms of a "damping" scenario in which the system under consideration tends toward some ultimate state of *stationary equilibrium* in which the relevant loss mechanisms cease to operate.

Strictly speaking, of course, it is impossible in an expanding universe to attain *exact* stationarity, which means invariance with respect to the action of a timelike Killing vector k^{μ}, i.e., a timelike solution of the Killing equation $\nabla_{(\mu}k_{\nu)}=0$. However, although a timelike Killing vector in this strict sense does not exist, the Friedmann models do have a *conformal* Killing vector of the kind discussed in the Appendix, which, in the simple case characterized by (5.23), will be expressible in terms of the unit flow vector u^{μ} of the cosmological background by

$$k^{\mu}=\frac{\Theta_0}{\Theta}u^{\mu}\ . \tag{7.1}$$

Over short time scales, during which the cosmological temperature Θ does not vary too much from its value Θ_0 at a chosen initial time, the conformal adjustment term on the right of the conformal Killing equation (A5) satisfied by this vector will be unimportant, so that k^{μ} will act to a good approximation as if it were a true Killing vector in the strict sense.

In particular, subject to a normalization convention fixed by the choice of the initial time at which the fixed temperature Θ_0 is evaluated at the epoch under consideration, the total quasi mass energy of the string loop, as defined in terms of the corresponding surface quasi-mass-energy current (A8), using the notation (6.24), by

$$\mathcal{M}=\oint \tilde{\rho}^{\mu} dl_{\mu} , \tag{7.2}$$

will be interpretable as a mass of the ordinary kind provided the relevant length and time scales are cosmologically small. Because of the effects of cosmological drag and expansion, not to mention losses by gravitational radiation, this mass will not be exactly conserved, but it can typically be expected that it will diminish toward a minimum in an ultimate approximately stationary state that is attained when the contribution to the right-hand side of (A9) from the effective force $\breve{F}_{\mu}$ given by (A10) ceases to be significant on dynamic time scales.

A scenario of this familiar kind cannot, however, occur in the exceptional case of a degenerate Goto-Nambu string model, for which an isolated loop in an approximately flat neighborhood cannot settle down toward any such equilibrium state for the simple reason that no such state exists, the implication being that it will ultimately lose all its energy and simply disappear. If gravitational radiation is the only loss mechanism involved, it is generally believed, on grounds provided by several mutually concordant studies [25–27], that the survival time scale for such a loop will be given roughly, in terms of its overall length scale l, say, by

$$\tau_{\rm gr} \approx 10^{-2} \frac{\hbar l}{G m^2} . \tag{7.3}$$

For a "heavyweight" string with $Gm^2/\hbar \approx 10^{-6}$, this time scale will be about 10^4 times larger than the light crossing time scale that is roughly of the same order as the fundamental dynamical oscillation time scale of the loop, which means that gravitational radiation reaction will be negligible on time scales comparable with or even many times longer than the dynamical time scale, but it nevertheless prevents the survival of the loop over cosmologically long time scales, thereby avoiding the danger that a distribution of relic loops might constitute a cosmological mass excess, while if other mechanisms were also effective, the survival time scale would correspondingly be shorter still.

For a general elastic string model of the kind characterized by (2.11)–(2.13), the situation is entirely different because in the generic case nonvanishing stationary equilibrium states *do* exist. In the context of superconducting cosmic string theory, the existence of centrifugally supported equilibrium states, and the consequent danger of a cosmologically catastrophic mass excess, was first pointed out by Davis and Shellard [28–30]. (It is to be mentioned that it had been pointed out even earlier that electromagnetically supported equilibrium states could also exist in principle [31–34], but in view of the weakness of the electromagnetic coupling constant $e^2/\hbar \approx \frac{1}{137}$, the electromagnetic support mechanism is of practical significance only as a minor correction effect [16,17]). As a conceivable mechanism for avoiding the buildup of a cosmological mass excess, it was pointed out at the outset [28] that a strictly conservative string model might cease to provide an accurate description in the long run due to quantum tunneling processes whose efficiency is sensitive dependent on provisionally unknown coupling constants in the underlying quantum field theory. However, even in the absence of any effective quantum or other decay process, it would appear [29–31,35,36], that the cosmological mass buildup would remain within observationally acceptable bounds if the strings in question were of the lightweight variety characterized by $Gm^2/\hbar \approx 10^{-32}$ that might have been formed as a by-product of electroweak symmetry breaking [15].

However that may be, our purpose in the present work is to consider the analogous centrifugally supported equilibrium states that exist for closed loops of the nonsuperconducting "warm" string model described in Sec. VI. This application does not entail any danger of cosmological mass excess even for strings of the "heavyweight" variety characterized by $Gm^2/\hbar \approx 10^{-6}$ because for such strings there is no need to speculate about hypothetical quantum decay processes in view of the undoubted existence of an efficient classical loss mechanism: The validity of the description by the perfectly elastic "warm" string model can be confidently expected to be limited to time scales of roughly the order given by the formula (7.3) or even less, i.e., short by cosmological standards, though long compared with the relevant dynamical time scales, since (although further detailed checking still needs to be done) it seems clear that (7.3) provides a rough upper limit on the time scale characterizing energy loss by gravitational radiation from the (thermal or other) distribution of microscopic wiggle modes whose averaged effect is described by the model.

As in the analogous superconducting examples, the consideration that the string loop is characterized by a pair of conserved contour integrals, namely, $[S]$ and $[\vartheta]$ in the case under consideration, obviously implies that within the time scale during which losses remain sufficiently small for the perfectly elastic model to remain valid as an accurate description, the loop cannot disappear through gravitational radiation. However, whether the loop will have time to get close to the stationary equilibrium state minimizing its mass energy $\mathcal{M}$ for the given values of $[S]$ and $[\vartheta]$ is less obvious than in the analogous superconducting case because in the wiggly string case the same kinds of drag and radiation loss mechanisms are involved both in the macroscopic damping process and in the microscopic delay process. All that can be said pending further detailed investigation is that the observed tendency [37,38] for radiation efficiency to be enhanced by highly salient features such as cusps and kinks suggests that evolution will tend to proceed in the direction of increasing smoothness and that a recent study [39] of the effect of the drag force in the "wiggly" string case reinforces the impression that they will tend toward the smoothest conceivable states, which are of course circular equilibrium configurations.

Before restricting our attention to such circular configurations (whose superconducting analogues were referred to as "vortons" [28–30]) on space and time scales sufficiently small for the background to be treated as flat, it is to be remarked that these were the only kind that was considered in the earlier studies of stationary cosmic string states for generic elastic string models

[8,40]. However, more recent and comprehensive studies of stationary equilibrium [41,42] have made it clear that more general "deformed" equilibrium states are also possible. What characterizes *all* stationary equilibrium states of an isolated string loop in any such conservative mode is the *transcharacteristic running* condition [43] $v=c_E$, where v is the running velocity of the global rest frame with respect to the intrinsically preferred rest frame of the string and c_E is the extrinsic characteristic speed as given by (4.13). (In a generic model, it is also necessary that the equilibrium state should be *longitudinally uniform,* meaning that physical quantities such as the rest frame energy density U and the tension T be constant throughout, but in the very special case of the mode under consideration here this further simplifying condition is not an obligatory requirement for equilibrium, though it will of course hold automatically in the circularly symmetric case.)

Since we are now considering only a limited approximately flat spacetime neighborhood, we can take it that k^μ is an exact Killing vector with components given simply by $k^\mu \mapsto \{1,0,0,0\}$ in its own rest frame. The transcharacteristic running condition means that with respect to a local orthonormal frame aligned not only with the Killing vector, but also with the direction of the string, the timelike and spacelike stress energy-momentum eigenvectors $\tilde{u}^\mu$ and $\tilde{v}^\mu$ will have the form

$$\tilde{u}^\mu \mapsto \gamma\{1,v,0,0\}, \quad \tilde{v}^\mu \mapsto \gamma\{v,1,0,0\} , \tag{7.4}$$

where, using the transcharacteristic condition,

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \quad v = \left[\frac{T}{U}\right]^{1/2} , \tag{7.5}$$

It follows in the present case that the corresponding values of the conserved integrals $[S]$ and $[\vartheta]$ will be given in terms of the total circumferential length

$$l = \oint dl, \quad dl^2 = dl_\mu dl^\mu = \eta_{\mu\nu} dx^\mu dx^\nu , \tag{7.6}$$

of the loop (which if it is circular will be given in terms of the corresponding radius r by $l=2\pi r$) by formulas of the very simple form

$$[S] = \left[\frac{2\pi}{3}\right]^{1/2} \frac{ml}{\hbar v}, \quad [\vartheta] = \left[\frac{3}{2\pi}\right]^{1/2} mlv . \tag{7.7}$$

Thus, from a knowledge just of the total entropy $[S]$ and of the thermal action $[\vartheta]$, it will be possible to predict both the total length l of the corresponding equilibrium configuration, which will be given by the product of the conserved numbers

$$l^2 = \frac{\hbar[S][\vartheta]}{m^2} , \tag{7.8}$$

and also the local state, which will be determined by (7.5) from the running velocity v, which will itself be given by the ratio of the two conserved quantities according to the formula

$$v^2 = \frac{2\pi}{3\hbar}\frac{[\vartheta]}{[S]} , \tag{7.9}$$

which by (6.30) will always give a satisfactorily non tachyonic solution $v^2 \leq 1$. Using the formulas of Sec. VI, we can immediately go on to evaluate all the other relevant local quantities including the string tension

$$T = \frac{m^2 v}{\hbar} , \tag{7.10}$$

the mass energy density eigenvalue

$$U = \frac{m^2}{\hbar v} , \tag{7.11}$$

the entropy density scalar

$$\tilde{s} = \left[\frac{2\pi}{3}\right]^{1/2} \frac{m}{\hbar\gamma v} , \tag{7.12}$$

and finally the effective temperature

$$\tilde{\Theta} = \frac{\tilde{\Theta}_{\max}}{\gamma} . \tag{7.13}$$

The latter is of crucial importance since it is presumably what will mainly determine the radiative energy loss rate that will govern the time scale over which the strictly stationary description remains valid. The requirement that the "effective temperature" $\tilde{\Theta}$ should remain low can be seen to entail that the relevant Lorentz factor should be high. Subject to the sign convention adopted in (6.30), it can be seen that the Lorentz factor γ appearing here will be expressible rather simply in terms of the conserved entropy contributions by

$$\gamma^2 = \frac{[S]}{2[S_-]} . \tag{7.14}$$

It also turns out that the value of the minimized mass energy $\mathcal{M}$ that is obtained in this case depends *only* on the "right"-moving contribution $[S_+]$, being derivable [8,36,41] as

$$\mathcal{M} = l(U+T) = \left[\frac{6}{\pi}\right]^{1/2} m[S_+] . \tag{7.15}$$

It is to be remarked that in order for the picture presented above to be valid it is of course necessary that the absolute values of $[\vartheta]$ and $[S]$ should be sufficiently large for the circumference l as given by (7.6) to be large compared with the relevant length scale characterizing the underlying spectrum of microscopic wiggles; i.e., we must have

$$l^2 \gg \frac{\tilde{\lambda}^2}{\gamma^2} , \tag{7.16}$$

where $\tilde{\lambda}$ is an estimate of mean wavelength of the wiggles with respect to their own local reference system as specified by $\tilde{u}^\mu$, which will, of course, be longer by a factor γ than what is observed with respect to the global reference system in which the loop is perceived to be stationary. If the spectrum really is of exactly or approximately thermal type, the typical value of the intrinsic wavelength will be given in order of magnitude simply by

$$\tilde{\lambda} \approx \frac{\hbar}{\tilde{\Theta}} , \tag{7.17}$$

and it will then follow from (7.13) and (6.12) that although $\tilde{\lambda}$ itself will be much longer (as is obviously necessary for the thin string description to be valid), the corresponding blueshifted wavelength $\tilde{\lambda}/\gamma$ that will be observed in the stationary rest frame of the loop will reduce to the same order of magnitude as the estimated value [9] of the thickness of the vacuum vortex; i.e., one will have just

$$\frac{\lambda}{\gamma} \approx \frac{\hbar}{m} . \tag{7.18}$$

It can thus be seen that provided $[S]$ and $[\vartheta]/\hbar$ are each reasonably large compared with unity the requirement (7.16) will automatically be satisfied since by (7.8) and (7.18) it will obviously be equivalent to the simple condition

$$[S][\vartheta] \gg \hbar . \tag{7.19}$$

However, it is clear from (6.13) and (7.13) that there is another, more delicate, condition, applying not to the product of $[S]$ to $[\vartheta]$, but to their ratio, which must also be satisfied in order for the description we are using to avoid breaking down as a result of "overheating," namely, that the Lorentz factor, as given by (7.14), should be sufficiently large,

$$\gamma^2 \gg 1 , \tag{7.20}$$

which implies that the value of the total thermal winding angle $[\vartheta]$ must be sufficiently close to the upper limit allowed by (6.30), which is equivalent to the requirement that one of the characteristic contributions (the "right" one with the sign convention we have been using) be very much larger than the other; i.e., we must have

$$[S_+] \gg [S_-] . \tag{7.21}$$

This means that the kind of "warm" quasistationary equilibrium state we are considering can be obtained only for loop in which there is a highly asymmetric noise distribution, i.e., one in which there is a heavy relative preponderance of "right-moving" over "left-moving" wiggles or vice versa (which, it is to be remarked, is in no way incompatible with approximate or even exact thermal equilibrium in the appropriate, sufficiently boosted, local frame with respect to which the distribution would appear to be symmetric).

In the cases for which this condition is satisfied, so that the thermal loss rate is sufficiently low for approximately stationary equilibrium to be maintained in the short run, the next question to be addressed is what will happen after that. In other words, we will address the issue whether in the long run when even this very low thermal loss rate starts to have a significant effect, the residual temporarily minimized mass energy $\mathcal{M}$ of the loop, as given by (7.15), will further diminish as the parameters $[\vartheta]$ and $[S]$ continue to undergo a slow "secular" evolution.

VIII. SECULAR EVOLUTION OF CIRCULAR EQUILIBRIUM STATES

The formulas in the previous section were based only on the assumption that the equilibrium state under consideration be longitudinally uniform in the sense of having constant tension (a condition that would be rigorously necessary for equilibrium for a more general equation of state, but that is an optical simplifying postulate in the present case), but did not depend on any presupposition as to its extrinsic geometry. In order to address the question posed at the end of the previous section, we now adopt the further postulate that the configuration under consideration is circular, which means that it maximizes the angular momentum $\mathcal{J}$ for the given values of the parameters $[S]$ and $[\vartheta]$ by which it is characterized at any given instant, the magnitude of this angular momentum being then given [8,36] [using the sign convention adopted in (6.30)] by

$$\mathcal{J} = \frac{[\vartheta][S]}{2\pi} = \frac{2\pi m^2 r^2}{\hbar} , \tag{8.1}$$

where

$$r = \frac{l}{2\pi} \tag{8.2}$$

is the corresponding loop radius. It may be commented that the situation is analogous to that in simple black hole equilibrium theory, in that the state is determined by just two parameters which may be taken to be the angular momentum $\mathcal{J}$ and the total entropy $[S]$, with respect to which the formula for infinitesimal mass variations takes a form precisely analogous to that of the standard "first law of black hole thermodynamics" [44]: In terms of the appropriately redshifted effective temperature $\bar{\Theta}$ as measured with respect to the stationary background, which is given in relation to the local temperature $\tilde{\Theta}$, as measured in the corotating frame, by

$$\bar{\Theta} = \frac{\tilde{\Theta}}{\gamma} , \tag{8.3}$$

and of the relevant angular velocity Ω, which is given by

$$\Omega = \frac{v}{r} , \tag{8.4}$$

the variation law obtained from the formulas derived above can be seen to conform to the general law

$$d\mathcal{M} = \bar{\Theta} d[S] + \Omega \, d\mathcal{J} , \tag{8.5}$$

with explicit values of the coefficients given, in the present case, by

$$\bar{\Theta} = \left[\frac{6}{\pi} \right]^{1/2} \frac{m[S_-]}{[S]} \tag{8.6}$$

and

$$\Omega = \frac{(2\pi)^{3/2} m}{\sqrt{3}\hbar [S]} . \tag{8.7}$$

Subject to the supposition that the subsequent "secular" evolution of the loop is describable approximately in terms of a slow variation through a succession of such circular quasiequilibrium states, it is possible to draw some clear-cut quantitative conclusions from the following qualitative considerations.

Since according to (7.16) the wiggles that give rise to the radiative losses are supposed to have wavelengths whose blueshifted magnitude $\tilde{\lambda}/\gamma$ is small compared with the circumference l of the loop, it follows that their relative propagation speed will be close to the unique characteristic speed (6.1). Since the rotation speed v of the ring configuration is also known [8,40] to be the same as the characteristic speed, it follows that the net speed of the relatively backward "leftward" propagating wiggles will be zero; i.e., they will be effectively static, with vanishing angular velocity

$$\Omega_- = 0 \,, \tag{8.8}$$

which presumably implies that they will not contribute at all to the radiation losses. This means that the radiation will be entirely attributable to the wiggles that propagate in the opposite, "rightward," i.e., relatively forward, direction. These forward propagating modes will all have an angular velocity that is close to the uniquely defined value Ω_+, say, that would simply be twice the corotation angular velocity (8.4) if the ordinary nonrelativistic velocity addition law were applicable. Since by (7.20) the relevant Lorentz factor γ must be large, it is necessary to use the relativistic velocity addition law, from which one deduces that the angular velocity Ω_+ of the radiating wiggle modes will be given by the rather lower value

$$\Omega_+ = \frac{2v}{r(1+v^2)} \,. \tag{8.9}$$

The foregoing conclusion allows us to apply the well-known principle that radiation from any rigidly rotating pattern gives rise to losses of mass energy $\mathcal{M}$ and angular momentum $\mathcal{J}$ that will be related by the simple differential formula [45]

$$d\mathcal{M} = \Omega_+ d\mathcal{J} \,, \tag{8.10}$$

where Ω_+ is the angular velocity of the pattern, which in the present case is obtainable from (8.9) in a very convenient simple form as

$$\Omega_+ = \frac{4\pi m^2}{\hbar \mathcal{M}} \,. \tag{8.11}$$

It is apparent that this enables (8.10) to be integrated immediately to give a relation of the form

$$\mathcal{M}^2 - \frac{8\pi m^2}{\hbar}\mathcal{J} = \mathcal{C}^2 \,, \tag{8.12}$$

from which one obtains

$$\mathcal{M} = \mathcal{C}(2\gamma^2 - 1) \,, \tag{8.13}$$

where $\mathcal{C}^2$ is a positive constant of integration.

The relation (8.12) shows how, as gravitational radiation from the microstructure proceeds, the corresponding slow decrease in the mass energy $\mathcal{M}$ will be accompanied by a corresponding slow decrease in the angular momentum $\mathcal{J}$ and, hence, as can be seen from (8.1), by a corresponding quasistationary contraction of the ring radius. It follows from (7.13) that the relation between the effective temperature $\tilde{\Theta}$ and the mass energy $\mathcal{M}$ of the slowly contracting ring will have the form

$$\tilde{\Theta}^2 = \frac{2C\tilde{\Theta}^2_{\max}}{\mathcal{M} + \mathcal{C}} \,. \tag{8.14}$$

It can also be seen that the effective entropy parameter $[S]$ and the thermal action $[\vartheta]$ of the loop will decrease together in such a way as to preserve the value of the "left"-moving effective entropy contribution $[S_-]$, for which one obtains the relation

$$[S_-] = \left\{\frac{\pi}{6}\right\}^{1/2} \frac{\mathcal{C}}{m} \,. \tag{8.15}$$

The conclusion that the "left"-moving contribution should thus be left constant might have been guessed from the consideration that the "left"-moving wiggles do not radiate, but its derivation as a precise mathematical result provides a reassuring check on the coherence of the scenario as a whole.

It is apparent from (8.14) that the circular string loop has a negative specific heat property analogous to that of self-gravitating stars and black holes: As it continues its slow contraction, the ring will gradually heat up, according to an approximate power law obtainable from (8.14) and (8.15) in view of (6.13) as

$$\tilde{\Theta} \simeq \left\{\frac{2\mathcal{C}}{\mathcal{M}}\right\}^{1/2} \tilde{\Theta}_{\max} \,, \tag{8.16}$$

until the effective temperature $\tilde{\Theta}$ (as measured locally in the corotating frame) becomes comparable with $\tilde{\Theta}_{\max}$ at which stage more rapid evolution will again take place, so that within a comparatively short time thereafter there will be nothing left at all. When converted into terms of the redshifted effective temperature $\bar{\Theta}$, as defined with respect to the stationary background by (8.3) and given by (8.6), this relation takes the form

$$\bar{\Theta} \simeq \frac{6m^2}{\pi} \frac{[S_-]}{\mathcal{M}} \,. \tag{8.17}$$

It is to be remarked that this behavior is similar to what occurs in the well-known black hole case, in which the effective Hawking [46] radiation temperature $\bar{\Theta}$ observable at large distance also varies inversely as the mass, being given by $\bar{\Theta} \approx \hbar / G\mathcal{M}$.

IX. CONCLUSION

It is rather reasonable to believe that in addition to the large scale structure, a string network is also characterized by the survival of microscopic substructure on scales possibly as low as the correlation lengths of the phase transition during which strings were formed. This has been our motivation for the work presented above, which was basically to provide an analytic description of the microscopic extrinsic world sheet perturbations, wiggles, as well as to analyze the effects of such relatively microscopic string substructure.

At early times, cosmic string dynamics are affected by the drag force caused by the relative motion with respect to the ambient cosmological background medium. Considering cosmic strings in a conformally expanding black-body radiation background, we have shown that the effect of dissipative drag damping is mathematically strictly conservative, in the sense that it can be represented by a variational principle expressible in terms of an action integral. The effect of the drag force is to multiply the usual Goto-Nambu action by an appropriate cosmological temperature-dependent conformal factor.

For the macroscopic averaged description of the surviving microscopic wiggles, we have used the warm string model, based on the intuition that this extra microscopic structure will not cause any dispersion between distinct longitudinal and transverse modes. Moreover, as a consequence of the one-dimensional spatial geometry of the string, the effective equation of state is such that the product of the effective energy density with the corresponding effective tension remains constant, unaffected by deviations from true thermal equilibrium.

Neglecting effects of both the ambient drag force on the string microscopic scales, as well as gravitational radiation back reaction, the internal evolution of the string can be characterized by a pair of conserved surface currents. These currents represent the left- and right-moving distributions of microscopic wiggles according to the strictly elastic warm string model. In addition, the sum of the pair of left- and right-moving characteristic currents forms the conserved effective entropy current.

In the long run, the effect of large scale damping caused by resistive drag forces, radiation reaction, or any other loss mechanism leads to an approximately stationary equilibrium state, in which any loss mechanism ceases to operate. Within the warm string model, such a quasistationary equilibrium state can be obtained only for a closed loop with a heavy relative predominance of left-over right-moving wiggles or vice versa.

When the thermal energy loss by gravitational radiation to the environment can no longer be neglected, such a quasiequilibrium state will follow a secular evolution characterized, in the case of a circular string loop, by a negative specific heat, while as it continues its slow contraction, the ring will monotonically increase its effective temperature.

Finally, we think that the analysis described in this paper should be incorporated in the numerical simulations of cosmic string evolution. According to our understanding, we have the appropriate formalism in order to include the survival of microscopic substructure and its effects in the discrete lattice representation of cosmic strings. In addition, it is conceivable that such numerical simulations are easier to carry out in a flat background, since the effect of expansion in a Friedmann universe can be obtained by a force renormalization with respect to the conformally related static background. Along these lines we are planning to pursue our future work.

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APPENDIX: QUASIENERGY AND QUASIFORCE IN CONFORMALLY STATIONARY COSMOLOGY

The familiar concept of energy as a useful conserved quantity is dependent on having a background that is stationary, so that although it is applicable on a planetary or even galactic scale, it is not globally well defined on a cosmological scale in an arbitrarily expanding universe. There is, however, a generalized definition of what may be qualified as *quasienergy* that goes over to ordinary energy on a local scale, but that is globally meaningful on a cosmological scale, in the very extensive category of cosmological models that can be characterized as *conformally stationary.* This category, which includes the standard homogeneous isotropic Robertson-Walker cosmological models (and hence in particular the familiar Friedmann solutions [20] of the Einstein equations) as a special subcategory, is defined by the condition that the "true" spacetime background metric $g_{\mu\nu}$ should be obtainable by a conformal transformation from an *auxiliary* metric $\breve{g}_{\mu\nu}$ that is stationarity in the ordinary sense. This means that there is a timelike vector field k^μ that generates an ordinary symmetry of the auxiliary metric, i.e., which satisfies the corresponding Killing equation

$$\breve{\nabla}_{(\mu}\breve{k}_{\nu)}=0 \qquad \text{(A1)}$$

(using parentheses to denote index symmetrization), where the breve symbol is used to indicate that the Riemannian covariant differentiation operation and the index lowering operation that are involved have been carried out not with respect to the "true" metric, but with respect to the auxiliary metric, so that in particular we have $\breve{k}_\mu=\breve{g}_{\mu\nu}k^\nu$. Using the symbol a to denote the conformal amplification factor, the "true" spacetime metric will be given by

$$g_{\mu\nu}=a^2\breve{g}_{\mu\nu} \ . \qquad \text{(A2)}$$

As far as the definition of conformal stationarity is concerned, a may be an arbitrarily variable function of time and space, so that even if the auxiliary metric were homogeneous (or even flat) it would not follow that the "true" metric would have to be homogeneous.

The conformal stationarity conditions (A1) and (A2) do not fix the amplification factor a uniquely, since it can be renormalized by an arbitrary constant factor that can be absorbed in a complementary renormalization of the auxiliary metric $\breve{g}_{\mu\nu}$, whose strict stationarity property (and other symmetries if any) will be unaffected thereby. Such a renormalization also has no effect on the logarithmic derivative of the amplification factor, which we shall refer to as the *Hubble covector,* namely,

$$H_\mu=\nabla_\mu(\ln a)=a^{-1}\nabla_\mu a \ . \qquad \text{(A3)}$$

The magnitude H of this covector, as given, assuming it is timelike, by

$$H^2=-H_\mu H^\mu \ , \qquad \text{(A4)}$$

is a generalization of the Hubble expansion rate that is familiar in the particular case of the Friedmann models.

With respect to the "true" metric $g_{\mu\nu}$, it can be seen that the vector k^μ will satisfy not a strict Killing equation, but a *conformal Killing equation,* which will be expressible in terms of the Hubble covector as

$$\nabla_{(\mu}k_{\nu)}=g_{\mu\nu}k^\rho H_\rho \ . \tag{A5}$$

The existence of a vector field k^μ satisfying an equation of this form can be considered as a definition of local conformal stationarity, since it is the Yano condition [47] that is sufficient for the local existence of an amplification factor a satisfying (A2) for a metric $\breve{g}_{\mu\nu}$ with respect to which (A1) holds.

It is to be noted that the conformal Killing condition (A5) is not by itself sufficient to determine the covector H_μ uniquely. If, as well as satisfying (A5), the vector k^μ also satisfies the hypersurface orthogonality condition

$$k_{[\mu}\nabla_\nu k_{\rho]}=0$$

(where square brackets denote antisymmetrization), the metric will be describable as conformally *static.* The conformal staticity condition is necessary, though not sufficient, for it to be possible to resolve the indeterminacy in the Hubble covector by taking it to be *aligned* with the conformal Killing vector in the sense that

$$H_{[\mu}k_{\nu]}=0 \ ,$$

in which case the metric may be described as conformally static in the *strong* sense. This special property of strong conformal staticity is exemplified by the Robertson-Walker models and hence in particular by the Friedmann solutions.

Not only in the weakly or strongly conformally static case, but quite generally for any locally conformally stationary space time, we can use the conformal Killing vector k^μ as if it were an ordinary Killing vector to define the corresponding conformal *quasienergy current* ρ^μ, say, associated with a stress momentum-energy density tensor $T^{\mu\nu}$ as

$$\rho^\mu=-T^\mu{}_\nu k^\nu \ . \tag{A6}$$

As in the case of an ordinary energy, defined in the same way with respect to a Killing vector in the strict sense, it will be possible to adjust the normalization of the quasienergy by an arbitrary constant multiplicative factor since the conformal Killing equation, like the ordinary Killing equation, will be preserved when the vector in question is multiplied by a constant factor. In a space-time that is asymptotically flat, this ambiguity can be resolved by taking k^μ to have unit normalization in the asymptotic limit, but in the generic case the choice of normalization will be arbitrary, both for ordinary energy and for its conformal generalization as introduced here.

The reason why ordinary energy is such a useful quantity to work with is that it is conserved whenever the relevant stress momentum-energy tensor satisfies the corresponding pseudoconservation law. In a similar way, though not quite to the same extent, conformal pseudoenergy will also be a useful quantity since it will also be conserved in suitable, though more restricted circumstances. To be specific, for a medium that is decoupled from all external influence other than gravity, so that a covariant pseudoconservation law of the form (3.8) is satisfied, i.e., $\nabla_\mu T^\mu{}_\nu=0$, it follows from (A5) that the corresponding conformal energy current as defined by (A6) will satisfy

$$\nabla_\mu \rho^\mu=-T^\nu{}_\nu k^\mu H_\mu \ . \tag{A7}$$

The term on the right-hand side here is interpretable as representing the residual effect of the inflation of the scale on which the quasienergy is measured after allowance for the work done pushing the cosmological expansion. Although it will not be conserved in general, it can be seen that the quasienergy *will* be conserved if the stress momentum-energy tensor is trace free, as will be the case for a simple ultrarelativistic radiation gas with equation of state given by (3.10), for which (A7) reduces simply to the form $\nabla_\mu\rho^\mu=0$.

In the case of a point particle or string, the energy current corresponding to that given in the continuum case by (A6) will be given in terms of the corresponding world sheet restricted tensor $\tilde{T}^\mu{}_\nu$ by

$$\tilde{\rho}^\mu=-\tilde{T}^\mu{}_\nu k^\nu \ . \tag{A8}$$

It can be seen that if the particle or string evolves in accordance with a force law of the standard form (3.1), then this world sheet energy current will satisfy a corresponding world sheet divergence relation of the form

$$\tilde{\nabla}_\mu\tilde{\rho}^\mu=-k^\mu\breve{F}_\mu \ , \tag{A9}$$

where what we shall refer to as the *quasiforce* covector $\breve{F}_\mu$ is given in terms of the "true force" covector F_μ, as introduced in (3.1), by the basic defining relation

$$\breve{F}_\mu=F_\mu+\tilde{T}^\nu{}_\nu H_\mu \ . \tag{A10}$$

This formalism will be very useful in the following by enabling us to study all the quantities relevant to string dynamics in Sec. V in a static and even flat background metric.

[1] D. Bennett and F. Bouchet, Phys. Rev. Lett. **60**, 257 (1988).
[2] A. Albrecht, in *Formation and Evolution of Cosmic Strings,* edited by G. Gibbons, S. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990), p. 403.
[3] B. Allen and E. P. S. Shellard, Phys. Rev. Lett. **64**, 119 (1990).
[4] M. Sakellariadou and A. Vilenkin, Phys. Rev. D **42**, 349 (1990).
[5] T. Vachaspati and A. Vilenkin, Phys. Rev. D **30**, 2036 (1984).
[6] D. Austin, E. J. Copeland, and T. W. B. Kibble, Phys. Rev. D **48**, 5594 (1993).
[7] B. Carter, Phys. Rev. D **41**, 3869 (1990).
[8] B. Carter, in *Formation and Evolution of Cosmic Strings* [2], p. 143.
[9] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).

[10] B. Carter, Phys. Lett. B **224**, 61 (1989).
[11] B. Carter, Phys. Lett. B **228**, 466 (1989).
[12] E. Witten, Nucl. Phys. **B249**, 557 (1985).
[13] T. W. B. Kibble, in *Formation and Evolution of Cosmic Strings* [2], p. 3.
[14] A. Vilenkin, Phys. Rev. D **43**, 1060 (1991).
[15] P. Peter, Phys. Rev. D **46**, 3322 (1992).
[16] P. Peter, Phys. Rev. D **46**, 3335 (1992).
[17] P. Peter, Phys. Lett. B **298**, 60 (1993).
[18] B. Carter, J. Class. Quantum Grav. **9**, 19 (1992).
[19] B. Carter, J. Geom. Phys. **8**, 53 (1992).
[20] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 472.
[21] A. Vilenkin, Phys. Rev. D **41**, 3038 (1990).
[22] E. P. S. Shellard and B. Allen, in *Formation and Evolution of Cosmic Strings* [2], p. 421.
[23] J. Hong, J. Kim, and P. Sikivie, Phys. Rev. Lett. **69**, 2611 (1992).
[24] B. Carter, Nucl. Phys. **B412**, 345 (1994).
[25] N. Turok, Nucl. Phys. **B242**, 520 (1984).
[26] T. Vachaspati and A. Vilenkin, Phys. Rev. D **31**, 3035 (1985).
[27] R. Durrer, Nucl. Phys. **B328**, 238 (1989).
[28] R. L. Davis and E. P. S. Shellard, Phys. Lett. B **209**, 485 (1988).
[29] R. L. Davis, Phys. Rev. D **38**, 3722 (1988).
[30] R. L. Davis and E. P. S. Shellard, Nucl. Phys. **B323**, 209 (1989).
[31] J. Ostriker, C. Thompson, and E. Witten, Phys. Lett. B **180**, 231 (1986).
[32] E. Copeland, M. Hindmarsh, and N. Turok, Phys. Rev. Lett. **56**, 910 (1987).
[33] D. Haws, M. Hindmarsh, and N. Turok, Phys. Lett. B **209**, 225 (1988).
[34] E. Copeland, D. Haws, M. Hindmarsh, and N. Turok, Nucl. Phys. **B306**, 908 (1988).
[35] B. Carter, in *Early Universe and Cosmic Structures,* Proceedings of the Xth Moriond Astrophysics Meeting, Les Arcs, France, 1990, edited by A. Blanchard and J. Tran Thanh Van (Editions Frontières, Gif-sur-Yvette, 1991), p. 213.
[36] B. Carter, Ann. N.Y. Acad. Sci. **647**, 758 (1991).
[37] M. Sakellariadou, Phys. Rev. D **42**, 354 (1990).
[38] F. Bouchet, in *Formation and Evolution of Cosmic Strings* [2], p. 360.
[39] J. Garriga and M. Sakellariadou, Phys. Rev. D **48**, 2502 (1993).
[40] B. Carter, Phys. Lett. B **238**, 166 (1990).
[41] B. Carter, V. P. Frolov, and O. Heinrich, Class. Quantum Grav. **8**, 135 (1991).
[42] B. Carter, *Relativistic Astrophysics and Cosmology,* Proceedings of the 10th Potsdam Seminar, 1991, edited by S. Gottlöber, J. P. Mücket, and V. Müller (World Scientific, Singapore, 1992), p. 300.
[43] B. Carter and X. Martin, Ann. Phys. (N.Y.) **227**, 151 (1993).
[44] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 181 (1983).
[45] B. Carter, in *General Relativity: an Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), p. 294.
[46] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[47] L. Defrise-Carter, Commun. Math. Phys. **40**, 273 (1975).