Differential geometry with SageMath

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- In 1969, during his PhD under Pirani supervision, Ray d'Inverno wrote `ALAM (Atlas Lisp Algebraic Manipulator)` and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, *Applications of SHEEP* (1994)]

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- Since then, many software tools for tensor calculus have been developed... A rather exhaustive list: http://www.xact.es/links.html

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The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Some advantages of SageMath

SageMath is free

Freedom means

1. everybody can use it, by downloading the software from `http://sagemath.org`
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- easy access for students
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SageMath is developing and spreading fast

...sustained by an enthusiastic community of developers

Object-oriented notation in Python

As an object-oriented language, Python (and hence SageMath) makes use of the following **postfix notation** (same in C++, Java, etc.):

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result = object.function(arguments)
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In a procedural language, this would be written as

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Examples

1. `riem = g.riemann()`
2. `lie_t_v = t.lie_der(v)`

NB: no argument in example 1

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The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement smooth manifolds of arbitrary dimension in SageMath and tensor calculus on them

In particular:

- one should be able to introduce an arbitrary number of coordinate charts on a given manifold, with the relevant transition maps
- tensor fields must be manipulated as such and not through their components with respect to a specific (possibly coordinate) vector frame

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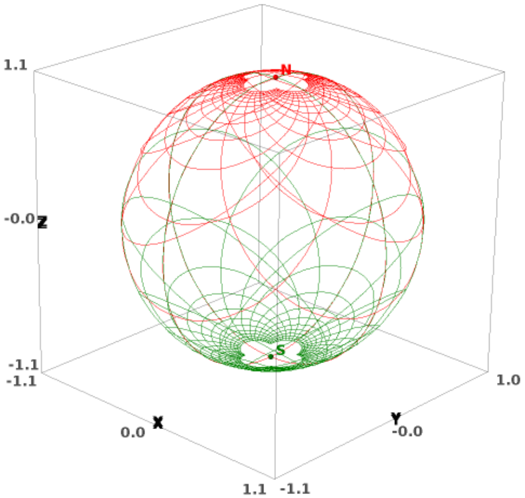
Concretely, the project amounts to creating new Python classes, such as `TopologicalManifold`, `DifferentiableManifold`, `Chart`, `TensorField` or `Metric`, within SageMath's **Parent/Element framework**.

The 2-sphere example



Stereographic coordinates on the 2-sphere

Two charts:

- X_1: $\mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- X_2: $\mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

← picture obtained via function `RealChart.plot()`

See the worksheet at http://sagemanifolds.obspm.fr/examples.html

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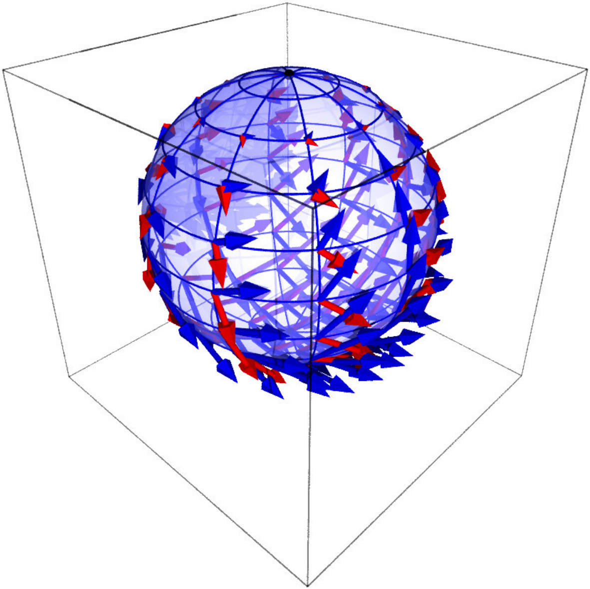

Vector frame associated with the stereographic coordinates (x, y) from the North pole

- $\frac{\partial}{\partial x}$
- $\frac{\partial}{\partial y}$

← picture obtained via the function `VectorField.plot()`

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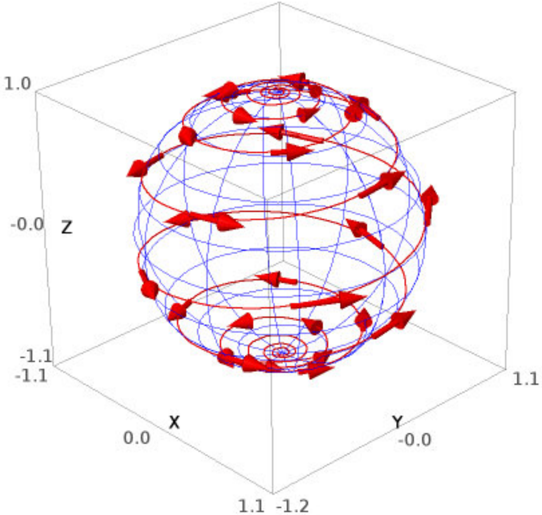
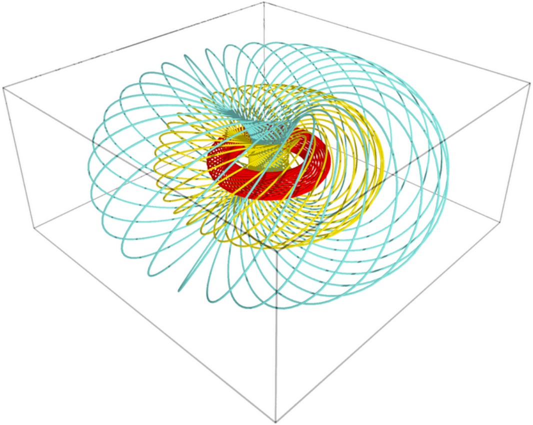


A curve in $\mathbb{S}^2$: a loxodrome and its tangent vector field

← picture obtained via the functions `DifferentiableCurve.plot()` and `VectorField.plot()`

See the worksheet at http://sagemanifolds.obspm.fr/examples.html

The 3-sphere example



Some fibers of the **Hopf fibration** of $\mathbb{S}^3$ viewed in stereographic coordinates

← picture obtained via the function `DifferentiableCurve.plot()`

See the worksheet at http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.0/SM_sphere_S3_Hopf.ipynb

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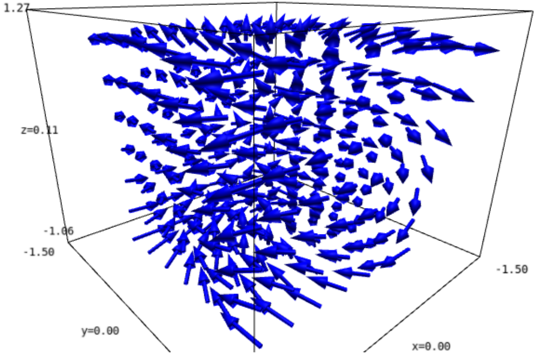


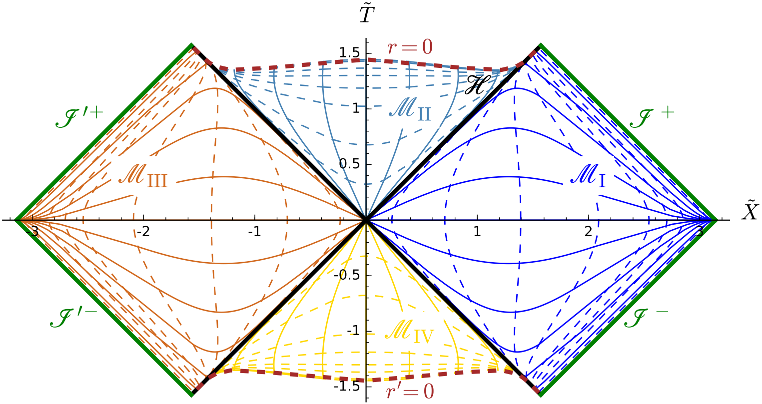
One of the vector fields of a left-invariant global vector frame of $\mathbb{S}^3$, viewed in stereographic coordinates

← picture obtained via the function `VectorField.plot()`

See the worksheet at http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.0/SM_sphere_S3_vectors.ipynb

Charts on Schwarzschild spacetime

The Carter-Penrose diagram



Two charts of standard Schwarzschild-Droste coordinates (t, r, θ, φ) plotted in terms of Frolov-Novikov compactified coordinates $(\tilde{T}, \tilde{X}, \theta, \varphi)$; see the worksheet at
http://luth.obspm.fr/~luthier/gourgoulhon/bh16/sage.html

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Summary

SageManifolds: extends the modern computer algebra system SageMath towards differential geometry and tensor calculus

- http://sagemanifolds.obspm.fr/
- free software (GPL), as SageMath
- $\sim$ 65,000 lines of Python code (including comments and doctests)
- submitted to SageMath community as a sequence of 14 tickets
  $\rightarrow$ first ticket accepted in March 2015,
  the 14th one in Nov. 2016
- 5 developers, 3 reviewers

SageManifolds 1.0 released on 11 Jan. 2017 and fully included in SageMath 7.5

SageManifolds 1.0.1 released on 25 March 2017 and fully incl. in SageMath 7.6

Current status

Already present (v1.0):

- topological manifolds: charts, open subsets, maps between manifolds, scalar fields
- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- taking into account any monoterm tensor symmetry
- exterior calculus (wedge product, exterior derivative, Hodge duality)
- Lie derivatives of tensor fields
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations, via the Python library `multiprocessing`

Current status

Future prospects:

- extrinsic geometry of pseudo-Riemannian submanifolds
- computation of geodesics (numerical integration via SageMath/GSL or `Gyoto`)
- integrals on submanifolds
- more graphical outputs
- more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
- connection with numerical relativity: using SageMath to explore numerically-generated spacetimes

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Want to join the project or simply to stay tuned?

visit http://sagemanifolds.obspm.fr/
(download, documentation, example worksheets, mailing list)