An introduction to polynomial interpolation

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Outline

1. Introduction
2. Interpolation on an arbitrary grid
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Basic idea: approximate functions $\mathbb{R} \rightarrow \mathbb{R}$ by polynomials

Polynomials are the only functions that a computer can evaluate exactly.

Two types of numerical methods based on polynomial approximations:

- **spectral methods**: high order polynomials on a single domain (or a few domains)
- **finite elements**: low order polynomials on many domains
**Basic idea:** approximate functions $\mathbb{R} \to \mathbb{R}$ by polynomials

Polynomials are the only functions that a computer can evaluate exactly.

Two types of numerical methods based on polynomial approximations:

- **spectral methods:** high order polynomials on a single domain (or a few domains)
- **finite elements:** low order polynomials on many domains
We consider real-valued functions on the compact interval $[-1,1]$: 

$$f : [-1,1] \rightarrow \mathbb{R}$$

We denote

- by $\mathbb{P}$ the set all real-valued polynomials on $[-1,1]$:

$$\forall p \in \mathbb{P}, \forall x \in [-1,1], \ p(x) = \sum_{i=0}^{n} a_i \ x^i$$

- by $\mathbb{P}_N$ (where $N$ is a positive integer), the subset of polynomials of degree at most $N$. 
Is it a good idea to approximate functions by polynomials?

For continuous functions, the answer is yes:

**Theorem (Weierstrass, 1885)**

\[ \mathbb{P} \text{ is a dense subspace of the space } C^0([−1, 1]) \text{ of all continuous functions on } [−1, 1], \text{ equiped with the uniform norm } \| \cdot \|_\infty. \]

\[ ^a \text{This is a particular case of the Stone-Weierstrass theorem} \]

The uniform norm or maximum norm is defined by \[ \| f \|_\infty = \max_{x \in [−1, 1]} |f(x)| \]

Other phrasings:

For any continuous function on \([-1, 1], f\), and any \(\epsilon > 0\), there exists a polynomial \(p \in \mathbb{P}\) such that \(\| f - p \|_\infty < \epsilon\).

For any continuous function on \([-1, 1], f\), there exists a sequence of polynomials \((p_n)_{n \in \mathbb{N}}\) which converges uniformly towards \(f\): \[ \lim_{n \to \infty} \| f - p_n \|_\infty = 0. \]
Best approximation polynomial

For a given continuous function: \( f \in C^0([-1, 1]) \), a best approximation polynomial of degree \( N \) is a polynomial \( p_N^*(f) \in \mathbb{P}_N \) such that

\[
\|f - p_N^*(f)\|_\infty = \min \{\|f - p\|_\infty, \ p \in \mathbb{P}_N\}
\]

Chebyshev's alternant theorem (or equioscillation theorem)

For any \( f \in C^0([-1, 1]) \) and \( N \geq 0 \), the best approximation polynomial \( p_N^*(f) \) exists and is unique. Moreover, there exists \( N + 2 \) points \( x_0, x_1, \ldots, x_{N+1} \) in \([-1,1]\) such that

\[
f(x_i) - p_N^*(f)(x_i) = (-1)^i \|f - p_N^*(f)\|_\infty, \quad 0 \leq i \leq N + 1
\]

or

\[
f(x_i) - p_N^*(f)(x_i) = (-1)^{i+1} \|f - p_N^*(f)\|_\infty, \quad 0 \leq i \leq N + 1
\]

Corollary: \( p_N^*(f) \) interpolates \( f \) in \( N + 1 \) points.
Illustration of Chebyshev’s alternant theorem

\[ N = 1 \]

\[ \| f - p_1^*(f) \|_\infty = \int_0^1 | f(x) - p_1^*(f)(x) | \, dx \]
Illustration of Chebyshev’s alternant theorem

\[ N = 1 \]

\[ \| f - p_1^*(f) \|_\infty = \| f - p_1^*(f) \|_\infty \]
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Interpolation on an arbitrary grid

Definition: given an integer $N \geq 1$, a grid is a set of $N + 1$ points $X = (x_i)_{0 \leq i \leq N}$ in $[-1,1]$ such that $-1 \leq x_0 < x_1 < \cdots < x_N \leq 1$. The $N + 1$ points $(x_i)_{0 \leq i \leq N}$ are called the nodes of the grid.

Theorem

Given a function $f \in C^0([-1,1])$ and a grid of $N + 1$ nodes, $X = (x_i)_{0 \leq i \leq N}$, there exist a unique polynomial of degree $N$, $I_N^X f$, such that

$$I_N^X f(x_i) = f(x_i), \quad 0 \leq i \leq N$$

$I_N^X f$ is called the interpolant (or the interpolating polynomial) of $f$ through the grid $X$. 
The interpolant $I_N^X f$ can be expressed in the \textit{Lagrange form}:

$$I_N^X f(x) = \sum_{i=0}^{N} f(x_i) \ell_i^X(x),$$

where $\ell_i^X(x)$ is the $i$-th \textbf{Lagrange cardinal polynomial} associated with the grid $X$:

$$\ell_i^X(x) := \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}, \quad 0 \leq i \leq N$$

The Lagrange cardinal polynomials are such that

$$\ell_i^X(x_j) = \delta_{ij}, \quad 0 \leq i, j \leq N$$
Interpolation on an arbitrary grid

Examples of Lagrange polynomials

Uniform grid $N = 8$  \( \ell_0^X(x) \)

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8$ \( \ell^X_1(x) \)

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8 \quad \ell_2^X(x)$
Examples of Lagrange polynomials

Uniform grid $N = 8$ \quad $\ell_3^X(x)$

Lagrange polynomials

![Graph of Lagrange polynomials

$\ell_3^X(x)$]
Examples of Lagrange polynomials

Uniform grid \( N = 8 \) \( \ell_4^X(x) \)

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid $N = 8$ \quad $\ell_5^X(x)$
Examples of Lagrange polynomials

Uniform grid $N = 8$ \quad $\ell_6^X(x)$
Examples of Lagrange polynomials

Uniform grid $N = 8$ \[ \ell_7^X(x) \]

Lagrange polynomials
Examples of Lagrange polynomials

Uniform grid \( N = 8 \) \( \ell_8^X(x) \)
Examples of Lagrange polynomials

Uniform grid $N = 8$

Lagrange polynomials
Let \( N \in \mathbb{N} \), \( X = (x_i)_{0 \leq i \leq N} \) a grid of \( N + 1 \) nodes and \( f \in C^0([−1, 1]) \).

Let us consider the interpolant \( I_X^N f \) of \( f \) through the grid \( X \).

The best approximation polynomial \( p_N^*(f) \) is also an interpolant of \( f \) at \( N + 1 \) nodes (in general different from \( X \)).

How does the error \( \| f - I_X^N f \|_\infty \) behave with respect to the smallest possible error \( \| f - p_N^*(f) \|_\infty \) ?

The answer is given by the formula:

\[
\| f - I_X^N f \|_\infty \leq (1 + \Lambda_N(X)) \| f - p_N^*(f) \|_\infty
\]

where \( \Lambda_N(X) \) is the **Lebesgue constant** relative to the grid \( X \):

\[
\Lambda_N(X) := \max_{x \in [-1,1]} \sum_{i=0}^{N} \left| \ell_i^X(x) \right|
\]
Interpolation on an arbitrary grid

Lebesgue constant

The Lebesgue constant contains all the information on the effects of the choice of \(X\) on \(\|f - I_N^X f\|_\infty\).

**Theorem (Erdős, 1961)**

For any choice of the grid \(X\), there exists a constant \(C > 0\) such that

\[
\Lambda_N(X) > \frac{2}{\pi} \ln(N + 1) - C
\]

Corollary: \(\Lambda_N(X) \to \infty\) as \(N \to \infty\)

In particular, for a uniform grid, \(\Lambda_N(X) \sim \frac{2^{N+1}}{eN \ln N}\) as \(N \to \infty\)!

This means that for any choice of type of sampling of \([-1, 1]\), there exists a continuous function \(f \in C^0([-1, 1])\) such that \(I_N^X f\) does not convergence uniformly towards \(f\).
Example: uniform interpolation of a “gentle” function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 4 \):

\[ \| f - I_4^X f \|_\infty \simeq 1.40 \]
Interpolation on an arbitrary grid

Example: uniform interpolation of a “gentle” function \( f(x) = \cos(2 \exp(x)) \) uniform grid \( N = 6 \): \( \| f - I_6^X f \|_\infty \simeq 1.05 \)

Interpolation of \( \cos(2 \exp(x)) \)
Example: uniform interpolation of a “gentle” function

\[ f(x) = \cos(2 \exp(x)) \text{ uniform grid } N = 8 : \| f - I_8^x f \|_\infty \simeq 0.13 \]
Example: uniform interpolation of a "gentle" function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 12 \) : \( \| f - I_{12}^X f \|_\infty \approx 0.13 \)
Example: uniform interpolation of a “gentle” function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 16 : \| f - I_{16}^X f \|_\infty \approx 0.025 \]
Example: uniform interpolation of a “gentle” function

\[ f(x) = \cos(2 \exp(x)) \] uniform grid \( N = 24 \) : \( \| f - I_{24}^X f \|_\infty \approx 4.6 \times 10^{-4} \)
$f(x) = \frac{1}{1 + 16x^2}$  

uniform grid $N = 4$ : $\| f - I^X_4 f \|_\infty \simeq 0.39$
$f(x) = \frac{1}{1 + 16x^2}$  

uniform grid $N = 6$ : $\| f - I_X^6 f \|_\infty \simeq 0.49$
Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{uniform grid } N = 8 : \| f - I_8^X f \|_\infty \simeq 0.73 \]
Interpolation on an arbitrary grid

Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{uniform grid } N = 12 : \|f - I_{12}^X f\|_{\infty} \approx 1.97 \]
Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{uniform grid } N = 16 : \|f - I_{16}^X f\|_\infty \simeq 5.9 \]
Interpolation on an arbitrary grid

Runge phenomenon

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{uniform grid } N = 24 : \| f - I_{24}^x f \|_\infty \simeq 62 \]
Evaluation of the interpolation error

Let us assume that the function $f$ is sufficiently smooth to have derivatives at least up to the order $N + 1$, with $f^{(N+1)}$ continuous, i.e. $f \in C^{N+1}([-1, 1])$.

Theorem (Cauchy)

If $f \in C^{N+1}([-1, 1])$, then for any grid $X$ of $N + 1$ nodes, and for any $x \in [-1, 1]$, the interpolation error at $x$ is

$$ f(x) - I_X^N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \omega_{N+1}^X(x) $$

where $\xi = \xi(x) \in [-1, 1]$ and $\omega_{N+1}^X(x)$ is the nodal polynomial associated with the grid $X$.

Definition: The nodal polynomial associated with the grid $X$ is the unique polynomial of degree $N + 1$ and leading coefficient 1 whose zeros are the $N + 1$ nodes of $X$:

$$ \omega_{N+1}^X(x) := \prod_{i=0}^{N} (x - x_i) $$
Example of nodal polynomial

Uniform grid \( N = 8 \)
In Eq. (1), we have no control on $f^{(N+1)}$, which can be large.
For example, for $f(x) = 1/(1 + \alpha^2 x^2)$, $\|f^{(N+1)}\|_{\infty} = (N + 1)! \alpha^{N+1}$.

Idea: choose the grid $X$ so that $\omega_{N+1}^X(x)$ is small, i.e. $\|\omega_{N+1}^X\|_{\infty}$ is small.

Notice: $\omega_{N+1}^X(x)$ has leading coefficient 1: $\omega_{N+1}^X(x) = x^{N+1} + \sum_{i=0}^{N} a_i x^i$.

**Theorem (Chebyshev)**

*Among all the polynomials of degree $N + 1$ and leading coefficient 1, the unique polynomial which has the smallest uniform norm on $[-1, 1]$ is the $(N + 1)$-th Chebyshev polynomial divided by $2^N$: $T_{N+1}(x)/2^N$.*

Since $\|T_{N+1}\|_{\infty} = 1$, we conclude that if we choose the grid nodes $(x_i)_{0 \leq i \leq N}$ to be the $N + 1$ zeros of the Chebyshev polynomial $T_{N+1}$, we have

$$\|\omega_{N+1}^X\|_{\infty} = \frac{1}{2^N}$$

and this is the smallest possible value.
The grid $X = (x_i)_{0 \leq i \leq N}$ such that the $x_i$'s are the $N + 1$ zeros of the Chebyshev polynomial of degree $N + 1$ is called the Chebyshev-Gauss (CG) grid. It has much better interpolation properties than the uniform grid considered so far. In particular, from Eq. (1), for any function $f \in C^{N+1}([-1, 1])$,

$$
\| f - I_{CG}f \|_\infty \leq \frac{1}{2^N(N+1)!} \| f^{(N+1)} \|_\infty
$$

If $f^{(N+1)}$ is uniformly bounded, the convergence of the interpolant $I_{CG}^nf$ towards $f$ when $N \to \infty$ is then extremely fast. Also the Lebesgue constant associated with the Chebyshev-Gauss grid is small:

$$
\Lambda_N(CG) \sim \frac{2}{\pi} \ln(N+1) \quad \text{as} \quad N \to \infty
$$

This is much better than uniform grids and close to the optimal value.
Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$\| f - I_4^{CG} f \|_\infty \approx 0.31$
Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1 + 16x^2} \)

\[ f(x) = \frac{1}{1 + 16x^2} \quad \text{CG grid } N = 6 : \| f - I_{6}^{CG} f \|_{\infty} \approx 0.18 \]
Interpolation on an arbitrary grid

Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1 + 16x^2} \)

\( f(x) = \frac{1}{1 + 16x^2} \) \quad \text{CG grid } N = 8 : \| f - I^\text{CG}_8 f \|_\infty \approx 0.10
Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1 + 16x^2} \)

CG grid \( N = 12 \): \( \| f - I_{12}^{CG} f \|_\infty \simeq 3.8 \times 10^{-2} \)
Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1+16x^2} \)

CG grid \( N = 16 \) : \( \| f - I_{16}^{CG} f \|_\infty \simeq 1.5 \times 10^{-2} \)
Interpolation on an arbitrary grid

Example: Chebyshev-Gauss interpolation of \( f(x) = \frac{1}{1+16x^2} \)

\[ f(x) = \frac{1}{1+16x^2} \]

CG grid \( N = 24 \): \( \| f - I_{24}^{CG} f \|_\infty \approx 2.0 \times 10^{-3} \)

no Runge phenomenon!
Interpolation on an arbitrary grid

Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

Variation of the interpolation error as $N$ increases
The Chebyshev polynomials, the zeros of which provide the Chebyshev-Gauss nodes, constitute a family of orthogonal polynomials, and the Chebyshev-Gauss nodes are associated to Gauss quadratures.
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Hilbert space $L^2_w(-1, 1)$

Framework: Let us consider the functional space

$$L^2_w(-1, 1) = \left\{ f : (-1, 1) \to \mathbb{R}, \quad \int_{-1}^{1} f(x)^2 w(x) \, dx < \infty \right\}$$

where $w : (-1, 1) \to (0, \infty)$ is an integrable function, called the weight function.

$L^2_w(-1, 1)$ is a Hilbert space for the scalar product

$$(f|g)_w := \int_{-1}^{1} f(x) g(x) w(x) \, dx$$

with the associated norm

$$\|f\|_w := (f|f)_w^{1/2}$$
Orthogonal polynomials

The set $\mathbb{P}$ of polynomials on $[-1, 1]$ is a subspace of $L^2_w(-1, 1)$. A family of orthogonal polynomials is a set $(p_i)_{i \in \mathbb{N}}$ such that

- $p_i \in \mathbb{P}$
- $\deg p_i = i$
- $i \neq j \Rightarrow (p_i | p_j)_w = 0$

$(p_i)_{i \in \mathbb{N}}$ is then a basis of the vector space $\mathbb{P}$: $\mathbb{P} = \text{span} \{ p_i, i \in \mathbb{N} \}$

Theorem

A family of orthogonal polynomial $(p_i)_{i \in \mathbb{N}}$ is a Hilbert basis of $L^2_w(-1, 1)$:

$$\forall f \in L^2_w(-1, 1), \quad f = \sum_{i=0}^{\infty} \tilde{f}_i p_i$$

with $\tilde{f}_i := \frac{(f | p_i)_w}{\|p_i\|_w^2}$.

The above infinite sum means

$$\lim_{N \to \infty} \left\| f - \sum_{i=0}^{N} \tilde{f}_i p_i \right\|_w = 0$$
**Jacobi polynomials**

Jacobi polynomials are orthogonal polynomials with respect to the weight

\[ w(x) = (1 - x)^\alpha (1 + x)^\beta \]

**Subcases:**

- **Legendre polynomials** \( P_n(x) \): \( \alpha = \beta = 0 \), i.e. \( w(x) = 1 \)
- **Chebyshev polynomials** \( T_n(x) \): \( \alpha = \beta = -\frac{1}{2} \), i.e. \( w(x) = \frac{1}{\sqrt{1 - x^2}} \)

Jacobi polynomials are eigenfunctions of the singular\(^1\) **Sturm-Liouville problem**

\[ -\frac{d}{dx} \left[ (1 - x^2) w(x) \frac{du}{dx} \right] = \lambda w(x) u, \quad x \in (-1, 1) \]

\(^1\) *singular* means that the coefficient in front of \( du/dx \) vanishes at the extremities of the interval \([-1, 1]\)
Legendre polynomials

Expansions onto orthogonal polynomials

Legendre polynomials

\[ w(x) = 1: \quad \int_{-1}^{1} P_i(x)P_j(x) \, dx = \frac{2}{2i + 1} \delta_{ij} \]

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2} (3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2} (5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) \]
\[ P_{i+1}(x) = \frac{2i+1}{i+1} x P_i(x) - \frac{i}{i+1} P_{i-1}(x) \]

Legendre polynomials up to N=8
Expansions onto orthogonal polynomials

Chebyshev polynomials

\[ w(x) = \frac{1}{\sqrt{1 - x^2}}: \quad \int_{-1}^{1} T_i(x)T_j(x) \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2} (1 + \delta_{0i}) \delta_{ij} \]

\[ T_0(x) = 1 \]
\[ T_1(x) = x \]
\[ T_2(x) = 2x^2 - 1 \]
\[ T_3(x) = 4x^3 - 3x \]
\[ T_4(x) = 8x^4 - 8x^2 + 1 \]

\[ \cos(n\theta) = T_n(\cos \theta) \]

\[ T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x) \]
Legendre and Chebyshev compared

[from Fornberg (1998)]
Let us consider \( f \in L^2_w(-1, 1) \) and a family \( (p_i)_{i \in \mathbb{N}} \) of orthogonal polynomials with respect to the weight \( w \).

Since \( (p_i)_{i \in \mathbb{N}} \) is a Hilbert basis of \( L^2_w(-1, 1) \), we have \( f(x) = \sum_{i=0}^{\infty} \tilde{f}_i \, p_i(x) \) with \( \tilde{f}_i := \frac{(f|p_i)_w}{\|p_i\|_w^2} \).

The truncated sum

\[
\Pi_N^w f(x) := \sum_{i=0}^{N} \tilde{f}_i \, p_i(x)
\]

is a polynomial of degree \( N \): it is the orthogonal projection of \( f \) onto the finite dimensional subspace \( \mathbb{P}_N \) with respect to the scalar product \( (\cdot|\cdot)_w \).

We have

\[
\lim_{N \to \infty} \|f - \Pi_N^w f\|_w = 0
\]

Hence \( \Pi_N^w f \) can be considered as a polynomial approximation of the function \( f \).
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

$$f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 4 : \|f - \Pi_4^w f\|_\infty \approx 0.66$$
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

\[ f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 6 : \| f - \Pi_6^w f \|_\infty \approx 0.30 \]
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

$f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 8 : \| f - \Pi_8^w f \|_\infty \approx 4.9 \times 10^{-2}$
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

\[
f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 12 : \|f - \Pi_{12}^w f\|_\infty \simeq 6.1 \times 10^{-3}
\]
Example: Chebyshev projection of $f(x) = \cos(2 \exp(x))$

Variation of the projection error $\|f - \Pi_N^w f\|_\infty$ as $N$ increases.
The coefficients $\tilde{f}_i$ of the orthogonal projection of $f$ are given by

$$
\tilde{f}_i := \frac{(f|p_i)_w}{\|p_i\|^2_w} = \frac{1}{\|p_i\|^2_w} \int_{-1}^{1} f(x) p_i(x) w(x) \, dx
$$

(2)

**Problem:** the above integral cannot be computed exactly; we must seek a numerical approximation.

**Solution:** Gaussian quadrature
Theorem (Gauss, Jacobi)

Let \((p_i)_{i \in \mathbb{N}}\) be a family of orthogonal polynomials with respect to some weight \(w\). For \(N > 0\), let \(X = (x_i)_{0 \leq i \leq N}\) be the grid formed by the \(N + 1\) zeros of the polynomial \(p_{N+1}\) and

\[
w_i := \int_{-1}^{1} \ell^X_i(x) w(x) \, dx
\]

where \(\ell^X_i\) is the \(i\)-th Lagrange cardinal polynomial of the grid \(X\).

Then

\[
\forall f \in \mathbb{P}_{2N+1}, \quad \int_{-1}^{1} f(x) w(x) \, dx = \sum_{i=0}^{N} w_i f(x_i)
\]

If \(f \not\in \mathbb{P}_{2N+1}\), the above formula provides a good approximation of the integral.
The nodes of the Gauss quadrature, being the zeros of $p_{N+1}$, do not encompass the boundaries $-1$ and $1$ of the interval $[-1,1]$. For numerical purpose, it is desirable to include these points in the boundaries.

This possible at the price of reducing by 2 units the degree of exactness of the Gauss quadrature.
Theorem (Gauss-Lobatto quadrature)

Let \((p_i)_{i \in \mathbb{N}}\) be a family of orthogonal polynomials with respect to some weight \(w\). For \(N > 0\), let \(X = (x_i)_{0 \leq i \leq N}\) be the grid formed by the \(N + 1\) zeros of the polynomial

\[q_{N+1} = p_{N+1} + \alpha p_N + \beta p_{N-1}\]

where the coefficients \(\alpha\) and \(\beta\) are such that \(x_0 = -1\) and \(x_N = 1\).

Let

\[w_i := \int_{-1}^{1} \ell_i^X(x) w(x) \, dx\]

where \(\ell_i^X\) is the \(i\)-th Lagrange cardinal polynomial of the grid \(X\).

Then

\[\forall f \in \mathbb{P}_{2N-1}, \quad \int_{-1}^{1} f(x) w(x) \, dx = \sum_{i=0}^{N} w_i f(x_i)\]

Notice: \(f \in \mathbb{P}_{2N-1}\) instead of \(f \in \mathbb{P}_{2N+1}\) for Gauss quadrature.
Remark: if the \((p_i)\) are Jacobi polynomials, i.e. if \(w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}\), then the Gauss-Lobatto nodes which are strictly inside \((-1, 1)\), i.e. \(x_1, \ldots, x_{N-1}\), are the \(N - 1\) zeros of the polynomial \(p'_N\), or equivalently the points where the polynomial \(p_N\) is extremal.

This of course holds for Legendre and Chebyshev polynomials. For Chebyshev polynomials, the Gauss-Lobatto nodes and weights have simple expressions:

\[
x_i = -\cos\left(\frac{\pi i}{N}\right), \quad 0 \leq i \leq N
\]

\[
w_0 = w_N = \frac{\pi}{2N}, \quad w_i = \frac{\pi}{N}, \quad 1 \leq i \leq N - 1
\]

Note: in the following, we consider only Gauss-Lobatto quadratures
The Gauss-Lobatto quadrature motivates the introduction of the following scalar product:

$$\langle f|g \rangle_N = \sum_{i=0}^{N} w_i f(x_i)g(x_i)$$

It is called the discrete scalar product associated with the Gauss-Lobatto nodes $X = (x_i)_{0 \leq i \leq N}$.

Setting $\gamma_i := \langle p_i|p_i \rangle_N$, the discrete coefficients associated with a function $f$ are given by

$$\hat{f}_i := \frac{1}{\gamma_i} \langle f|p_i \rangle_N, \quad 0 \leq i \leq N$$

which can be seen as approximate values of the coefficients $\tilde{f}_i$ provided by the Gauss-Lobatto quadrature [cf. Eq. (2)]
Let $I_{N}^{\text{GL}} f$ be the interpolant of $f$ at the Gauss-Lobatto nodes $X = (x_i)_{0 \leq i \leq N}$. Being a polynomial of degree $N$, it is expandable as

$$I_{N}^{\text{GL}} f(x) = \sum_{i=0}^{N} a_i p_i(x)$$

Then, since $I_{N}^{\text{GL}} f(x_j) = f(x_j)$,

$$\hat{f}_i = \frac{1}{\gamma_i} \langle f | p_i \rangle_N = \frac{1}{\gamma_i} \langle I_{N}^{\text{GL}} f | p_i \rangle_N = \frac{1}{\gamma_i} \sum_{j=0}^{N} a_j \langle p_j | p_i \rangle_N$$

Now, if $j = i$, $\langle p_j | p_i \rangle_N = \gamma_i$ by definition. If $j \neq i$, $p_j p_i \in \mathbb{P}_{2N-1}$ so that the Gauss-Lobatto formula holds and gives $\langle p_j | p_i \rangle_N = (p_j | p_i)_w = 0$. Thus we conclude that $\langle p_j | p_i \rangle_N = \gamma_i \delta_{ij}$ so that the above equation yields $\hat{f}_i = a_i$, i.e. the discrete coefficients are nothing but the coefficients of the expansion of the interpolant at the Gauss-Lobatto nodes.
In a spectral method, the numerical representation of a function $f$ is through its interpolant at the Gauss-Lobatto nodes:

$$I_{GL}^N f(x) = \sum_{i=0}^{N} \hat{f}_i p_i(x)$$

The discrete coefficients $\hat{f}_i$ are computed as

$$\hat{f}_i = \frac{1}{\gamma_i} \sum_{j=0}^{N} w_j f(x_j) p_i(x_j)$$

$I_{GL}^N f(x)$ is an approximation of the truncated series $\Pi_{N}^w f(x) = \sum_{i=0}^{N} \tilde{f}_i p_i(x)$, which is the orthogonal projection of $f$ onto the polynomial space $\mathbb{P}_N$. $\Pi_{N}^w f$ should be the true spectral representation of $f$, but in general it is not computable exactly.

The difference between $I_{GL}^N f$ and $\Pi_{N}^w f$ is called the aliasing error.
Example: aliasing error for \( f(x) = \cos(2 \exp(x)) \)

\[
f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 4
\]

red: \( f \); blue: \( \Pi_N^w f \); green: \( I^G_L N f \)
Example: aliasing error for $f(x) = \cos(2 \exp(x))$

$$f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 6$$

red: $f$; blue: $\Pi_N^w f$; green: $I_N^{GL} f$
Expansions onto orthogonal polynomials

Example: aliasing error for $f(x) = \cos(2 \exp(x))$

\[
f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 8
\]

red: $f$; blue: $\Pi_N^w f$; green: $I_N^{\text{GL}} f$
Example: aliasing error for $f(x) = \cos(2 \exp(x))$

\[ f(x) = \cos(2 \exp(x)) \quad w(x) = (1 - x^2)^{-1/2} \quad N = 12 \]
Aliasing error = contamination by high frequencies

Aliasing of a $\sin(x)$ wave by a $\sin(5x)$ wave on a 4-points grid
Outline

1. Introduction
2. Interpolation on an arbitrary grid
3. Expansions onto orthogonal polynomials
4. Convergence of the spectral expansions
5. References
Let us consider a function \( f \in C^m([-1, 1]) \), with \( m \geq 0 \).

The **Sobolev norm** of \( f \) with respect to some weight function \( w \) is

\[
\| f \|_{H^m_w} := \left( \sum_{k=0}^{m} \| f^{(k)} \|_w^2 \right)^{1/2}
\]
Convergence of the spectral expansions

Convergence rates for \( f \in C^m([-1, 1]) \)

**Chebyshev expansions:**
- Truncation error:
  \[
  \|f - \Pi_N^w f\|_w \leq \frac{C_1}{N^m} \|f\|_{H_w^m} \quad \text{and} \quad \|f - \Pi_N^w f\|_\infty \leq \frac{C_2 (1 + \ln N)}{N^m} \sum_{k=0}^m \|f^{(k)}\|_\infty
  \]
- Interpolation error:
  \[
  \|f - I_N^{GL} f\|_w \leq \frac{C_3}{N^m} \|f\|_{H_w^m} \quad \text{and} \quad \|f - I_N^{GL} f\|_\infty \leq \frac{C_4}{N^{m-1/2}} \|f\|_{H_w^m}
  \]

**Legendre expansions:**
- Truncation error:
  \[
  \|f - \Pi_N^w f\|_w \leq \frac{C_1}{N^m} \|f\|_{H_w^m} \quad \text{and} \quad \|f - \Pi_N^w f\|_\infty \leq \frac{C_2}{N^{m-1/2}} V(f^{(m)})
  \]
- Interpolation error:
  \[
  \|f - I_N^{GL} f\|_w \leq \frac{C_3}{N^{m-1/2}} \|f\|_{H_w^m}
  \]
If \( f \in C^\infty([-1, 1]) \), the error of the spectral expansions \( \Pi_N^w f \) or \( I_N^{GL} f \) decays more rapidly than any power of \( N \).

In practice: **exponential decay**

This error is called **evanescent**.
Convergence of the spectral expansions

For non-smooth functions: Gibbs phenomenon

Extreme case: $f$ discontinuous
References