Special relativity from an accelerated observer perspective

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1. Introduction
2. Accelerated observers in special relativity
3. Kinematics
4. Physics in an accelerated frame
5. Physics in a rotating frame
Outline

1. Introduction
2. Accelerated observers in special relativity
3. Kinematics
4. Physics in an accelerated frame
5. Physics in a rotating frame
A brief history of special relativity

- 1898: H. Poincaré: *simultaneity* must result from some *convention*
- 1900: H. Poincaré: synchronization of clocks by exchange of light signals
- 1905: A. Einstein: funding article based on 2 axioms, both related to *inertial observers*: (i) the relativity principle, (ii) the constancy of the velocity of light
- 1905: H. Poincaré: mathematical use of time as a fourth dimension
- 1907: A. Einstein: first mention of an *accelerated observer* (uniform acceleration)
- 1908: H. Minkowsky: 4-dimensional spacetime, generic accelerated observer
- 1909: M. Born: detailed study of uniformly accelerated motion
- 1909: P. Ehrenfest: paradox on the circumference of a disk set to rotation
- 1911: A. Einstein, P. Langevin: round-trip motion and differential aging (*twin paradox*)
- 1911: M. Laue: prediction of the *Sagnac effect* within special relativity
- 1956: J. L. Synge: fully geometrical exposure of special relativity
Standard textbook presentations of special relativity are based on inertial observers.

For these privileged observers, there exists a global 3+1 decomposition of spacetime, i.e. a split between some time and some 3-dimensional Euclidean space. This could make people comfortable to think in a “Newtonian way”.

Special relativity differs then from Newtonian physics only in the manner one moves from one inertial observer to another one:

Lorentz transformations $\leftrightarrow$ Galilean transformations
Some drawback of this approach: the twin paradox

In most textbooks the twin paradox is presented by means of a reference inertial observer and his twin who is "piecewise inertial", yielding the result

\[ T' = T \sqrt{1 - \frac{V^2}{c^2}} \leq T \]
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This requires some infinite acceleration episodes.

A (very) skeptical physicist may say that the infinite acceleration spoils the explanation.
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A more satisfactory presentation would require an accelerated observer.
The real world is made of accelerated / rotating observers.
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Well known relativistic effects arise for accelerated observers: *Thomas precession, Sagnac effect.*
Other arguments for considering accelerated observers

- The real world is made of accelerated / rotating observers.
- Well known relativistic effects arise for accelerated observers: *Thomas precession, Sagnac effect.*
- Explaining the above effects by relying only on inertial observers is tricky; it seems *logically more appropriate* to introduce *generic (accelerated) observers* first, considering inertial observers as a special subcase.
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Well known relativistic effects arise for accelerated observers: *Thomas precession*, *Sagnac effect*.

Explaining the above effects by relying only on inertial observers is tricky; it seems *logically more appropriate* to introduce *generic (accelerated) observers* first, considering inertial observers as a special subcase.

Often students learning *general* relativity discover notions like *Fermi-Walker transport* or *Rindler horizon* which have nothing to do with spacetime curvature and actually pertain to the realm of *special* relativity.
When limiting the discussion to inertial observers, one can stick to a 3+1 point of view and avoid to refer to Minkowsky spacetime.

On the contrary, the appropriate framework for introducing accelerated observers is **Minkowsky spacetime**, that is the quadruplet $(\mathcal{E}, g, \mathcal{I}^+, \epsilon)$ where:

- $\mathcal{E}$ is a 4-dimensional affine space on $\mathbb{R}$ (associate vector space $E$)
- $g$ is the **metric tensor**, i.e. a bilinear form on $E$ that is symmetric, non-degenerate and has signature $(-, +, +, +)$
- $\mathcal{I}^+$ is one of the two sheets of $g$’s null cone, defining the **time orientation** of spacetime
- $\epsilon$ is the **Levi-Civita alternating tensor**, i.e. a quadrilinear form on $E$ that is antisymmetric and results in $\pm 1$ when applied to any vector basis which is orthonormal with respect to $g$
**The null cone and vector gender**

$E$: space of vectors on spacetime (4-vectors)

**Metric tensor:**

\[
g : E \times E \rightarrow \mathbb{R} \\
(\vec{u}, \vec{v}) \mapsto g(\vec{u}, \vec{v}) = : \vec{u} \cdot \vec{v}
\]

A vector $\vec{v} \in E$ is

- **spacelike** iff $\vec{v} \cdot \vec{v} > 0$
- **timelike** iff $\vec{v} \cdot \vec{v} < 0$
- **null** iff $\vec{v} \cdot \vec{v} = 0$
Physical interpretation of the metric tensor 1:

Proper time along a (massive) particle worldline = length given by the metric tensor:

\[ d\tau = \frac{1}{c} \sqrt{-g(d\vec{x}, d\vec{x})} \]

4-velocity \( \vec{u} \) = unit timelike future-directed tangent to the worldline:

\[ \vec{u} := \frac{1}{c} \frac{d\vec{x}}{d\tau}, \quad g(\vec{u}, \vec{u}) = -1 \]
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\]

Physical interpretation of the metric tensor 2:
The worldline of massless particles (e.g. photons) are null
lines of \( g \) (i.e. straight lines with a null tangent vector)
Observer $\mathcal{O}$ of worldline $\mathcal{L}_0$

$A$ event on $\mathcal{L}_0$, $B$ distant event

Using only proper times measured by $\mathcal{O}$ and a round-trip light signal:

Einstein-Poincaré definition of simultaneity

$B$ is simultaneous with $A$ $\iff$ $t = \frac{1}{2}(t_1 + t_2)$

$t$: proper time of $A$

$t_1$ (resp. $t_2$): proper time of signal emission (resp. reception)
Accelerated observers in special relativity

Einstein-Poincaré simultaneity

Observer $\mathcal{O}$ of worldline $L_0$

$A$ event on $L_0$, $B$ distant event

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**Geometrical characterization**

If $B$ is “closed” to $\mathcal{O}$’s worldline,

$B$ is simultaneous with $A \iff \vec{u}(A) \cdot \overrightarrow{AB} = 0$
Local rest space of an observer

Observer $O$: worldline $L_0$, 4-velocity $\vec{u}$, proper time $t$

Given an event $A \in L_0$ of proper time $t$,

- hypersurface of simultaneity of $A$ for $O$: set $\Sigma_u(t)$ of all events simultaneous to $A$ according to $O$
- local rest space of $O$: hyperplane $E_u(t)$ tangent to $\Sigma_u(t)$ at $A$

According to the geometrical characterization of Einstein-Poincaré simultaneity:

$E_u(t)$ is the spacelike hyperplane orthogonal to $\vec{u}(t)$

Notation: $E_u(t) = 3$-dimensional vector space associated with the affine space $E_u(t); E_u(t)$ is a subspace of $E$
Local frame of an observer

An observer is defined not only by its wordline, but also by an orthonormal basis \((\vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t))\) of its local rest space \(E_u(t)\) at each instant \(t\).

\[ (\vec{e}_\alpha(t)) = (\vec{u}(t), \vec{e}_1(t), \vec{e}_2(t), \vec{e}_3(t)) \]

is then an orthonormal basis of \(E\): it is \(O\)'s local frame.
Accelerated observers in special relativity

Coordinates associated with an observer

Observer $O$:
- proper time $t$
- local frame $(\vec{e}_\alpha(t))$

$M \in \mathcal{E}$ “close” to $O$’s worldline $\mathcal{L}_0$

Coordinates $(t, x^1, x^2, x^3)$ of $M$ with respect to $O$:
- $t$ defined by $M \in \mathcal{E}_u(t)$
- $(x^1, x^2, x^3)$ defined by $O(t)M = x^i \vec{e}_i(t)$

Misner, Thorne & Wheeler’s generalization (1973) of coordinates introduced by Synge (1956) (called by him Fermi coordinates)
Reference space of observer $\mathcal{O}$

3-dim. Euclidean space $R_\mathcal{O}$ with mapping

$$\varphi : \mathcal{E} \quad \longmapsto \quad R_\mathcal{O}$$

$$M(t, x^i) \quad \longmapsto \quad \vec{x} = x^i \vec{e}_i$$
Variation of the local frame (1/2)

Expand $d\vec{e}_\alpha/dt$ on the basis $(\vec{e}_\alpha)$: $\frac{d\vec{e}_\alpha}{dt} = \Omega^\beta_\alpha \vec{e}_\beta$

Introduce $\Omega$ endomorphism of $E$ whose matrix in the $(\vec{e}_\alpha)$ basis is $(\Omega^\alpha_\beta)$. Then

$$\frac{d\vec{e}_\alpha}{dt} = \Omega(\vec{e}_\alpha)$$

From the property $\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$ and $d\eta_{\alpha\beta}/dt = 0$ one gets immediately

$$\Omega(\vec{e}_\alpha) \cdot \vec{e}_\beta = -\vec{e}_\alpha \cdot \Omega^\beta_\alpha \vec{e}_\beta$$

$\Rightarrow$ the bilinear form $\Omega$ defined by $\forall (\vec{v}, \vec{w}) \in E^2$, $\Omega(\vec{v}, \vec{w}) := \vec{v} \cdot \Omega(\vec{w})$ is antisymmetric, i.e. $\Omega$ is a 2-form.
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\(\implies\) the bilinear form \(\Omega\) defined by \(\forall(\vec{v}, \vec{w}) \in E^2\), \(\Omega(\vec{v}, \vec{w}) := \vec{v} \cdot \Omega(\vec{w})\) is antisymmetric, i.e. \(\Omega\) is a 2-form.

\(\implies\) \(\exists\) a unique 1-form \(a\) and a unique vector \(\vec{\omega}\) such that

\[
\Omega = c \, u \otimes a - c \, a \otimes u - \epsilon(\vec{u}, \vec{\omega}, ..), \quad a \cdot \vec{u} = 0, \quad \vec{\omega} \cdot \vec{u} = 0
\]

This is similar to the electric / magnetic decomposition of the electromagnetic field tensor \(F\) with respect to an observer:

\[
F = u \otimes E - E \otimes u + \epsilon(\vec{u}, c\vec{B}, ..), \quad E \cdot \vec{u} = 0, \quad \vec{B} \cdot \vec{u} = 0
\]
Accordingly

\[
\frac{d\tilde{e}_\alpha}{dt} = c(\tilde{a} \cdot \tilde{e}_\alpha) \tilde{u} - c(\tilde{u} \cdot \tilde{e}_\alpha) \tilde{a} + \tilde{\omega} \times \tilde{u} \tilde{e}_\alpha
\]

with \( \tilde{v} \times_u \tilde{w} := \epsilon(\tilde{u}, \tilde{v}, \tilde{w}, .) \)
Accordingly

\[ \frac{d\vec{e}_\alpha}{dt} = c(\vec{a} \cdot \vec{e}_\alpha) \vec{u} - c(\vec{u} \cdot \vec{e}_\alpha) \vec{a} + \vec{\omega} \times \vec{u} \vec{e}_\alpha \]  

(1)

with \( \vec{v} \times_u \vec{w} := \vec{\epsilon}(\vec{u}, \vec{v}, \vec{w}, .) \)

- Since \( \vec{a} \cdot \vec{u} = 0 \), \( \vec{u} \cdot \vec{u} = -1 \) and \( \vec{\omega} \times_u \vec{u} = 0 \), applying (1) to \( \vec{e}_0 = \vec{u} \) yields

\[ \frac{d\vec{u}}{dt} = c \vec{a} \]

\( \vec{a} \) is thus the 4-acceleration of observer \( O \)
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- The vector \( \vec{\omega} \) is called the **4-rotation** of observer \( \mathcal{O} \)
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\( \vec{a} \) is thus the \textbf{4-acceleration} of observer \( O \)

- The vector \( \vec{\omega} \) is called the \textbf{4-rotation} of observer \( O \)

As for the 4-velocity, the 4-acceleration and the 4-rotation are \textit{absolute quantities}

\( O \text{ inertial observer} \iff \vec{a} = 0 \text{ and } \vec{\omega} = 0 \iff \frac{d\vec{e}_\alpha}{dt} = 0 \)
The local frame of observer $O$ is valid within a range

$$r \ll a^{-1} = \|\vec{a}\|_g^{-1} = (\vec{a} \cdot \vec{a})^{-1/2}$$

$$a = \gamma/c^2$$ with $\gamma$ acceleration of $O$ relative to a tangent inertial observer

$$\gamma = 10 \text{ m s}^{-2} \implies c^2/\gamma \approx 9 \times 10^{15} \text{ m} \approx 1 \text{ light-year}$$
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Lorentz factor of $\mathcal{P}$ with respect to $\mathcal{O}$: \[ \Gamma := \frac{dt}{dt'} \]

One can show \[ \Gamma = -\frac{\vec{u} \cdot \vec{u}'}{1 + \vec{a} \cdot \overrightarrow{OM}} \]
Relative velocity

Monitoring the motion of particle $P$ within $O$’s local coordinates $(t, x^i)$:

$$O(t)M(t) = x^i(t) \vec{e}_i(t)$$

The velocity of $P$ relative to $O$ is

$$\vec{V}(t) := \frac{dx^i}{dt} \vec{e}_i(t)$$

By construction $\vec{V}(t) \in E_u(t)$: $\vec{u} \cdot \vec{V} = 0$

The 4-velocity of $P$ is expressible in terms of $\Gamma$ and $\vec{V}$ as

$$\vec{u}' = \Gamma \left[ (1 + \vec{a} \cdot \vec{OM}) \vec{u} + \frac{1}{c} \left( \vec{V} + \vec{\omega} \times \vec{u} \vec{OM} \right) \right] \quad (2)$$

The normalization relation $\vec{u}' \cdot \vec{u}' = -1$ is then equivalent to

$$\Gamma = \left[ (1 + \vec{a} \cdot \vec{OM})^2 - \frac{1}{c^2} \left( \vec{V} + \vec{\omega} \times \vec{u} \vec{OM} \right) \cdot \left( \vec{V} + \vec{\omega} \times \vec{u} \vec{OM} \right) \right]^{-1/2} \quad (3)$$
Relative acceleration

The acceleration of \( \mathcal{P} \) relative to \( \mathcal{O} \) is

\[
\overset{\cdot}{\gamma}(t) := \frac{d^2 x^i}{dt^2} \overset{\cdot}{e}_i(t)
\]

By construction \( \overset{\cdot}{\gamma}(t) \in E_u(t) \): \( \overset{\cdot}{u} \cdot \overset{\cdot}{\gamma} = 0 \)

The 4-acceleration of \( \mathcal{P} \) reads

\[
\overset{\cdot}{a}' = \frac{\Gamma^2}{c^2} \left\{ \overset{\cdot}{\gamma} + \overset{\cdot}{\omega} \times_u (\overset{\cdot}{\omega} \times_u \overset{\cdot}{O \bar{M}}) + 2\overset{\cdot}{\omega} \times_u \overset{\cdot}{V} + \frac{d\overset{\cdot}{\omega}}{dt} \times_u \overset{\cdot}{O \bar{M}} \\
+ c^2 (1 + \overset{\cdot}{a} \cdot \overset{\cdot}{O \bar{M}}) \overset{\cdot}{a} + \frac{1}{\Gamma} \frac{d\Gamma}{dt} \left( \overset{\cdot}{V} + \overset{\cdot}{\omega} \times_u \overset{\cdot}{O \bar{M}} \right) \\
+ c \left[ 2\overset{\cdot}{a} \cdot (\overset{\cdot}{V} + \overset{\cdot}{\omega} \times_u \overset{\cdot}{O \bar{M}}) + \frac{d\overset{\cdot}{a}}{dt} \cdot \overset{\cdot}{O \bar{M}} + \frac{1}{\Gamma} \frac{d\Gamma}{dt} (1 + \overset{\cdot}{a} \cdot \overset{\cdot}{O \bar{M}}) \right] \overset{\cdot}{u} \right\}.
\]
If $\mathcal{O}$ is inertial, $\vec{a} = 0$, $\vec{\omega} = 0$, and we recover well known formulæ:

$$\vec{u}' = \Gamma \left( \vec{u} + \frac{1}{c} \vec{V} \right)$$

$$\Gamma = \left( 1 - \frac{1}{c^2} \vec{V} \cdot \vec{V} \right)^{-1/2}$$

$$\vec{a}' = \frac{\Gamma^2}{c^2} \left[ \vec{\gamma} + \frac{\Gamma^2}{c^2} (\vec{\gamma} \cdot \vec{V}) (\vec{V} + c\vec{u}) \right]$$

$$\vec{a}' = \frac{1}{c^2} \vec{\gamma} \quad (\vec{V} = 0)$$
**Definition:** the observer $O$ is **uniformly accelerated** iff

- its worldline stays in a plane $\Pi \subset \mathcal{E}$
- the norm of its 4-acceleration is constant $a := \|\vec{a}\|_g = \sqrt{\vec{a} \cdot \vec{a}} = \text{const}$
- its 4-rotation vanishes: $\vec{\omega} = 0$

Worldline in terms of the coordinates $(ct_*, x_*, y_*, z_*)$ associated with an inertial observer $O_*$:

$$
\begin{align*}
ct_* &= a^{-1} \sinh(\text{act}) \\
x_* &= a^{-1} [\cosh(\text{act}) - 1] \\
y_* &= 0 \\
z_* &= 0.
\end{align*}
$$

$$(ax_* + 1)^2 - (act_*^2) = 1$$

$\vec{u}(t) = \cosh(\text{act}) \vec{e}_0^* + \sinh(\text{act}) \vec{e}_1^*$

$\vec{a}(t) = a [\sinh(\text{act}) \vec{e}_0^* + \cosh(\text{act}) \vec{e}_1^*]$
Coordinates associated with the accelerated observer

Relation between the coordinates \((t, x, y, z)\) associated with \(O\) and the inertial coordinates \((t_*, x_*, y_*, z_*)\):

\[
\begin{align*}
ct_* &= (x + a^{-1}) \sinh(act) \\
x_* &= (x + a^{-1}) \cosh(act) - a^{-1} \\
y_* &= y \\
z_* &= z.
\end{align*}
\]

with \(x > -a^{-1}\)

The coordinates \((t, x, y, z)\) are called Rindler coordinates
Observer $O'$ at rest with respect to $O$, located at coord. $(x, y, z) = (x_0, 0, 0)$

$$\Rightarrow \vec{V} = 0$$

(3) \ \Rightarrow \ \Gamma = \left[ 1 + \tilde{a}(t) \cdot \overrightarrow{O(t)O'(t')} \right]^{-1}

(2) \ \Rightarrow \ \tilde{u}'(t') = \tilde{u}(t)

\Rightarrow \text{the local rest spaces of } O \text{ and } O' \text{ coincide: } \mathcal{E}_{\tilde{u}}(t') = \mathcal{E}_{\tilde{u}}(t)

$$\tilde{a}(t) = a \tilde{e}_1(t) \text{ and } \overrightarrow{O(t)O'(t')} = x_0 \tilde{e}_1(t)$$

$$\Rightarrow \Gamma = (1 + ax_0)^{-1} \& \ dt' = (1 + ax_0) \ dt$$

Since $x_0 = \text{const}$, this relation can be integrated:

$$t' = (1 + ax_0) t$$
Observer $O'$ at rest with respect to $O$, located at coord. $(x, y, z) = (x_0, 0, 0)$

$\Rightarrow \vec{V} = 0$

(3) $\Rightarrow \Gamma = \left[1 + \vec{a}(t) \cdot \overrightarrow{O(t)O'(t')}\right]^{-1}$

(2) $\Rightarrow \vec{u}'(t') = \vec{u}(t)$

$\Rightarrow$ the local rest spaces of $O$ and $O'$ coincide: $E_{\vec{u}'}(t') = E_{\vec{u}}(t)$

$\vec{a}(t) = a \vec{e}_1(t)$ and $\overrightarrow{O(t)O'(t')} = x_0 \vec{e}_1(t)$

$\Rightarrow \Gamma = (1 + ax_0)^{-1}$ & $dt' = (1 + ax_0) dt$

Since $x_0 = \text{const}$, this relation can be integrated:

$$t' = (1 + ax_0) t$$

Analogous to Einstein effect in general relativity
Null geodesics in terms of inertial coordinates:

\[ ct_* = \pm (x_* - b), \quad b \in \mathbb{R} \]

in terms of \( O \)'s coordinates:

\[ ct = \pm a^{-1} \ln \left( \frac{1 + ax}{1 + ab} \right) \]

\[ x = -a^{-1} : \text{Rindler horizon} \]
Redshift

Reception by $O$ of a photon emitted by $O'$ at $t' = 0$

If $\vec{p}$ is the photon 4-momentum, the energy measured by $O$ is

$$E_{\text{rec}} = -c \vec{p} \cdot \vec{u}(t_{\text{rec}})$$

with

$$\vec{p} = \frac{E_{\text{em}}}{c} (\vec{u}'(0) + \vec{n}') = \frac{E_{\text{em}}}{c} (\vec{e}^*_0 - \vec{e}^*_1)$$

$$\vec{u}(t_{\text{rec}}) = \cosh(\alpha t_{\text{rec}}) \vec{e}^*_0 + \sinh(\alpha t_{\text{rec}}) \vec{e}^*_1$$

$$c t_{\text{rec}} = a^{-1} \ln(1 + a x_{\text{em}})$$

$$\implies E_{\text{rec}} = E_{\text{em}} (1 + a x_{\text{em}})$$
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with

$$\vec{p} = \frac{E_{\text{em}}}{c} (\vec{u}'(0) + \vec{n}') = \frac{E_{\text{em}}}{c} (\vec{e}_0^* - \vec{e}_1^*)$$

$$\vec{u}(t_{\text{rec}}) = \cosh(act_{\text{rec}}) \vec{e}_0^* + \sinh(act_{\text{rec}}) \vec{e}_1^*$$

$$ct_{\text{rec}} = a^{-1} \ln(1 + ax_{\text{em}})$$

$$\implies E_{\text{rec}} = E_{\text{em}}(1 + ax_{\text{em}})$$

$\implies$ spectral shift

$$z = \frac{1}{1 + ax_{\text{em}}} - 1$$

$$\left\{ \begin{array}{ll}
z > 0 & \text{for } x_{\text{em}} < 0 \\
z < 0 & \text{for } x_{\text{em}} > 0 \\
\end{array} \right.$$
Thomas precession

\( \mathcal{O}_* \) = inertial observer ; proper time \( t_* \); (local) frame \( (\vec{e}_{\alpha}) \)

\( \mathcal{O} \) = accelerated observer \textit{without rotation}; proper time \( t \); local frame \( (\vec{e}_{\alpha}(t)) \)

\( S_t \) : the boost from \( \vec{e}_0^* \) to \( \vec{e}_0(t) \):

\[
\vec{e}_0(t) = S_t(\vec{e}_0^*)
\]

Let

\[
\vec{e}_\alpha(t_*) := S_t^{-1}(\vec{e}_\alpha(t))
\]

\[
\vec{e}_\alpha(t) = S_t(\vec{e}_\alpha(t_*))
\]

\( \vec{e}_0 = \vec{e}_0^* \)

\( (\vec{e}_i) \) = triad in \( \mathcal{O}_* \)'s rest space which is “quasi-parallel” to the triad \( (\vec{e}_i) \) of \( \mathcal{O} \)'s local rest frame.
Evolution of $\mathcal{O}$’s local rest frame:

$$\vec{e}_\alpha(t + dt) = \Lambda(\vec{e}_\alpha)$$

According to (1) with $\vec{\omega} = 0$, $\Lambda(\vec{e}_\alpha) = \vec{e}_\alpha + c dt[(\vec{a} \cdot \vec{e}_\alpha) \vec{u} - (\vec{u} \cdot \vec{e}_\alpha) \vec{a}]$

$\Lambda$ is an infinitesimal boost

Hence

$$\vec{e}_\alpha(t + dt) = \Lambda \circ S_t(\vec{e}_\alpha(t_*))$$

Now in general, the composition of the boosts $\Lambda$ and $S_t$ is a boost times a rotation — **Thomas rotation**:

$$\Lambda \circ S_t = S'_t \circ R$$

In the present case, $R(\vec{e}_{0*}) = \vec{e}_{0*}$, so that necessarily $S' = S_{t+dt}$. Hence

$$\vec{e}_\alpha(t + dt) = S_{t+dt} \circ R(\vec{e}_\alpha(t_*))$$

$$\Rightarrow \quad \vec{e}_\alpha(t_* + dt_*) = R(\vec{e}_\alpha(t_*))$$
Thomas precession

Thus

$$\frac{d\vec{e}_i}{dt} = \vec{\omega}_T \times e_0^* \vec{e}_i$$

The following expression can be established for the rotation vector:

$$\vec{\omega}_T = \frac{\Gamma^2}{c^2(1 + \Gamma)} \vec{\gamma} \times e_0^* \vec{V}$$

with

\[ \vec{V} \] = velocity of \( O \) with respect to \( O_* \)
\[ \vec{\gamma} \] = acceleration of \( O \) with respect to \( O_* \)
\[ \Gamma \] = Lorentz factor of \( O \) with respect to \( O_* \)

Remark: if \( O \) is a uniformly accelerated observer, \( \vec{V} \) and \( \vec{\gamma} \) are parallel, so that
\( \vec{\omega}_T = 0 \)
Outline

1 Introduction

2 Accelerated observers in special relativity

3 Kinematics

4 Physics in an accelerated frame

5 Physics in a rotating frame
Observer $O$ in **uniform rotation**:
\[ \vec{a} = 0 \text{ and } \vec{\omega} = \text{const} \]

Local frame of $O$:
\[
\begin{align*}
\vec{e}_0(t) &= \vec{e}_0^* \\
\vec{e}_1(t) &= \cos \omega t \vec{e}_1^* + \sin \omega t \vec{e}_2^* \\
\vec{e}_2(t) &= -\sin \omega t \vec{e}_1^* + \cos \omega t \vec{e}_2^* \\
\vec{e}_3(t) &= \vec{e}_3^* = \omega^{-1} \vec{\omega}
\end{align*}
\]

with $(\vec{e}_\alpha^*)$ reference frame of inertial observer $O^*_\alpha$

Coordinate system of $O$ : $(t, x, y, z)$ such that
\[
\begin{align*}
x^*_\alpha &= x \cos \omega t - y \sin \omega t \\
y^*_\alpha &= x \sin \omega t + y \cos \omega t \\
z^*_\alpha &= z
\end{align*}
\]
Corotating observer

Observer $O'$ at rest with respect to $O$, i.e. at fixed values of $x = r \cos \varphi$ and $y = r \cos \varphi$ ($z = 0$)

Worldline in term of inertial coordinates:

\[
\begin{align*}
x_*(t) &= r \cos(\omega t + \varphi) \\
y_*(t) &= r \sin(\omega t + \varphi) \\
z_*(t) &= 0.
\end{align*}
\]

Velocity of $O'$ w.r.t. $O_*$:

\[ \vec{V} = r \omega \vec{n}, \quad \text{with} \quad \vec{n} := -\sin \varphi \vec{e}_1 + \cos \varphi \vec{e}_2 \]

4-acceleration of $O'$:

\[ \vec{a}' = \frac{\Gamma^2}{c^2} r \omega^2 \vec{e}'_2, \quad \vec{e}'_2 = -\cos \varphi \vec{e}_1 - \sin \varphi \vec{e}_2 \]
The problem of clock synchronization

1-parameter family of corotating observers $O'(\lambda)$

Moving from $O(\lambda)$ to $O'(\lambda+d\lambda)$

$A(\lambda)$: event on $O'(\lambda)$'s worldline

$A(\lambda+d\lambda)$: event on $O'(\lambda+d\lambda)$'s worldline simultaneous to $A(\lambda)$ for $O'(\lambda)$:

$$\vec{u}'(\lambda) \cdot \overrightarrow{A(\lambda)A(\lambda+d\lambda)} = 0 \quad (4)$$

with $\overrightarrow{A(\lambda)A(\lambda+d\lambda)} = c\, dt\, \vec{u} + d\vec{\ell} + dt\, \vec{V}$

$d\vec{\ell} := dx^i \, \vec{e}_i(t)$, separation between $O'(\lambda)$ and $O'(\lambda+d\lambda)$ from the point of view of $O$

Expanding (4) yields

$$dt = \Gamma^2 \frac{\vec{V} \cdot d\vec{\ell}}{c^2}$$
The problem of clock synchronization

Integrating on a closed contour

Desynchronization lapse:

\[ \Delta t'_{\text{desync}} = \frac{1}{c^2 \Gamma_0} \oint_C \Gamma^2 \mathbf{V} \cdot d\mathbf{\ell} \]

Synchronization helix
Two signals of *same velocity* w.r.t. $\mathcal{O}$

After a round trip, discrepancy between the two arrival times ($t'$: proper time of emitter $\mathcal{O}'$):

$$\Delta t' := t'_+ - t'_- = 2\Delta t'_{\text{desync}}$$

$$\implies \Delta t' = \frac{2}{c^2 \Gamma(0)} \oint_{\mathcal{C}} \Gamma^2 \vec{V} \cdot d\vec{\ell}$$

Sagnac delay
Phase shift:
\[ \Delta \phi = \frac{4\pi f}{c^2 \Gamma(0)} \oint \Gamma^2 \vec{V} \cdot d\vec{\ell} \]

Slow rotation limit \((r\omega \ll c)\):
\[ \Delta \phi = \frac{8\pi f}{c^2} \vec{\omega} \cdot \vec{A} \]

Application: gyrometers
Bibliography