Modelling black holes as trapping horizons

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based on a collaboration with

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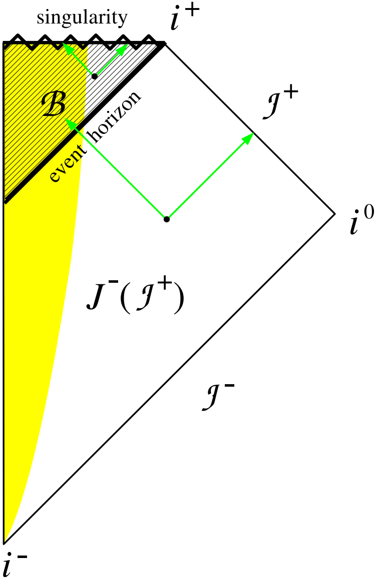
Post Newton 2008
Jena, 11-14 juin 2008

Plan

1. Local approaches to black holes
2. Viscous fluid analogy
3. Angular momentum and area evolution laws
4. Applications to numerical relativity

Outline

Classical definition of a black hole



black hole: $\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)$

i.e. the region of spacetime where light rays cannot escape to infinity

- $(\mathscr{M}, \boldsymbol{g})$ = *asymptotically flat* manifold
- $\mathscr{I}^+$ = future null infinity
- $J^-(\mathscr{I}^+)$ = causal past of $\mathscr{I}^+$

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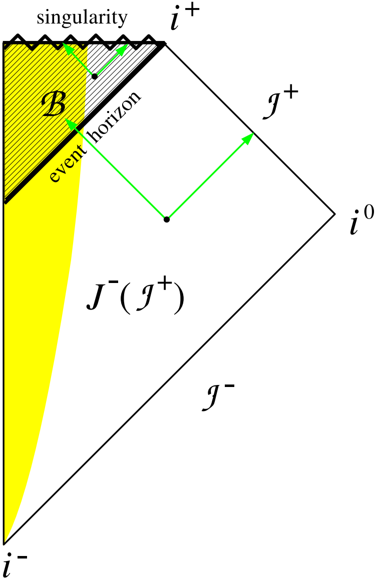

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event horizon: $\mathcal{H} := \dot{J}^-(\mathscr{I}^+)$
(boundary of $J^-(\mathscr{I}^+)$)

$\mathcal{H}$ smooth $\Longrightarrow$ $\mathcal{H}$ null hypersurface

Drawbacks of the classical definition

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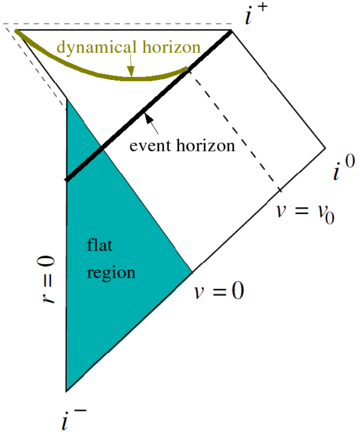
Determination of $\dot{J}^-(\mathscr{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is *not locally linked with the notion of strong gravitational field:*



Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

with

$m(v) = 0$	for $v < 0$
$dm/dv > 0$	for $0 \leq v \leq v_0$
$m(v) = M_0$	for $v > v_0$

[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

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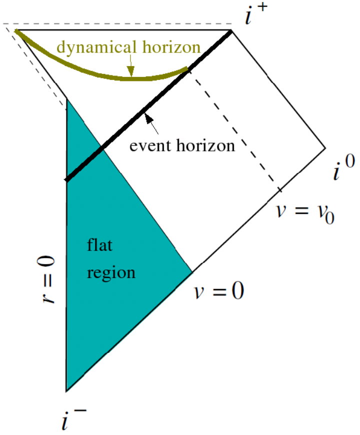

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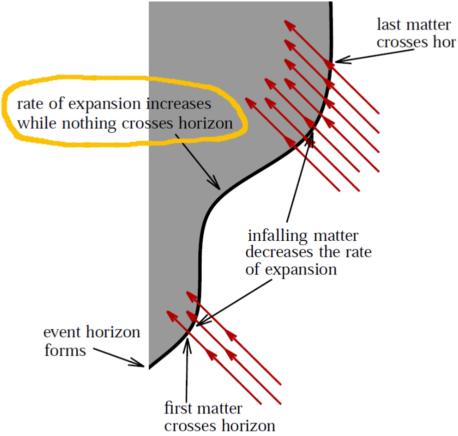
$\Rightarrow$ no local physical experiment whatsoever can locate the event horizon

Another non-local feature: teleological nature of event horizons



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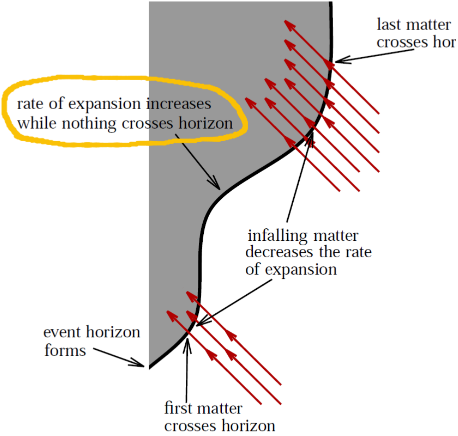

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To deal with black holes as ordinary physical objects, a **local** definition would be desirable
$\rightarrow$ quantum gravity, numerical relativity

Local characterizations of black holes

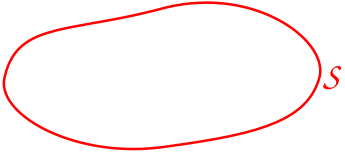
Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)
- slowly evolving horizons (Booth and Fairhurst 2004)

All these concepts are **local** and are based on the notion of trapped surfaces

Definition of a trapped surface

$\mathcal{S}$: closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathscr{M}, \boldsymbol{g})$



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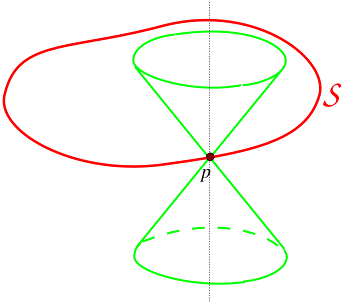

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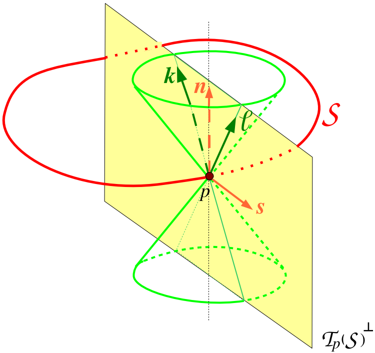

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$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$

$\boldsymbol{k}$ = ingoing, expansion $\theta^{(\boldsymbol{k})}$

In flat space, $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} > 0$

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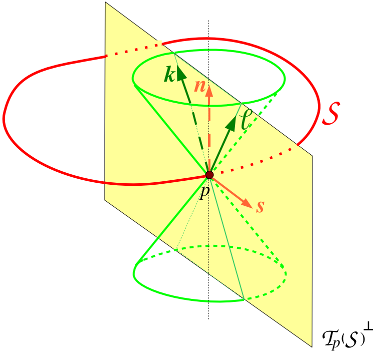

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$\mathcal{S}$ is trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} < 0$ [Penrose 1965]

$\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$

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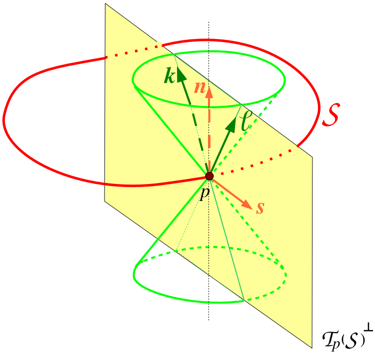

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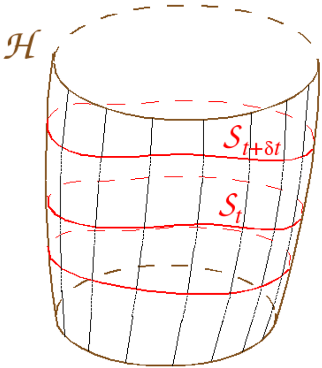
trapped surface = **local** concept characterizing very strong gravitational fields

Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be



- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces ($\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$)
  (ii) $\boldsymbol{\mathcal{L}}_{\boldsymbol{k}}\, \theta^{(\boldsymbol{\ell})} < 0$ (locally outermost trapped surf.)
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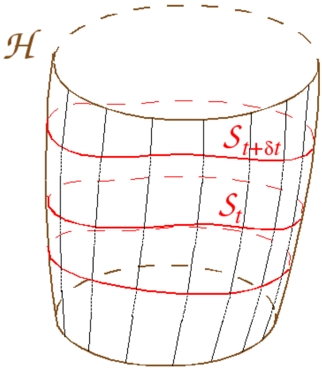

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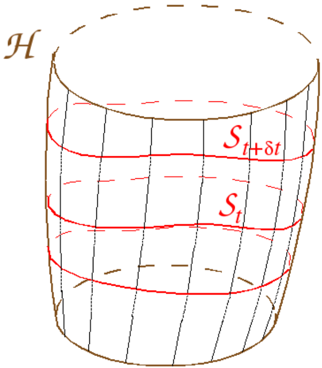

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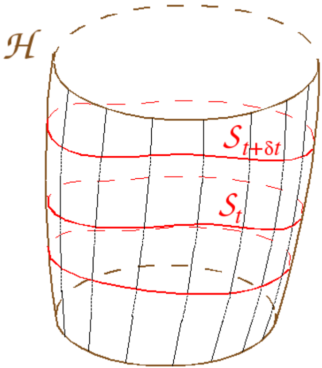

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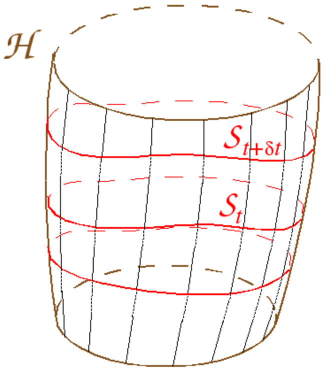

BH in equilibrium = IH
(e.g. Kerr)
BH out of equilibrium = DH
generic BH = FOTH

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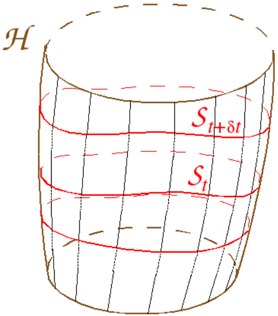
Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.
In particular, one can establish for them

- existence and (partial) uniqueness theorems
  [Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
  [Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
  [Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy ("membrane paradigm", as for the event horizon)
  [EG, PRD **72**, 104007 (2005)], [EG & Jaramillo, PRD **74**, 087502 (2006)]

Reviews: [Ashtekar & Krishnan, Liv. Rev. Relat. **7**, 10 (2004)], [Booth, Can. J. Phys. **83**, 1073 (2005)], [EG & Jaramillo, Phys. Rep. **423**, 159 (2006)], [Krishnan, CQG **25**, 114005 (2008)]

Foliation of a hypersurface by spacelike 2-surfaces



hypersurface $\mathcal{H}$ = submanifold of spacetime $(\mathscr{M}, \boldsymbol{g})$ of codimension 1

$\mathcal{H}$ can be $\begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$

$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

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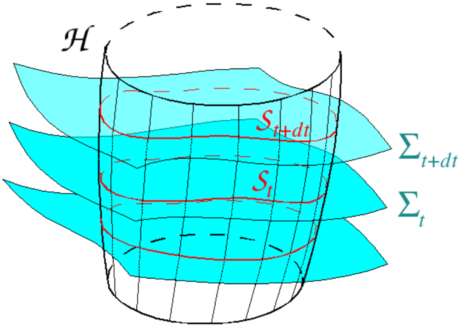

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$\Longleftarrow$ 3+1 perspective

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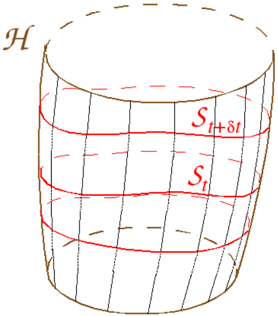

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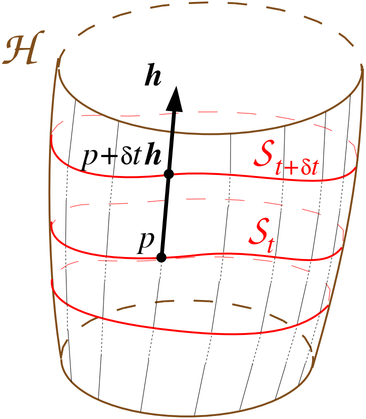
intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

$\boldsymbol{q}$: induced metric on $\mathcal{S}_t$ (positive definite)

$\boldsymbol{\mathcal{D}}$: connection associated with $\boldsymbol{q}$

Evolution vector on the horizon



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\mathcal{L}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

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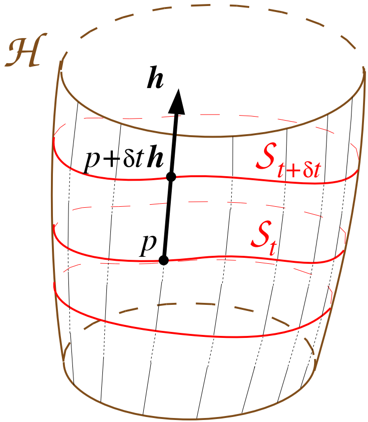

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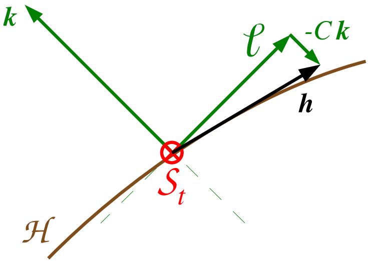
Define $C := \frac{1}{2}\,\boldsymbol{h} \cdot \boldsymbol{h}$

$\mathcal{H}$ is spacelike	$\iff$	$C > 0$	$\iff$	$\boldsymbol{h}$ is spacelike
$\mathcal{H}$ is null	$\iff$	$C = 0$	$\iff$	$\boldsymbol{h}$ is null
$\mathcal{H}$ is timelike	$\iff$	$C < 0$	$\iff$	$\boldsymbol{h}$ is timelike.

Normal null frame associated with the evolution vector



The foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame $(\boldsymbol{\ell}, \boldsymbol{k})$, via the evolution vector $\boldsymbol{h}$: there exists a unique normal null frame $(\boldsymbol{\ell}, \boldsymbol{k})$ such that

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k} \quad \text{and} \quad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$$

Normal fundamental form: $\boxed{\boldsymbol{\Omega}^{(\ell)} := -\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\,\boldsymbol{\ell}}$ or $\Omega^{(\ell)}_\alpha := -k_\mu \nabla_\nu \ell^\mu\, q^\nu{}_\alpha$

Evolution of $\boldsymbol{h}$ along itself: $\boxed{\boldsymbol{\nabla}_{\boldsymbol{h}}\boldsymbol{h} = \kappa\,\boldsymbol{\ell} + (C\kappa - \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, C)\boldsymbol{k} - \boldsymbol{\mathcal{D}}C}$

NB: null limit : $C = 0$, $\boldsymbol{h} = \boldsymbol{\ell} \Longrightarrow \boldsymbol{\nabla}_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa\,\boldsymbol{\ell} \Longrightarrow \kappa$ = surface gravity

Outline

Concept of black hole viscosity

- Hartle and Hawking (1972, 1973): introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Longrightarrow$ *shear viscosity* and *bulk viscosity*
- Thorne and Price (1986): **membrane paradigm** for black holes

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Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. future outer trapping horizons and dynamical horizons ?

NB: *event horizon* = null hypersurface
future outer trapping horizon = null or spacelike hypersurface
dynamical horizon = spacelike hypersurface

Original Damour-Navier-Stokes equation

Hyp: $\mathcal{H}$ = null hypersurface (particular case: black hole **event horizon**)
Then $\boldsymbol{h} = \boldsymbol{\ell}$ ($C = 0$)

Damour (1979) has derived from Einstein equation the relation

$$ {}^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)} \boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\ell)} + \frac{1}{2} \boldsymbol{\mathcal{D}}\theta^{(\ell)} + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{\ell} $$

or equivalently

$$ {}^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\pi} + \theta^{(\ell)} \boldsymbol{\pi} = -\boldsymbol{\mathcal{D}} P + 2\mu \boldsymbol{\mathcal{D}} \cdot \vec{\boldsymbol{\sigma}}^{(\ell)} + \zeta \boldsymbol{\mathcal{D}}\theta^{(\ell)} + \boldsymbol{f} \qquad (*) $$

with $\boldsymbol{\pi} := -\frac{1}{8\pi} \boldsymbol{\Omega}^{(\ell)}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\mu := \frac{1}{16\pi}$ shear viscosity

$\zeta := -\frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

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$P := \frac{\kappa}{8\pi}$ pressure

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$\zeta := -\frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

$(*)$ is identical to a 2-dimensional Navier-Stokes equation

to avoid any misunderstanding...

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because

"A black hole is not a water fall" (Clifford Will)

Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation, $\boxed{\zeta = -\frac{1}{16\pi} < 0}$

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid

It is in agreement with the tendency of a null hypersurface to continually contract or expand

The event horizon is stabilized by the teleological condition imposing its expansion to vanish in the far future (equilibrium state reached)

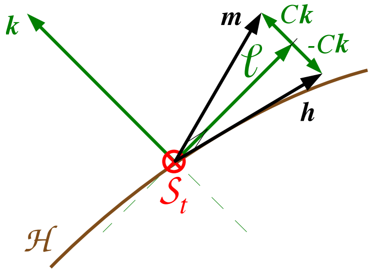
Generalization to the non-null case

Starting remark: in the null case (event horizon), ℓ plays two different roles:

- evolution vector along $\mathcal{H}$ (e.g. term ${}^{\mathcal{S}}\mathcal{L}_{\ell}$)
- normal to $\mathcal{H}$ (e.g. term $\vec{q}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $\boldsymbol{h}$
- vector normal to $\mathcal{H}$: a natural choice is $\boxed{\boldsymbol{m} := \boldsymbol{\ell} + C\boldsymbol{k}}$



Generalized Damour-Navier-Stokes equation

From the contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$: $(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu)\, q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$ and using Einstein equation to express $R_{\mu\nu}$, one gets an evolution equation for $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ along $\mathcal{H}$:

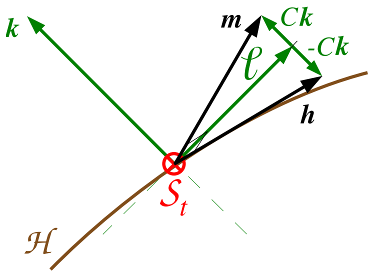
$$ {}^{\mathcal{S}}\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})}\boldsymbol{\mathcal{D}}C + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{m} $$

- $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$: normal fundamental form of $\mathcal{S}_t$ associated with null normal $\boldsymbol{\ell}$
- $\theta^{(\boldsymbol{h})}$, $\theta^{(\boldsymbol{m})}$ and $\theta^{(\boldsymbol{k})}$: expansion scalars of $\mathcal{S}_t$ along the vectors $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{k}$ respectively
- $\boldsymbol{\mathcal{D}}$: covariant derivative within $(\mathcal{S}_t, \boldsymbol{q})$
- κ : component of $\boldsymbol{\nabla}_{\boldsymbol{h}}\boldsymbol{h}$ along $\boldsymbol{\ell}$
- $\boldsymbol{\sigma}^{(\boldsymbol{m})}$: shear tensor of $\mathcal{S}_t$ along the vector $\boldsymbol{m}$
- C : half the scalar square of $\boldsymbol{h}$

Null limit (event horizon)

If $\mathcal{H}$ is a null hypersurface,

$$\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell} \qquad \text{and} \qquad C = 0$$



and we recover the original Damour-Navier-Stokes equation:

$$^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{\Omega}^{(\ell)} + \theta^{(\ell)}\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\ell)} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\ell)} + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$$

Case of future trapping horizons

Definition [Hayward, PRD **49**, 6467 (1994)] :
$\mathcal{H}$ is a **future trapping horizon** iff $\theta^{(\ell)} = 0$ and $\theta^{(k)} < 0$.
The generalized Damour-Navier-Stokes equation reduces then to

$$ {}^{\mathcal{S}}\!\mathcal{L}_{h}\,\boldsymbol{\Omega}^{(\ell)} + \theta^{(h)}\,\boldsymbol{\Omega}^{(\ell)} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(m)} - \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(h)} - \theta^{(k)}\boldsymbol{\mathcal{D}}C + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{m} $$

[EG, PRD **72**, 104007 (2005)]

NB: Notice the change of sign in the $-\frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(h)}$ term with respect to the original Damour-Navier-Stokes equation

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The explanation: it is $\theta^{(m)}$ which appears in the general equation and

$$ \theta^{(m)} + \theta^{(h)} = 2\theta^{(\ell)} \Longrightarrow \begin{cases} \text{event horizon } (\boldsymbol{m} = \boldsymbol{h}) & : \quad \theta^{(m)} = \theta^{(\ell)} \\ \text{trapping horizon } (\theta^{(\ell)} = 0) & : \quad \theta^{(m)} = -\theta^{(h)} \end{cases} $$

Viscous fluid form

$$ {}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,\boldsymbol{\pi} + \theta^{(\boldsymbol{h})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + \frac{1}{8\pi}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{m})} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})} + \boldsymbol{f} $$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{m})}$ shear stress tensor

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Similar to the Damour-Navier-Stokes equation for an event horizon, except

- the **Newtonian-fluid** relation between *stress* and *strain* does not hold: $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi \neq 2\mu\boldsymbol{\sigma}^{(\boldsymbol{h})}$, rather $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi = [\boldsymbol{\sigma}^{(\boldsymbol{h})} + 2C\boldsymbol{\sigma}^{(\boldsymbol{k})}]/8\pi$

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  a post-Newtonian fluid ?
- positive bulk viscosity

This positive value of bulk viscosity shows that FOTHs and DHs behave as "ordinary" physical objects, in perfect agreement with their local nature

Outline

Angular momentum of trapping horizons

Definition [Booth & Fairhurst, CQG **22**, 4545 (2005)]: Let φ be a vector field on $\mathcal{H}$ which

- is tangent to $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\varphi} = 0$ (weaker than being a Killing vector of $(\mathcal{S}_t, \boldsymbol{q})$!)

For dynamical horizons, $\theta^{(\boldsymbol{h})} \neq 0$ and there is a unique choice of φ as the generator (conveniently normalized) of the curves of constant $\theta^{(\boldsymbol{h})}$

[Hayward, PRD **74**, 104013 (2006)]

The *generalized angular momentum associated with* φ is then defined by

$$J(\varphi) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \boldsymbol{\Omega}^{(\boldsymbol{\ell})}, \boldsymbol{\varphi} \rangle \, {}^{\mathcal{S}}\boldsymbol{\epsilon},$$

Remark 1: does not depend upon the choice of null vector $\boldsymbol{\ell}$, thanks to the divergence-free property of φ

Remark 2:

- coincides with Ashtekar & Krishnan's definition for a dynamical horizon
- coincides with Brown-York angular momentum if $\mathcal{H}$ is timelike and φ a Killing vector

Angular momentum flux law

Under the supplementary hypothesis that φ is transported along the evolution vector $\boldsymbol{h}$: $\mathcal{L}_{\boldsymbol{h}}\,\varphi = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt} J(\varphi) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m}, \varphi)\, {}^{\mathcal{S}}\boldsymbol{\epsilon} - \frac{1}{16\pi} \oint_{\mathcal{S}_t} \left[\boldsymbol{\sigma}^{(\boldsymbol{m})} : \mathcal{L}_{\varphi}\, \boldsymbol{q} - 2\theta^{(\boldsymbol{k})} \varphi \cdot \boldsymbol{\mathcal{D}} C \right] {}^{\mathcal{S}}\boldsymbol{\epsilon}$$

[EG, PRD **72**, 104007 (2005)]

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Area evolution law for an event horizon

$A(t)$: area of the 2-surface $\mathcal{S}_t$; ${}^{\mathcal{S}}\boldsymbol{\epsilon}$: volume element of $\mathcal{S}_t$; $\bar{\kappa}(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$

Integrating the null Raychaudhuri equation on $\mathcal{S}_t$, one gets

$$\frac{d^2 A}{dt^2} - \bar{\kappa} \frac{dA}{dt} = - \int_{\mathcal{S}_t} \left[8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell}) + \boldsymbol{\sigma}^{(\boldsymbol{\ell})} : \boldsymbol{\sigma}^{(\boldsymbol{\ell})} - \frac{(\theta^{(\boldsymbol{\ell})})^2}{2} + (\bar{\kappa} - \kappa)\theta^{(\boldsymbol{\ell})} \right] {}^{\mathcal{S}}\boldsymbol{\epsilon} \quad (1)$$

[Damour, 1979]

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Simplified analysis : assume $\bar{\kappa} = \text{const} > 0$:

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- Cauchy problem $\Longrightarrow$ diverging solution of the homogeneous equation: $\frac{dA}{dt} = \alpha \exp(\bar{\kappa} t)$!
- correct treatment: impose $\frac{dA}{dt} = 0$ at $t = +\infty$ (teleological !)

$$\frac{dA}{dt} = \int_t^{+\infty} D(u) e^{\bar{\kappa}(t-u)} \, du \qquad D(t) : \text{r.h.s. of Eq. (1)}$$

Non causal evolution

Area evolution law for a dynamical horizon

Dynamical horizon : $C > 0$; $\kappa' := \kappa - \mathcal{L}_{\boldsymbol{h}} \ln C$; $\bar{\kappa}'(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa' \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$

From the $(\boldsymbol{m}, \boldsymbol{h})$ component of Einstein equation, one gets

$$\frac{d^2 A}{dt^2} + \bar{\kappa}' \frac{dA}{dt} = \int_{\mathcal{S}_t} \left[8\pi \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) + \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{(\theta^{(\boldsymbol{h})})^2}{2} + (\bar{\kappa}' - \kappa')\theta^{(\boldsymbol{h})} \right] {}^{\mathcal{S}}\boldsymbol{\epsilon} \quad (2)$$

[EG & Jaramillo, PRD **74**, 087502 (2006)]

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[EG & Jaramillo, PRD **74**, 087502 (2006)]

Simplified analysis : assume $\bar{\kappa}' = \mathrm{const} > 0$
(OK for small departure from equilibrium [Booth & Fairhurst, PRL **92**, 011102 (2004)]):
Standard Cauchy problem :

$$\frac{dA}{dt} = \left. \frac{dA}{dt} \right|_{t=0} + \int_0^t D(u) e^{\bar{\kappa}'(u-t)} \, du \qquad D(t) : \text{r.h.s. of Eq. (2)}$$

Causal evolution, in agreement with local nature of dynamical horizons

Outline

Applications to numerical relativity

- **Initial data**: isolated horizons (helical symmetry)
  [EG, Grandclément & Bonazzola, PRD **65**, 044020 (2002)]
  [Grandclément, EG & Bonazzola, PRD **65**, 044021 (2002)]
  [Cook & Pfeiffer, PRD **70**, 104016 (2004)]

- **A posteriori analysis**: estimating mass, linear and angular momentum of formed black holes
  [Schnetter, Krishnan & Beyer, PRD **74**, 024028 (2006)]
  [Cook & Whiting, PRD **76**, 041501 (2007)]
  [Krishnan, Lousto & Zlochower, PRD **76**, 081501(R) (2007)]

- **Numerical construction of spacetime**: inner boundary conditions for a constrained scheme with "black hole excision"
  [Jaramillo, EG, Cordero-Carrión, & J.M. Ibáñez, PRD **77**, 047501 (2008)]

A few words about the history of 3+1 formalism

- G. Darmois (1927): 3+1 Einstein equations in terms of (γ_{ij}, K_{ij}) with $\alpha = 1$ and $\beta^i = 0$ (Gaussian normal coordinates)
  Cauchy problem well posed for *analytic* initial data
- A. Lichnerowicz (1939) : $\alpha \neq 1$ and $\beta^i = 0$ (normal coordinates)
- Y. Choquet-Bruhat (1948) : $\alpha \neq 1$ and $\beta^i \neq 0$ (general coordinates)
- Y. Choquet-Bruhat (1952) : Cauchy problem well posed for *smooth* (i.e. generic) initial data
- R. Arnowitt, S. Deser & C.W. Misner (1962) : *Hamiltonian formulation* of GR based on a 3+1 decomposition in terms of (γ_{ij}, π^{ij})
  NB: spatial projection of *Einstein tensor* instead of *Ricci tensor* in previous works
- J. Wheeler (1964) : coined the terms *lapse* and *shift*
- J.W. York (1979) : modern 3+1 decomposition based on spatial projection of *Ricci tensor*