Pictures of the interior of a Kerr black hole

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What we learn at school (1/2)

Kerr metric, expressed in **Boyer-Lindquist coordinates** (t, r, θ, φ):

$$\mathrm{d}s^2 = -\left(1 - \frac{2mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4amr\sin^2\theta}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2,$$

m: mass; $a = J/m$ reduced angular momentum

$\rho^2 := r^2 + a^2\cos^2\theta$, $\Delta := r^2 - 2mr + a^2 = (r - r_-)(r - r_+)$

$r_\pm := m \pm \sqrt{m^2 - a^2}$

- $a = 0 \Longrightarrow$ reduces to Schwarzschild metric
- black hole $\iff 0 \leq a \leq m$, naked singularity $\iff a > m$
- $\Delta = 0$: coordinate singularity (of Boyer-Lindquist coordinates) coincides with 2 horizons: $\mathcal{H}$ $(r = r_+)$ and $\mathcal{H}_{\mathrm{in}}$ $(r = r_-)$
- $\rho = 0$: curvature singularity
  $\rho = 0 \iff r = 0$ and $\theta = \pi/2$: it is a ring, not a point — Ah bon?

What we learn at school (2/2)

Kerr metric, expressed in **Kerr-Schild coordinates** $(\tilde{t}, x, y, z)$:

$$\begin{aligned} \mathrm{d}s^2 = \; & -\mathrm{d}\tilde{t}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ & + \frac{2mr^3}{r^4 + a^2 z^2} \left(\mathrm{d}\tilde{t} + \frac{rx + ay}{r^2 + a^2} \, \mathrm{d}x + \frac{ry - ax}{r^2 + a^2} \, \mathrm{d}y + \frac{z}{r} \, \mathrm{d}z \right)^2 \end{aligned}$$

with $r = r(x, y, z)$ such that $\dfrac{x^2 + y^2}{r^2 + a^2} + \dfrac{z^2}{r^2} = 1$

- $g_{\alpha\beta} = f_{\alpha\beta} + 2H k_\alpha k_\beta$ with $f_{\alpha\beta}$ = Minkowski metric, $H := mr^3/(r^4 + a^2 z^2)$ and null vector $k^\alpha = \left(1, -\dfrac{rx + ay}{r^2 + a^2}, -\dfrac{ry - ax}{r^2 + a^2}, -\dfrac{z}{r}\right)$
- Kerr-Schild coordinates are regular at $\mathcal{H}$ and $\mathcal{H}_{\rm in}$
- curvature singularity: $z = 0$ and $x^2 + y^2 = a^2$ (looks indeed like a ring)

A metric is not a spacetime

Spacetime: pair $(\mathscr{M}, \boldsymbol{g})$, where $\mathscr{M}$ is a smooth manifold and $\boldsymbol{g}$ a Lorentzian metric on $\mathscr{M}$

Spacetime manifold:
Boyer-Lindquist coordinates (t, r, θ, φ) are *not* spherical-type coordinates on $\mathbb{R}^4$
They are rather coordinates on the Cartesian product

$$\underbrace{\mathbb{R}^2}_{(t,r)} \times \underbrace{\mathbb{S}^2}_{(\theta,\varphi)}$$

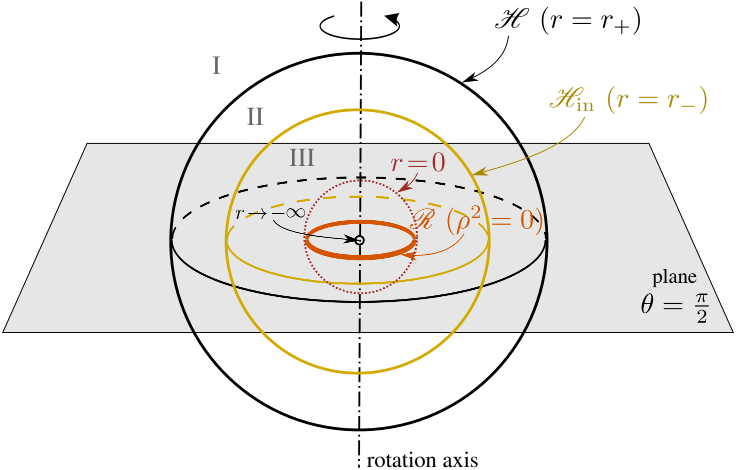
More precisely, the Kerr spacetime manifold is

$$\boxed{\mathscr{M} = \mathbb{R}^2 \times \mathbb{S}^2 \setminus \mathscr{R}}$$

with

$$\mathscr{R} = \left\{ p \in \mathbb{R}^2 \times \mathbb{S}^2, \quad r(p) = 0 \text{ and } \theta(p) = \frac{\pi}{2} \right\}$$

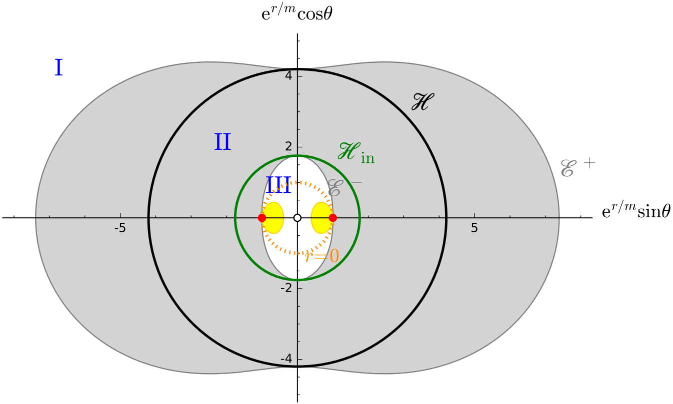
Hence on $\mathscr{M}$, $t \in (-\infty, +\infty)$, $\boxed{r \in (-\infty, +\infty)}$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$.



View of a slice $t = \mathrm{const}$ of Kerr spacetime in O'Neill coordinates (R, θ, φ), which are related to Boyer-Lindquist coordinates by $R = \mathrm{e}^r$

NB: the subset $r = 0$ is a 2-sphere, as for any constant value of r
the ring singularity is the equator $(\theta = \pi/2)$ of that sphere

The $\mathbb{R}^2 \times \mathbb{S}^2$ manifold



View of a meridional slice $t = \text{const}$ and $\varphi = 0$ or π of Kerr spacetime in O'Neill coordinates for $a/m = 0.90$

- in grey: the ergoregion: Killing vector $\dfrac{\partial}{\partial t}$ is spacelike
- in yellow: the Carter time machine: Killing vector $\dfrac{\partial}{\partial \varphi}$ is timelike

The subset $r = 0$

The metric induced by $\boldsymbol{g}$ on the subset $r = 0$ is

$$\left.\mathrm{d}s^2\right|_{r=0} = -\mathrm{d}t^2 + a^2\left(\cos^2\theta\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

This is a flat metric, as the change of coordinates $X = a\sin\theta\cos\varphi$, $Y = a\sin\theta\sin\varphi$ reveals: $\left.\mathrm{d}s^2\right|_{r=0} = -\mathrm{d}t^2 + \mathrm{d}X^2 + \mathrm{d}Y^2$

For $a > 0$, the subset $t = \mathrm{const}$, $r = 0$ of Kerr spacetime $\mathscr{M}$ is made of 2 connected components: the Northern and Southern (open) hemispheres of the sphere $t = \mathrm{const}$, $r = 0$ of $\mathbb{R}^2 \times \mathbb{S}^2$, which are actually two flat open disks of metric radius a. The equator of that sphere ($\theta = \pi/2$) would correspond to the curvature singularity and is excluded from $\mathscr{M}$.

Kerr coordinates

Kerr coordinates $(v, r, \theta, \tilde{\varphi})$ are the coordinates in which Roy Kerr obtained his solution to Einstein equation (1963); they are related to Boyer-Lindquist coordinates (t, r, θ, φ) by

$$v = t + r + \frac{m}{\sqrt{m^2 - a^2}} \left(r_+ \ln \left| \frac{r - r_+}{2m} \right| - r_- \ln \left| \frac{r - r_-}{2m} \right| \right)$$

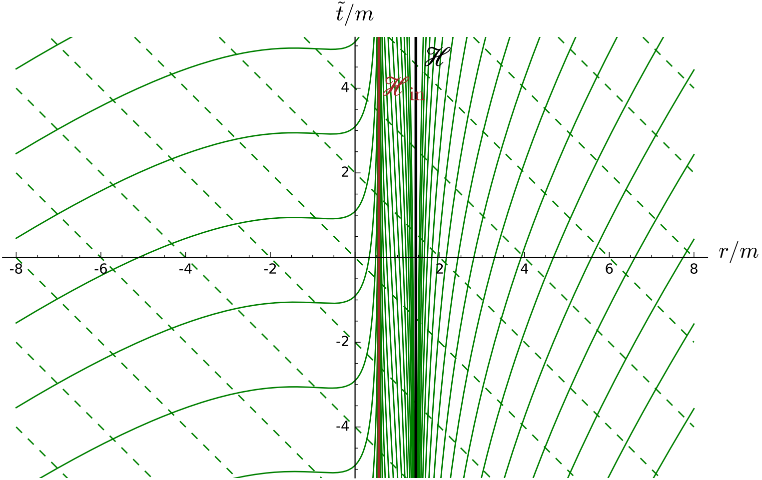
$$\tilde{\varphi} = \varphi + \frac{a}{2\sqrt{m^2 - a^2}} \ln \left| \frac{r - r_+}{r - r_-} \right|$$

Kerr metric in Kerr coordinates:

$$\begin{aligned} \mathrm{d}s^2 = \; & -\left(1 - \frac{2mr}{\rho^2}\right) \mathrm{d}v^2 + 2\mathrm{d}v\,\mathrm{d}r - \frac{4amr\sin^2\theta}{\rho^2}\,\mathrm{d}v\,\mathrm{d}\tilde{\varphi} \\ & -2a\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\tilde{\varphi} + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\tilde{\varphi}^2 \end{aligned}$$

- Kerr coordinates reduce to Eddington-Finkelstein coord. when $a = 0$
- They are regular on both Killing horizons $\mathcal{H}$ and $\mathcal{H}_{\rm in}$
- They are such that the curves $(v, \theta, \tilde{\varphi}) = \mathrm{const}$ are the ingoing principal null geodesics of Kerr spacetime

Principal null geodesics



Ingoing (dashed) and outgoing (solid) principal null geodesics of Kerr spacetime with $a/m = 0.90$ viewed in coordinates $(\tilde{t}, r)$ related to Kerr coordinates by $\tilde{t} = v - r$.

From Kerr coordinates to Kerr-Schild ones

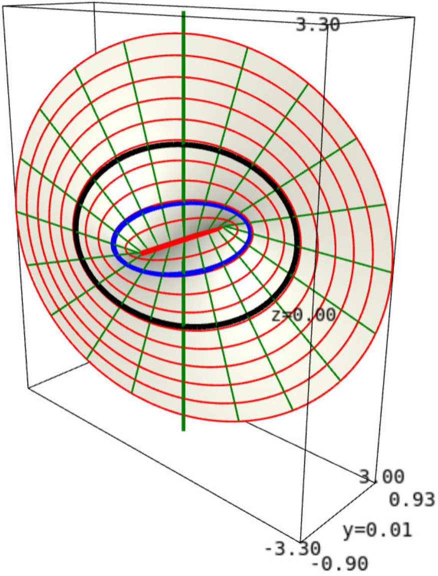
Kerr-Schild coordinates $(\tilde{t}, x, y, z)$ have been introduced in 1963 by Roy Kerr in the very same paper announcing the discovery of Kerr metric, via the following transformation from Kerr coordinates $(v, r, \theta, \tilde{\varphi})$:

$$\begin{aligned}
\tilde{t} &= v - r \\
x &= (r\cos\tilde{\varphi} - a\sin\tilde{\varphi})\sin\theta \\
y &= (r\sin\tilde{\varphi} + a\cos\tilde{\varphi})\sin\theta \\
z &= r\cos\theta
\end{aligned}$$

The null vector $\boldsymbol{k}$ entering in the Kerr-Schild form of the metric $(\boldsymbol{g} = \boldsymbol{f} + 2H\underline{\boldsymbol{k}}\otimes\underline{\boldsymbol{k}})$ is then nothing but the vector $\boldsymbol{k} = -\frac{\partial}{\partial r}$ tangent to the ingoing principal null geodesics.

Kerr-Schild coordinates, with $(\tilde{t}, x, y, z) \in \mathbb{R}^4$, cover only the part $r \geq 0$ of Kerr spacetime $\mathscr{M}$. Another Kerr-Schild patch is required to cover the part $r \leq 0$. Moreover Kerr-Schild coordinates are singular at $r = 0$: the points of Kerr coordinates $(v, 0, \theta, \tilde{\varphi})$ and $(v, 0, \pi - \theta, \tilde{\varphi})$ have the same Kerr-Schild coordinates $(\tilde{t}, x, y, z) = (v, -a\sin\theta\sin\tilde{\varphi}, a\sin\theta\cos\theta, 0)$.

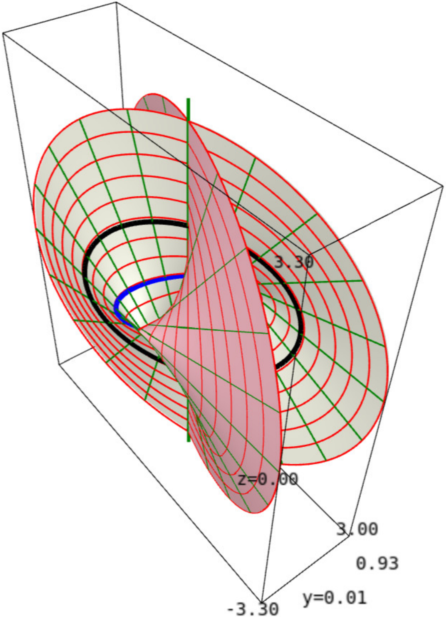
Meridional slice viewed in Kerr-Schild coordinates



Surface $\tilde{t} = \mathrm{const}$, $\tilde{\varphi} \in \{0, \pi\}$ and $r \geq 0$ of the $a/m = 0.90$ Kerr spacetime depicted in terms of the Kerr-Schild coordinates (x, y, z). Red lines are curves $r = \mathrm{const}$, while the green ones are curves $\theta = \mathrm{const}$, which can be thought of as the traces of the ingoing principal null geodesics. The thick black curve marks $\mathscr{H}$ and the thick blue curve $\mathscr{H}_{\mathrm{in}}$. The thick red segment along the y-axis corresponds to $r = 0$.

See the SageMath notebook https://nbviewer.jupyter.org/github/egourgoulhon/BHLectures/blob/master/sage/Kerr_Schild.ipynb for an interactive 3D view.

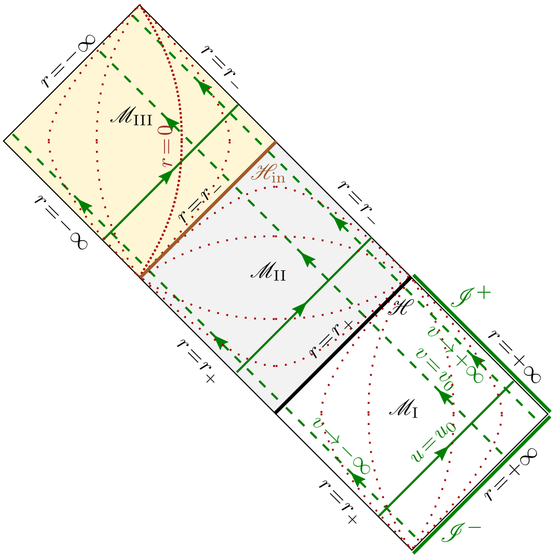
Immersion of a full meridional slice in Euclidean space



Immersion of the full $\tilde{t} = \mathrm{const}$ and $\tilde{\varphi} \in \{0, \pi\}$ surface of the $a/m = 0.90$ Kerr spacetime in the Euclidean space $\mathbb{R}^3$, using Kerr-Schild coordinates (x, y, z) for the $r \geq 0$ part (drawn in grey) and Kerr-Schild coordinates (x', y', z') for the $r \leq 0$ part (drawn in pink).

See the SageMath notebook `https://nbviewer.jupyter.org/github/egourgoulhon/BHLectures/blob/master/sage/Kerr_Schild.ipynb` for an interactive 3D view.

Carter-Penrose diagram of Kerr spacetime

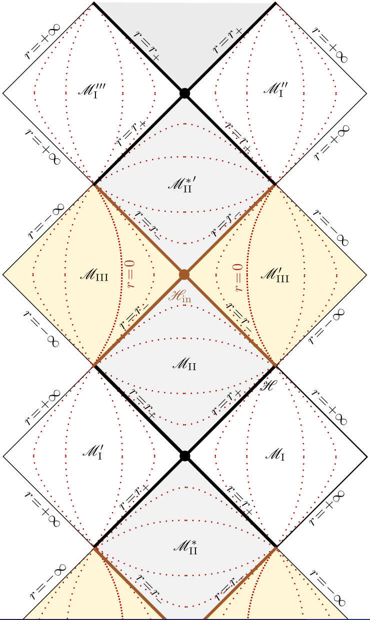


Conformal diagram of the Kerr spacetime $(\mathscr{M}, \boldsymbol{g})$, with $\mathscr{M} = \mathbb{R}^2 \times \mathbb{S}^2 \setminus \mathscr{R}$

- dashed green lines: ingoing principal null geodesics
- solid green lines: outgoing principal null geodesics
- dotted red curves: hypersurfaces $r = \text{const}$

The outgoing principal null geodesics are not complete (they end at some finite value of their affine parameter r) $\Longrightarrow$ the spacetime can be extended

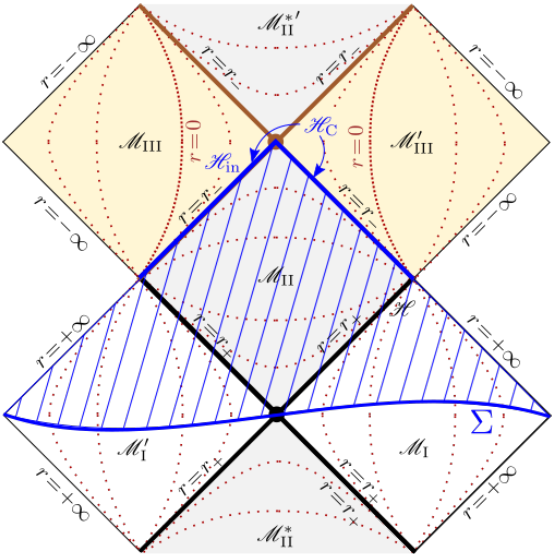
Maximal analytic extension of Kerr spacetime



Conformal diagram of maximal analytic extension of Kerr spacetime

- dotted red curves: hypersurfaces $r = \text{const}$
- black or light brown dots: bifurcation spheres of bifurcate Killing horizons

The inner horizon as a Cauchy horizon



Partial Cauchy surface Σ and its future Cauchy development $D^+(\Sigma)$ (hatched)

$\mathcal{H}_{\rm C}$: Cauchy horizon

More figures?

Lecture notes with more details and figures, as well as the Inkscape and SageMath sources of the figures shown here, can be found at

https://luth.obspm.fr/~luthier/gourgoulhon/bh16/