A geometrical approach to relativistic magnetohydrodynamics

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1. Relativistic MHD with exterior calculus

2. Stationary and axisymmetric electromagnetic fields in general relativity

3. Stationary and axisymmetric MHD
Outline

1 Relativistic MHD with exterior calculus

2 Stationary and axisymmetric electromagnetic fields in general relativity

3 Stationary and axisymmetric MHD
General framework and notations

Spacetime:

- $\mathcal{M}$: four-dimensional orientable real manifold
- $g$: Lorentzian metric on $\mathcal{M}$, $\text{sign}(g) = (-, +, +, +)$
- $\epsilon$: Levi-Civita tensor (volume element 4-form) associated with $g$: for any orthonormal basis $(\vec{e}_\alpha),
  \epsilon(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3) = \pm 1$

Notations:

- $\vec{v}$ vector $\implies \nu$ linear form associated to $\vec{v}$ by the metric tensor:
  $\nu := g(\vec{v}, .)$ \quad [$\nu = \nu^b$] \quad [$u_\alpha = g_{\alpha\mu} u^\mu$]

- $\vec{v}$ vector, $T$ multilinear form (valence $n$) $\implies \vec{v} \cdot T$ and $T \cdot \vec{v}$ multilinear forms (valence $n - 1$) defined by
  $\vec{v} \cdot T := T(\vec{v}, . , . , .)$ \quad $[(\vec{v} \cdot T)_{\alpha_1 \ldots \alpha_{n-1}} = \nu^\mu T_{\mu \alpha_1 \ldots \alpha_{n-1}}]$
  $T \cdot \vec{v} := T(. , . , . , \vec{v})$ \quad $[(T \cdot \vec{v})_{\alpha_1 \ldots \alpha_{n-1}} = T_{\alpha_1 \ldots \alpha_{n-1} \mu} \nu^\mu]$
Electromagnetic field in $\mathcal{M}$: 2-form $F$ which obeys to Maxwell equations:

\[ \text{d} F = 0 \]
\[ \text{d} \star F = \mu_0 \star j \]

- $\text{d} F$ : exterior derivative of $F$: \((\text{d} F)_{\alpha \beta \gamma} = \partial_\alpha F_{\beta \gamma} + \partial_\beta F_{\gamma \alpha} + \partial_\gamma F_{\alpha \beta}\)
- $\star F$ : Hodge dual of $F$: \(\star F_{\alpha \beta} := \frac{1}{2} \epsilon_{\alpha \beta \mu \nu} F^{\mu \nu}\)
- $\star j$ 3-form Hodge-dual of the 1-form $j$ associated to the electric 4-current $\vec{j}$: \(\star j := \epsilon(\vec{j},,,,.\,)\)
- $\mu_0$ : magnetic permeability of vacuum
**Fluid** : congruence of worldlines $\Rightarrow$ 4-velocity $\vec{u}$

- **Electric field** in the fluid frame: 1-form $e = F \cdot \vec{u}$
- **Magnetic field** in the fluid frame: vector $\vec{b}$ such that $e$ and $\vec{b}$ are orthogonal to $\vec{u}$: $e \cdot \vec{u} = 0$ and $\vec{b} \cdot \vec{u} = 0$

$$F = \underline{u} \wedge e + \epsilon(\vec{u}, \vec{b}, .., .)$$

$$\star F = -\underline{u} \wedge \vec{b} + \epsilon(\vec{u}, \vec{e}, .., .)$$
Fluid is a perfect conductor $\iff \vec{e} = 0 \iff \vec{F} \cdot \vec{u} = 0$

From now on, we assume that the fluid is a perfect conductor (ideal MHD)
The electromagnetic field is then entirely expressible in terms of vectors $\vec{u}$ and $\vec{b}$:

$$\vec{F} = \varepsilon(\vec{u}, \vec{b}, \ldots)$$

$$\star \vec{F} = \vec{b} \wedge \vec{u}$$
Alfvén’s theorem

Cartan’s identity applied to the 2-form $F$:

$$\mathcal{L}_{\vec{u}} F = \vec{u} \cdot dF + d(\vec{u} \cdot F)$$

Now $dF = 0$ (Maxwell eq.) and $\vec{u} \cdot F = 0$ (perfect conductor)

Hence the electromagnetic field is preserved by the flow:

$$\mathcal{L}_{\vec{u}} F = 0$$

Application:

$$\frac{d}{d\tau} \oint_{C(\tau)} A = 0$$

- $\tau$ : fluid proper time
- $C(\tau)$ = closed contour dragged along by the fluid
- $A$ : electromagnetic 4-potential : $F = dA$

Proof:

$$\frac{d}{d\tau} \oint_{C(\tau)} A = \frac{d}{d\tau} \int_{S(\tau)} \frac{dA}{F} = \frac{d}{d\tau} \int_{S(\tau)} F = \int_{S(\tau)} \mathcal{L}_{\vec{u}} F = 0$$

Non-relativistic limit: magnetic flux freezing : $\int_S \vec{b} \cdot d\vec{S} = \text{const}$ (Alfvén’s theorem)
We have obviously \( F \cdot \vec{b} = \epsilon(\vec{u}, \vec{b}, .., \vec{b}) = 0 \)

In addition, \( \mathcal{L}_{\vec{b}} F = \vec{b} \cdot dF + d(\vec{b} \cdot F) = 0 \)

Hence

\[
F \cdot \vec{b} = 0 \quad \text{and} \quad \mathcal{L}_{\vec{b}} F = 0
\]

similarly to

\[
F \cdot \vec{u} = 0 \quad \text{and} \quad \mathcal{L}_{\vec{u}} F = 0
\]
From $\mathcal{L}_\vec{u} F = 0$ and $F = \epsilon(\vec{u}, \vec{b}, ..)$, we get

$$
(\mathcal{L}_\vec{u} \epsilon)(\vec{u}, \vec{b}, ..) + \epsilon(\mathcal{L}_\vec{u} \vec{u}, \vec{b}, ..) + \epsilon(\vec{u}, \mathcal{L}_\vec{u} \vec{b}, ..) = 0
$$

Now $\mathcal{L}_\vec{u} \epsilon = (\nabla \cdot \vec{u}) \epsilon$, hence $\epsilon \left( \vec{u}, (\nabla \cdot \vec{u}) \vec{b} + \mathcal{L}_\vec{u} \vec{b}, .. \right) = 0$

This implies

$$
\mathcal{L}_\vec{u} \vec{b} = \alpha \vec{u} - (\nabla \cdot \vec{u}) \vec{b}
$$

(1)
Magnetic induction equation (2/2)

From $\mathcal{L}_\vec{u} \vec{F} = 0$ and $\vec{F} = \epsilon(\vec{u}, \vec{b}, ..,)$, we get

$$(\mathcal{L}_\vec{u} \epsilon)(\vec{u}, \vec{b}, .., ) + \epsilon(\mathcal{L}_\vec{u} \vec{u}, \vec{b}, .., ) + \epsilon(\vec{u}, \mathcal{L}_\vec{u} \vec{b}, .., ) = 0$$

Now $\mathcal{L}_\vec{u} \epsilon = (\nabla \cdot \vec{u})\epsilon$, hence $\epsilon \left( \vec{u}, (\nabla \cdot \vec{u})\vec{b} + \mathcal{L}_\vec{u} \vec{b}, .., \right) = 0$

This implies

$$\mathcal{L}_\vec{u} \vec{b} = \alpha \vec{u} - (\nabla \cdot \vec{u})\vec{b} \quad (1)$$

Similarly, the property $\mathcal{L}_\vec{b} \vec{F} = 0$ leads to $\epsilon \left( (\nabla \cdot \vec{b})\vec{u} + \mathcal{L}_\vec{b} \vec{u}, \vec{b}, .., \right) = 0$

which implies

$$\mathcal{L}_\vec{b} \vec{u} = -\mathcal{L}_\vec{u} \vec{b} = -(\nabla \cdot \vec{b})\vec{u} + \beta \vec{b} \quad (2)$$

Comparison of (1) and (2) leads to

$$\mathcal{L}_\vec{u} \vec{b} = (\nabla \cdot \vec{b})\vec{u} - (\nabla \cdot \vec{u})\vec{b} \quad (3)$$

Non-relativistic limit: $\frac{\partial \vec{b}}{\partial t} = \text{curl}(\vec{v} \times \vec{b})$ (induction equation)
Some simple consequences

We have $\mathbf{u} \cdot \mathbf{b} = 0 \implies \mathcal{L}_{\mathbf{u}} \mathbf{u} \cdot \mathbf{b} + \mathbf{u} \cdot \mathcal{L}_{\mathbf{u}} \mathbf{b} = 0$

Now $\mathcal{L}_{\mathbf{u}} \mathbf{u} = \mathbf{a}$ with $\mathbf{a} := \nabla \cdot \mathbf{u}$ (fluid 4-acceleration)

and $\mathbf{u} \cdot \mathcal{L}_{\mathbf{u}} \mathbf{b} = (\nabla \cdot \mathbf{b}) \mathbf{u} \cdot \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{u} \cdot \mathbf{b} = -\nabla \cdot \mathbf{b}$

Hence

$$\nabla \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$$
Some simple consequences

We have \( \mathbf{u} \cdot \mathbf{b} = 0 \implies \mathcal{L}\mathbf{u} \cdot \mathbf{b} + \mathbf{u} \cdot \mathcal{L}\mathbf{u} \mathbf{b} = 0 \)

Now \( \mathcal{L}\mathbf{u} = \mathbf{a} \) with \( \mathbf{a} := \nabla \times \mathbf{u} \) (fluid 4-acceleration)

and \( \mathbf{u} \cdot \mathcal{L}\mathbf{u} \mathbf{b} = (\nabla \cdot \mathbf{b}) \mathbf{u} \cdot \mathbf{u} - (\nabla \cdot \mathbf{u}) \mathbf{u} \cdot \mathbf{b} = -\nabla \cdot \mathbf{b} \)

Hence

\[ \nabla \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} \]

If we invoke **baryon number conservation**

\[ \nabla \cdot (n \mathbf{u}) = 0 \iff \nabla \cdot \mathbf{u} = -\frac{1}{n} \mathcal{L}\mathbf{u} n \]

the magnetic induction equation (3) leads to a simple equation for the vector \( \mathbf{b}/n \):

\[ \mathcal{L}\mathbf{u} \left( \frac{\mathbf{b}}{n} \right) = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{n} \right) \mathbf{u} \]
From now on, we assume that the fluid is a perfect one: its energy-momentum tensor is

\[ T_{\text{fluid}} = (\varepsilon + p)u \otimes u + pg \]

**Simple fluid model:** all thermodynamical quantities depend on

- \( s \): entropy density in the fluid frame,
- \( n \): baryon number density in the fluid frame

Equation of state: \( \varepsilon = \varepsilon(s, n) \)

\[ \begin{cases} 
T := \frac{\partial \varepsilon}{\partial s} & \text{temperature} \\
\mu := \frac{\partial \varepsilon}{\partial n} & \text{baryon chemical potential}
\end{cases} \]

First law of thermodynamics \( \implies p = -\varepsilon + Ts + \mu n \)

\( \implies \) enthalpy per baryon: \( h = \frac{\varepsilon + p}{n} = \mu + TS \), with \( S := \frac{s}{n} \) (entropy per baryon)
Conservation of energy-momentum

Conservation law for the total energy-momentum:

$$\nabla \cdot (T^\text{fluid} + T^\text{em}) = 0$$  \hspace{1cm} \text{(4)}

- from Maxwell equations, $\nabla \cdot T^\text{em} = - F \cdot \vec{j}$
- using the baryon number conservation, $\nabla \cdot T^\text{fluid}$ can be decomposed in two parts:
  - along $\vec{u}$: $\vec{u} \cdot \nabla \cdot T^\text{fluid} = - nT \vec{u} \cdot dS$
  - orthogonal to $\vec{u}$: $\perp \vec{u} \nabla \cdot T^\text{fluid} = n(\vec{u} \cdot d(\mu h) - T dS)$

(Synge 1937) [Lichnerowicz 1941] [Taub 1959] [Carter 1979]

Since $\vec{u} \cdot F \cdot \vec{j} = 0$, Eq. (4) is equivalent to the system

$$\vec{u} \cdot dS = 0$$ \hspace{1cm} \text{(5)}

$$\vec{u} \cdot d(\mu h) - T dS = \frac{1}{n} F \cdot \vec{j}$$  \hspace{1cm} \text{(6)}

Eq. (6) is the MHD-Euler equation in canonical form.
Example of application: Kelvin’s theorem

$C(\tau)$: closed contour dragged along by the fluid (proper time $\tau$)

**Fluid circulation around** $C(\tau)$: $C(\tau) := \oint_{C(\tau)} h\mathbf{u}$

Variation of the circulation as the contour is dragged by the fluid:

\[
\frac{dC}{d\tau} = \frac{d}{d\tau} \oint_{C(\tau)} h\mathbf{u} = \oint_{C(\tau)} \mathcal{L}_{\mathbf{u}} (h\mathbf{u}) = \oint_{C(\tau)} \mathbf{u} \cdot d(h\mathbf{u}) + \oint_{C(\tau)} d(h\mathbf{u} \cdot \mathbf{u})^{-1}
\]

where the last equality follows from Cartan’s identity

Now, since $C(\tau)$ is closed, $\oint_{C(\tau)} dh = 0$

Using the MHD-Euler equation (6), we thus get

\[
\frac{dC}{d\tau} = \oint_{C(\tau)} \left( T dS + \frac{1}{n} F \cdot \mathbf{j} \right)
\]

If $\mathbf{F} \cdot \mathbf{j} = 0$ (force-free MHD) and $T = \text{const}$ or $S = \text{const}$ on $C(\tau)$, then $C$ is conserved (Kelvin’s theorem)
1 Relativistic MHD with exterior calculus

2 Stationary and axisymmetric electromagnetic fields in general relativity

3 Stationary and axisymmetric MHD
Assume that \((\mathcal{M}, g)\) is endowed with two symmetries:

1. **Stationarity**: \(\exists\) a group action of \((\mathbb{R}, +)\) on \(\mathcal{M}\) such that
   - the orbits are timelike curves
   - \(g\) is invariant under the \((\mathbb{R}, +)\) action:
     \[
     \mathcal{L}_{\xi} g = 0
     \]  
     (7)

2. **Axisymmetry**: \(\exists\) a group action of \(SO(2)\) on \(\mathcal{M}\) such that
   - the set of fixed points is a 2-dimensional submanifold \(\Delta \subset \mathcal{M}\) (called the rotation axis)
   - \(g\) is invariant under the \(SO(2)\) action:
     \[
     \mathcal{L}_{\chi} g = 0
     \]  
     (8)

(7) and (8) are equivalent to **Killing equations**:

\[
\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \quad \text{and} \quad \nabla_\alpha \chi_\beta + \nabla_\beta \chi_\alpha = 0
\]
No generality is lost by considering that the stationary and axisymmetric actions commute [Carter 1970]:

$$(\mathcal{M}, g)$$ is invariant under the action of the Abelian group $(\mathbb{R}, +) \times \text{SO}(2)$, and not only under the actions of $(\mathbb{R}, +)$ and $\text{SO}(2)$ separately. It is equivalent to say that the Killing vectors commute:

$$[\vec{\xi}, \vec{\chi}] = 0$$
No generality is lost by considering that the stationary and axisymmetric actions commute \cite{Carter1970}:

\((\mathcal{M}, g)\) is invariant under the action of the Abelian group \((\mathbb{R}, +) \times \text{SO}(2)\), and not only under the actions of \((\mathbb{R}, +)\) and \text{SO}(2) separately. It is equivalent to say that the Killing vectors commute:

\[
[\vec{\xi}, \vec{\chi}] = 0
\]

\implies \exists \text{ coordinates } (x^\alpha) = (t, x^1, x^2, \varphi) \text{ on } \mathcal{M} \text{ such that } \vec{\xi} = \frac{\partial}{\partial t} \text{ and } \vec{\chi} = \frac{\partial}{\partial \varphi}

Within them, \(g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2)\)

Adapted coordinates are not unique:

\[
\begin{align*}
t' &= t + F_0(x^1, x^2) \\
x'^1 &= F_1(x^1, x^2) \\
x'^2 &= F_2(x^1, x^2) \\
\varphi' &= \varphi + F_3(x^1, x^2)
\end{align*}
\]
Assume that the electromagnetic field is both stationary and axisymmetric:

\[ \mathcal{L}_\xi F = 0 \quad \text{and} \quad \mathcal{L}_\chi F = 0 \]  \hspace{1cm} (9)

Cartan identity and Maxwell eq. \[ \implies \mathcal{L}_\xi F = \bar{\xi} \cdot \frac{dF}{0} + d(\bar{\xi} \cdot F) = d(\bar{\xi} \cdot F) \]

Hence (9) is equivalent to

\[ d(\bar{\xi} \cdot F) = 0 \quad \text{and} \quad d(\bar{\chi} \cdot F) = 0 \]

Poincaré lemma \[ \implies \exists \text{ locally two scalar fields } \Phi \text{ and } \Psi \text{ such that} \]

\[ \bar{\xi} \cdot F = -d\Phi \quad \text{and} \quad \bar{\chi} \cdot F = -d\Psi \]
Assume that the electromagnetic field is both stationary and axisymmetric:

\[ \mathcal{L}_\xi F = 0 \quad \text{and} \quad \mathcal{L}_\chi F = 0 \] (9)

Cartan identity and Maxwell eq. \[ \implies \mathcal{L}_\xi F = \xi \cdot \underbrace{\text{d}F}_0 + \text{d}(\xi \cdot F) = \text{d}(\xi \cdot F) \]

Hence (9) is equivalent to

\[ \text{d}(\xi \cdot F) = 0 \quad \text{and} \quad \text{d}(\chi \cdot F) = 0 \]

Poincaré lemma \[ \implies \exists \text{ locally two scalar fields } \Phi \text{ and } \Psi \text{ such that} \]

\[ \xi \cdot F = -\text{d}\Phi \quad \text{and} \quad \chi \cdot F = -\text{d}\Psi \]

Link with the 4-potential \( A \): one may use the gauge freedom on \( A \) to set

\[ \Phi = A \cdot \xi = A_t \quad \text{and} \quad \Psi = A \cdot \chi = A_\phi \]
Symmetries of the scalar potentials

From the definitions of $\Phi$ and $\Psi$:

- $\mathcal{L}_\xi \Phi = \xi \cdot d\Phi = -F(\xi, \xi) = 0$
- $\mathcal{L}_\chi \Psi = \chi \cdot d\Psi = -F(\chi, \chi) = 0$
- $\mathcal{L}_\chi \Phi = \chi \cdot d\Phi = -F(\xi, \chi)$
- $\mathcal{L}_\xi \Psi = \xi \cdot d\Psi = -F(\chi, \xi) = F(\xi, \chi)$
Symmetries of the scalar potentials

From the definitions of $\Phi$ and $\Psi$:
- $\mathcal{L}_\xi \Phi = \xi \cdot d\Phi = -\mathcal{F}(\xi, \xi) = 0$
- $\mathcal{L}_\chi \Psi = \chi \cdot d\Psi = -\mathcal{F}(\chi, \chi) = 0$
- $\mathcal{L}_\chi \Phi = \chi \cdot d\Phi = -\mathcal{F}(\xi, \chi)$
- $\mathcal{L}_\xi \Psi = \xi \cdot d\Psi = -\mathcal{F}(\chi, \xi) = \mathcal{F}(\xi, \chi)$

We have $d[\mathcal{F}(\xi, \chi)] = d[\xi \cdot d\Psi] = \mathcal{L}_\xi d\Psi - \xi \cdot dd\Psi = \mathcal{L}_\xi (\mathcal{F} \cdot \chi) = 0$

Hence $\mathcal{F}(\xi, \chi) = \text{const}$

Assuming that $\mathcal{F}$ vanishes somewhere in $\mathcal{M}$ (for instance at spatial infinity), we conclude that $\mathcal{F}(\xi, \chi) = 0$

Then $\mathcal{L}_\xi \Phi = \mathcal{L}_\chi \Phi = 0$ and $\mathcal{L}_\xi \Psi = \mathcal{L}_\chi \Psi = 0$

i.e. the scalar potentials $\Phi$ and $\Psi$ obey to the two spacetime symmetries
Most general stationary-axisymmetric electromagnetic field

\[ F = d\Phi \wedge \xi^* + d\Psi \wedge \chi^* + \frac{I}{\sigma} \epsilon(\vec{\xi}, \vec{\chi}, \ldots) \]  

(10)

\[ \star F = \epsilon(\vec{\nabla} \Phi, \vec{\xi}^*, \ldots) + \epsilon(\vec{\nabla} \Psi, \vec{\chi}^*, \ldots) - \frac{I}{\sigma} \xi \wedge \chi \]  

(11)

with

- \( \xi^* := \frac{1}{\sigma} (-X \xi + W \chi) \), \( \chi^* := \frac{1}{\sigma} (W \xi + V \chi) \)
- \( V := -\xi \cdot \vec{\xi}, \ W := \xi \cdot \vec{\chi}, \ X := \chi \cdot \vec{\chi} \)
- \( \sigma := VX + W^2 \)
- \( I := \star F(\vec{\xi}, \vec{\chi}) \) ← the only non-trivial scalar, apart from \( F(\vec{\xi}, \vec{\chi}) \), one can form from \( F, \vec{\xi} \) and \( \vec{\chi} \)

\( (\xi^*, \chi^*) \) is the dual basis of \( (\vec{\xi}, \vec{\chi}) \) in the 2-plane \( \Pi := \text{Vect}(\vec{\xi}, \vec{\chi}) \):

- \( \xi^* \cdot \vec{\xi} = 1, \ \xi^* \cdot \vec{\chi} = 0, \ \chi^* \cdot \vec{\xi} = 0, \ \chi^* \cdot \vec{\chi} = 1 \)
- \( \forall \vec{v} \in \Pi^\perp, \ \xi^* \cdot \vec{v} = 0 \) and \( \chi^* \cdot \vec{v} = 0 \)
Most general stationary-axisymmetric electromagnetic field

The proof

Consider the 2-form \( H := F - d\Phi \wedge \xi^* - d\Psi \wedge \chi^* \)

It satisfies

\[
H(\vec{\xi},..) = F(\vec{\xi},..) - (\vec{\xi} \cdot d\Phi)\xi^* + (\xi^* \cdot \vec{\xi})d\Phi - (\vec{\xi} \cdot d\Psi)\chi^* + (\chi^* \cdot \vec{\xi})d\Psi = 0
\]

Similarly \( H(\vec{\chi},..) = 0 \). Hence \( H|_\Pi = 0 \)

On \( \Pi^\perp \), \( H|_{\Pi^\perp} \) is a 2-form. Another 2-form on \( \Pi^\perp \) is \( \epsilon(\vec{\xi},\vec{\chi},..)|_{\Pi^\perp} \)

Since \( \text{dim} \Pi^\perp = 2 \) and \( \epsilon(\vec{\xi},\vec{\chi},..)|_{\Pi^\perp} \neq 0 \), \( \exists \) a scalar field \( I \) such that

\[
H|_{\Pi^\perp} = \frac{I}{\sigma} \epsilon(\vec{\xi},\vec{\chi},..)|_{\Pi^\perp}
\]

Because both \( H \) and \( \epsilon(\vec{\xi},\vec{\chi},..) \) vanish on \( \Pi \), we can extend the equality to all space:

\[
H = \frac{I}{\sigma} \epsilon(\vec{\xi},\vec{\chi},..)
\]

Thus \( F \) has the form (10). Taking the Hodge dual gives the form (11) for \( \star F \), on which we readily check that \( I = \star F(\vec{\xi},\vec{\chi}) \), thereby completing the proof.
Example: Kerr-Newman electromagnetic field

Using Boyer-Lindquist coordinates \((t, r, \theta, \varphi)\), the electromagnetic field of the Kerr-Newman solution (charged rotating black hole) is

\[
F = \frac{\mu_0 Q}{4\pi (r^2 + a^2 \cos^2 \theta)^2} \left\{ \frac{(r^2 - a^2 \cos^2 \theta)}{r^2 + a^2 \cos^2 \theta} dr - a^2 r \sin 2\theta \, d\theta \right\} \wedge dt + \frac{a (a^2 \cos^2 \theta - r^2)}{r^2 + a^2 \cos^2 \theta} \sin^2 \theta \, dr + ar (r^2 + a^2) \sin 2\theta \, d\theta \right\} \wedge d\varphi
\]

\(Q\): total electric charge, \(a := J/M\): reduced angular momentum

For Kerr-Newman, \(\xi^* = dt\) and \(\chi^* = d\varphi\); comparison with (10) leads to

\[
\Phi = -\frac{\mu_0 Q}{4\pi} \frac{r}{r^2 + a^2 \cos^2 \theta}, \quad \Psi = \frac{\mu_0 Q}{4\pi} \frac{ar \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}, \quad I = 0
\]

Non-rotating limit \((a = 0)\): Reissner-Nordström solution: \(\Phi = -\frac{\mu_0 Q}{4\pi r}, \Psi = 0\)
Maxwell equations

First Maxwell equation: $dF = 0$

It is automatically satisfied by the form (10) of $F$

Second Maxwell equation: $d \star F = \mu_0 \star \vec{j}$

It gives the electric 4-current:

$$\mu_0 \vec{j} = a \vec{\xi} + b \vec{\chi} + \frac{1}{\sigma} \vec{\epsilon}(\vec{\xi}, \vec{\chi}, \nabla I, \ldots)$$  \hspace{1cm} (12)

with

- $a := \nabla_\mu \left( \frac{X}{\sigma} \nabla^\mu \Phi - \frac{W}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[ -X \mathcal{C}(\vec{\xi}) + W \mathcal{C}(\vec{\chi}) \right]$
- $b := -\nabla_\mu \left( \frac{W}{\sigma} \nabla^\mu \Phi + \frac{V}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[ W \mathcal{C}(\vec{\xi}) + V \mathcal{C}(\vec{\chi}) \right]$
- $\mathcal{C}(\vec{\xi}) := \star(\vec{\xi} \wedge \vec{\chi} \wedge d\vec{\xi}) = \epsilon^{\mu \nu \rho \sigma} \xi_\mu \chi_\nu \nabla_\rho \xi_\sigma$ (circularity factor)
- $\mathcal{C}(\vec{\chi}) := \star(\vec{\xi} \wedge \vec{\chi} \wedge d\vec{\chi}) = \epsilon^{\mu \nu \rho \sigma} \xi_\mu \chi_\nu \nabla_\rho \chi_\sigma$ (circularity factor)

Remark: $\vec{j}$ has no meridional component (i.e. $\vec{j} \in \Pi \iff dI = 0$)
Simplification for circular spacetimes

Spacetime \((\mathcal{M}, g)\) is circular \(\iff\) the planes \(\Pi^\perp\) are integrable in 2-surfaces
\[ C(\vec{\xi}) = C(\vec{\chi}) = 0 \]

**Generalized Papapetrou theorem** [Papapetrou 1966] [Kundt & Trümper 1966] [Carter 1969]:
a stationary and axisymmetric spacetime ruled by the Einstein equation is circular iff the total energy-momentum tensor \(T\) obeys to
\[
\xi^\mu T_\mu^{[\alpha \xi^\beta \chi^\gamma]} = 0 \\
\chi^\mu T_\mu^{[\alpha \xi^\beta \chi^\gamma]} = 0
\]

Examples:

- **circular spacetimes**: Kerr-Newman, rotating star, magnetized rotating star with either purely poloidal magnetic field or purely toroidal magnetic field
- **non-circular spacetimes**: rotating star with meridional flow, magnetized rotating star with mixed magnetic field

In what follows, we do **not** assume that \((\mathcal{M}, g)\) is circular
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Perfect conductor hypothesis (1/2)

with the fluid 4-velocity decomposed as

\[ \vec{u} = \lambda (\vec{\xi} + \Omega \vec{\chi}) + \vec{w}, \quad \vec{w} \in \Pi^\perp \]

\( \vec{w} \) is the meridional flow

\( u \cdot \vec{u} = -1 \iff \lambda = \sqrt{\frac{1 + w \cdot \vec{w}}{V - 2\Omega W - \Omega^2 X}} \)
Perfect conductor hypothesis (1/2)

\[ F \cdot \vec{u} = 0 \]

with the fluid 4-velocity decomposed as
\[ \vec{u} = \lambda (\vec{\xi} + \Omega \vec{\chi}) + \vec{w}, \quad \vec{w} \in \Pi^\perp \]
\( \vec{w} \) is the meridional flow
\[ \vec{u} \cdot \vec{u} = -1 \iff \lambda = \sqrt{\frac{1 + \vec{w} \cdot \vec{w}}{V - 2\Omega W - \Omega^2 X}} \]

We have
\[ \mathcal{L}_{\vec{u}} \Phi = 0 \quad \text{and} \quad \mathcal{L}_{\vec{u}} \Psi = 0, \]
(13)
i.e. the scalar potentials \( \Phi \) and \( \Psi \) are constant along the fluid lines.

**Proof:** \( \mathcal{L}_{\vec{u}} \Phi = \vec{u} \cdot \text{d}\Phi = -F(\vec{\xi}, \vec{u}) = 0 \) by the perfect conductor property.

**Corollary:** since we had already \( \mathcal{L}_{\vec{\xi}} \Phi = \mathcal{L}_{\vec{\chi}} \Phi = 0 \) and \( \mathcal{L}_{\vec{\xi}} \Psi = \mathcal{L}_{\vec{\chi}} \Psi = 0 \), it follows from (13) that
\[ \vec{w} \cdot \text{d}\Phi = 0 \quad \text{and} \quad \vec{w} \cdot \text{d}\Psi = 0 \]
Expressing the condition $F \cdot \vec{u} = 0$ with the general form of a stationary-axisymmetric electromagnetic field yields

$$
\begin{align*}
&\left(\xi^* \cdot \vec{u}\right) \, d\Phi - \left(d\Phi \cdot \vec{u}\right) \xi^* + \left(\chi^* \cdot \vec{u}\right) d\Psi - \left(d\Psi \cdot \vec{u}\right) \chi^* + \frac{I}{\sigma} \epsilon(\xi, \chi, .., \vec{u}) = 0 \\
&\lambda \quad \lambda \Omega \\
&0 \quad 0
\end{align*}
$$

Hence

$$d\Phi = -\Omega d\Psi + \frac{I}{\sigma \lambda} \epsilon(\vec{\xi}, \vec{\chi}, \vec{w}, ..)$$

(14)
Case $d\Psi \neq 0$

$d\Psi \neq 0 \implies \dim \text{Vect}(\vec{\xi}, \vec{\chi}, \nabla \Psi) = 3$

Consider the 1-form $q := \epsilon(\vec{\xi}, \vec{\chi}, \nabla \Psi, .)$. It obeys

\[ q \cdot \vec{\xi} = 0, \quad q \cdot \vec{\chi} = 0, \quad q \cdot \nabla \Psi = 0 \]

Besides

\[ w \cdot \vec{\xi} = 0, \quad w \cdot \vec{\chi} = 0, \quad w \cdot \nabla \Psi = 0 \]

Hence the 1-forms $q$ and $w$ must be proportional: $\exists$ a scalar field $a$ such that

\[ w = a \epsilon(\vec{\xi}, \vec{\chi}, \nabla \Psi, .) \quad (15) \]

A consequence of the above relation is

\[ \epsilon(\vec{\xi}, \vec{\chi}, \vec{w}, .) = a\sigma d\Psi \]

\[ a = 0 \iff \text{no meridional flow} \]
Inserting $\epsilon(\vec{\xi}, \vec{\chi}, \vec{w}, .) = a\sigma \, d\Psi$ into the perfect conductor relation (14) yields

$$d\Phi = -\omega \, d\Psi$$

(16)

with

$$\omega := \Omega - \frac{a I}{\lambda}$$

(16) implies

$$d\omega \wedge d\Psi = 0$$

from which we deduce that $\omega$ is a function of $\Psi$:

$$\omega = \omega(\Psi)$$

**Remark:** for a pure rotating flow ($a = 0$), $\omega = \Omega$
Expression of the electromagnetic field with $d\Psi \neq 0$

\[ F = d\Psi \wedge (\chi^* - \omega \xi^*) + \frac{I}{\sigma} \epsilon(\vec{\xi}, \vec{\chi}, ..,.) \]

\[ *F = \epsilon(\vec{\nabla} \Psi, \vec{\chi}^* - \omega \vec{\xi}^*, ..,.) - \frac{I}{\sigma} \vec{\xi} \wedge \vec{\chi} \]
Conservation of baryon number

Taking the Lie derivative along $\vec{u}$ of the relation $\epsilon(\vec{\xi}, \vec{\chi}, \vec{u}, .) = a\sigma \, d\Psi$ and using $\mathcal{L}_{\vec{u}} \epsilon = (\nabla \cdot \vec{u})\epsilon$ yields

$$\mathcal{L}_{\vec{u}} (a\sigma) - a\sigma \nabla \cdot \vec{u} = 0$$

Invoking the baryon number conservation

$$\nabla \cdot \vec{u} = -\frac{1}{n} \mathcal{L}_{\vec{u}} n$$

leads to

$$\mathcal{L}_{\vec{u}} K = 0$$

where

$$K := an\sigma$$

$K$ is thus constant along the fluid lines.
Conservation of baryon number

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$K$ is thus constant along the fluid lines.

Moreover, we have

$$dK \cdot \vec{\xi} = 0, \quad dK \cdot \vec{\chi} = 0, \quad dK \cdot \vec{w} = 0$$

$$d\Psi \cdot \vec{\xi} = 0, \quad d\Psi \cdot \vec{\chi} = 0, \quad d\Psi \cdot \vec{w} = 0$$

Hence $dK \propto d\Psi$ and

$$K = K(\Psi)$$
Comparison with previous work
Bekenstein & Oron (1978)

[Bekenstein & Oron (1978)] have shown that the quantity

\[ C := \frac{F_{31}}{\sqrt{-gnu^2}} \]

is conserved along the fluid lines.

We have \[ C = \frac{1}{K} \]

Remark: for a purely rotational fluid motion (\( \vec{w} = 0 \iff a = 0 \iff K = 0 \)),

\[ C \to \infty \]
Helical vector

Let us introduce \( \vec{k} = \vec{\xi} + \omega \vec{\chi} \).

Since in general \( \omega \) is not constant, \( \vec{k} \) is not a Killing vector. However:

- \( \nabla \cdot \vec{k} = 0 \)
- for any scalar field \( f \) that obeys to spacetime symmetries, \( \mathcal{L}_{\vec{k}} f = 0 \)
- \( \mathcal{L}_{\vec{k}} \vec{u} = 0 \)

All these properties are readily verified.

Moreover, \( \vec{k} \cdot F = \vec{\xi} \cdot F + \omega \vec{\chi} \cdot F = 0 \):

\[
- d\Phi - d\Psi
\]

\( \vec{k} \cdot F = 0 \)
A conserved quantity from the MHD-Euler equation

From now on, we make use of the MHD-Euler equation (6):

\[ \vec{u} \cdot d(h\vec{u}) - T dS = \frac{1}{n} F \cdot \vec{j} \]

Let us apply this equality between 1-forms to the helical vector \( \vec{k} \):

\[ \vec{u} \cdot d(h\vec{u}) \cdot \vec{k} - T \vec{k} \cdot dS = \frac{1}{n} F(\vec{k}, \vec{j}) \]

Now, from previously listed properties of \( \vec{k} \), \( \vec{k} \cdot dS = 0 \) and \( F(\vec{k}, \vec{j}) = 0 \).

Hence there remains

\[ \vec{k} \cdot d(h\vec{u}) \cdot \vec{u} = 0 \]  \hspace{1cm} (17)
Besides, via Cartan’s identity,

\[ \vec{k} \cdot d(hu) = \mathcal{L}_{\vec{k}} (hu) - d(hu \cdot \vec{k}) = \mathcal{L}_\xi (hu) + \omega \mathcal{L}_\chi (hu) + (hu \cdot \vec{\chi}) d\omega - d(hu \cdot \vec{k}) \]

Hence Eq. (17) becomes

\[ (hu \cdot \vec{\chi}) \overline{u} \cdot d\omega - \overline{u} \cdot d(hu \cdot \vec{k}) = 0 \]

Thus we conclude

\[ \mathcal{L}_{\overline{u}} D = 0 \]

where

\[ D := hu \cdot \vec{k} \]
Restart previous computation with $\vec{\xi}$ instead of $\vec{k}$:

MHD-Euler equation $\implies \vec{u} \cdot d(hu) \cdot \vec{\xi} - T \underbrace{\vec{\xi} \cdot dS}_{0} = \frac{1}{n} \underbrace{F(\vec{\xi}, \vec{j})}_{-d\Phi \cdot \vec{j}}$

Since $d\Phi = -\omega d\Psi$, we get

$$\vec{u} \cdot d(hu) \cdot \vec{\xi} = \frac{\omega}{n} \vec{j} \cdot d\Psi$$

Cartan ident. $\implies \vec{\xi} \cdot d(hu) = \mathcal{L}_\vec{\xi}(hu) - d(hu \cdot \vec{\xi}) \implies d(hu) \cdot \vec{\xi} = d(hu \cdot \vec{\xi})$

Hence

$$\mathcal{L}_\vec{u}(hu \cdot \vec{\xi}) = \frac{\omega}{n} \vec{j} \cdot d\Psi \quad (18)$$

There remains to evaluate the term $\vec{j} \cdot d\Psi$
Another conserved quantity from the MHD-Euler equation

From the expression (12) for \( \vec{j} \) along with the properties \( \vec{\xi} \cdot d\Psi = 0 \) and \( \vec{\chi} \cdot d\Psi = 0 \), we get

\[
\vec{j} \cdot d\Psi = \frac{1}{\mu_0 \sigma} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla}I, \vec{\nabla}\Psi) = -\frac{1}{\mu_0 \sigma} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla}\Psi, \vec{\nabla}I)
\] (19)
Another conserved quantity from the MHD-Euler equation

From the expression (12) for $\vec{j}$ along with the properties $\vec{\xi} \cdot d\Psi = 0$ and $\vec{\chi} \cdot d\Psi = 0$, we get

$$\vec{j} \cdot d\Psi = \frac{1}{\mu_0 \sigma} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla} \Psi) = -\frac{1}{\mu_0} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} \Psi, \vec{\nabla} I) \quad (19)$$

Two cases must be considered:

(i) $a = 0$ ($\vec{\omega} = 0$):

$$\vec{u} = \lambda (\vec{\xi} + \Omega \vec{\chi}) \implies \mathcal{L}_\vec{u} (h\vec{u} \cdot \vec{\xi}) = 0.$$

Eqs. (18) and (19) then yield

$$\epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} \Psi, \vec{\nabla} I) = 0$$

from which we deduce

$$dI \propto d\Psi$$

and

$$I = I(\Psi)$$
Another conserved quantity from the MHD-Euler equation

(ii) \( a \neq 0 \) (\( \vec{w} \neq 0 \)): then Eq. (15) gives

\[
\epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} \Psi, \cdot) = \frac{1}{a} \vec{w}
\]

and we may write (19) as

\[
\vec{j} \cdot d\Psi = -\frac{1}{\mu_0 a\sigma} \vec{w} \cdot \vec{\nabla} I = -\frac{1}{\mu_0 a\sigma} \vec{w} \cdot dI = -\frac{1}{\mu_0 a\sigma} \vec{u} \cdot dI = -\frac{1}{\mu_0 a\sigma} \mathcal{L} \vec{u} \cdot I
\]

Thus Eq. (18) becomes, using \( K = a n \sigma \),

\[
\mathcal{L} \vec{u} (h \vec{u} \cdot \vec{\xi}) = -\frac{\omega}{\mu_0 K} \mathcal{L} \vec{u} \cdot I
\]

Since \( \mathcal{L} \vec{u} \omega = 0 \) and \( \mathcal{L} \vec{u} K = 0 \), we obtain

\[
\mathcal{L} \vec{u} E = 0,
\]

with

\[
E := -h \vec{u} \cdot \vec{\xi} - \frac{\omega I}{\mu_0 K}
\] (20)
Another conserved quantity from the MHD-Euler equation

Similarly, using $\vec{\chi}$ instead of $\vec{\xi}$, we arrive at

$$\mathcal{L}_{\vec{u}} (h\vec{u} \cdot \vec{\chi}) = \frac{1}{\mu_0 n_\sigma} \epsilon(\vec{\xi}, \vec{\chi}, \nabla \Psi, \nabla I)$$

Again we have to distinguish two cases:

(i) $a = 0 (\vec{w} = 0)$: then $\mathcal{L}_{\vec{u}} (h\vec{u} \cdot \vec{\chi}) = 0$ and we recover $I = I(\Psi)$ as above

(ii) $a \neq 0 (\vec{w} \neq 0)$: we obtain then

$$\mathcal{L}_{\vec{u}} L = 0,$$

with

$$L := h\vec{u} \cdot \vec{\chi} - \frac{I}{\mu_0 K}$$

(21)

Remark: the conserved quantities $D$, $E$ and $L$ are not independent since

$$D = -E + \omega L$$
Summary

- **For purely rotational fluid motion** \((a = 0)\): any scalar quantity which obeys to the spacetime symmetries is conserved along the fluid lines.

- **For a fluid motion with meridional components** \((a \neq 0)\): there exist four scalar quantities which are constant along the fluid lines: \(\omega, K, E, L\)

\((D\) being a combination of \(\omega, E\) and \(L\))

If there is no electromagnetic field, \(E = -hu \cdot \xi\) and the constancy of \(E\) along the fluid lines is the **relativistic Bernoulli theorem**.
The constancy of $\omega$, $K$, $D$, $E$ and $L$ along the fluid lines has been shown first by Bekenstein & Oron (1978).

Bekenstein & Oron have provided coordinate-dependent definitions of $\omega$ and $K$, namely

$$\omega := -\frac{F_{01}}{F_{31}} \quad \text{and} \quad K^{-1} := \frac{F_{31}}{\sqrt{-gnu^2}}$$

Besides, they have obtained expressions for $E$ and $L$ slightly more complicated than (20) and (21), namely

$$E = - \left( h + \frac{|b|^2}{\mu_0 n} \right) u \cdot \vec{\xi} - \frac{1}{\mu_0 K} (u \cdot \vec{k})(b \cdot \vec{\xi})$$

$$L = \left( h + \frac{|b|^2}{\mu_0 n} \right) u \cdot \vec{\chi} + \frac{1}{\mu_0 K} (u \cdot \vec{k})(b \cdot \vec{\chi})$$

It can be shown that these expressions are equivalent to (20) and (21).


A. Lichnerowicz: *Relativistic hydrodynamics and magnetohydrodynamics*, Benjamin (New York) (1967)