A geometrical approach to relativistic magnetohydrodynamics

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based on a collaboration with

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Plan

Outline

General framework and notations

Spacetime:

- $\mathscr{M}$: four-dimensional orientable real manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$, $\mathrm{sign}(\boldsymbol{g}) = (-,+,+,+)$
- $\boldsymbol{\epsilon}$: Levi-Civita tensor (volume element 4-form) associated with $\boldsymbol{g}$: for any orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$,

$$\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = \pm 1$$

Notations:

- $\vec{\boldsymbol{v}}$ vector $\Longrightarrow$ $\underline{\boldsymbol{v}}$ linear form associated to $\vec{\boldsymbol{v}}$ by the metric tensor:

$$\underline{\boldsymbol{v}} := \boldsymbol{g}(\vec{\boldsymbol{v}}, .) \qquad [\underline{\boldsymbol{v}} = \boldsymbol{v}^\flat] \qquad [u_\alpha = g_{\alpha\mu} u^\mu]$$

- $\vec{\boldsymbol{v}}$ vector, $\boldsymbol{T}$ multilinear form (valence n) $\Longrightarrow$ $\vec{\boldsymbol{v}} \cdot \boldsymbol{T}$ and $\boldsymbol{T} \cdot \vec{\boldsymbol{v}}$ multilinear forms (valence $n-1$) defined by

$$\vec{\boldsymbol{v}} \cdot \boldsymbol{T} := \boldsymbol{T}(\vec{\boldsymbol{v}}, ., \ldots, .) \qquad [(\vec{\boldsymbol{v}} \cdot \boldsymbol{T})_{\alpha_1 \cdots \alpha_{n-1}} = v^\mu T_{\mu \alpha_1 \cdots \alpha_{n-1}}]$$

$$\boldsymbol{T} \cdot \vec{\boldsymbol{v}} := \boldsymbol{T}(., \ldots, ., \vec{\boldsymbol{v}}) \qquad [(\boldsymbol{T} \cdot \vec{\boldsymbol{v}})_{\alpha_1 \cdots \alpha_{n-1}} = T_{\alpha_1 \cdots \alpha_{n-1} \mu} v^\mu]$$

Maxwell equations

Electromagnetic field in $\mathscr{M}$: 2-form $\boldsymbol{F}$ which obeys to Maxwell equations:

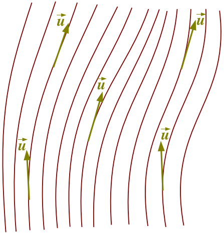
$$\boxed{\mathbf{d}\boldsymbol{F} = 0}$$

$$\boxed{\mathbf{d}\star\boldsymbol{F} = \mu_0 \star\underline{\boldsymbol{j}}}$$

- $\mathbf{d}\boldsymbol{F}$: exterior derivative of $\boldsymbol{F}$: $(\mathbf{d}\boldsymbol{F})_{\alpha\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}$
- $\star\boldsymbol{F}$: Hodge dual of $\boldsymbol{F}$: $\star F_{\alpha\beta} := \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$
- $\star\underline{\boldsymbol{j}}$ 3-form Hodge-dual of the 1-form $\underline{\boldsymbol{j}}$ associated to the electric 4-current $\vec{\boldsymbol{j}}$: $\star\underline{\boldsymbol{j}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{j}}, ., ., .)$
- μ_0 : magnetic permeability of vacuum

Electric and magnetic fields in the fluid frame

Fluid : congruence of worldlines $\Longrightarrow$ 4-velocity $\vec{u}$



- **Electric field** in the fluid frame: 1-form $\boxed{e = F \cdot \vec{u}}$
- **Magnetic field** in the fluid frame: vector $\vec{b}$ such that $\boxed{\underline{b} = \vec{u} \cdot \star F}$

e and $\vec{b}$ are orthogonal to $\vec{u}$: $e \cdot \vec{u} = 0$ and $\underline{b} \cdot \vec{u} = 0$

$$F = \underline{u} \wedge e + \epsilon(\vec{u}, \vec{b}, ., .)$$

$$\star F = -\underline{u} \wedge \underline{b} + \epsilon(\vec{u}, \vec{e}, ., .)$$

Perfect conductor

Fluid is a perfect conductor $\iff$ $\vec{e} = 0$ $\iff$ $\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0}$

From now on, we assume that the fluid is a perfect conductor (ideal MHD)

The electromagnetic field is then entirely expressible in terms of vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{b}}$:

$$\boxed{\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .)}$$

$$\boxed{\star\boldsymbol{F} = \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{u}}}$$

Alfvén's theorem

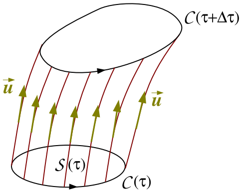
Cartan's identity applied to the 2-form $\boldsymbol{F}$:

$$\mathcal{L}_{\vec{u}}\, \boldsymbol{F} = \vec{u} \cdot \mathbf{d}\boldsymbol{F} + \mathbf{d}(\vec{u} \cdot \boldsymbol{F})$$

Now $\mathbf{d}\boldsymbol{F} = 0$ (Maxwell eq.) and $\vec{u} \cdot \boldsymbol{F} = 0$ (perfect conductor)
Hence the electromagnetic field is preserved by the flow:

$$\boxed{\mathcal{L}_{\vec{u}}\, \boldsymbol{F} = 0}$$

Application: $\boxed{\frac{d}{d\tau} \oint_{\mathcal{C}(\tau)} \boldsymbol{A} = 0}$



- τ : fluid proper time
- $\mathcal{C}(\tau)$ = closed contour dragged along by the fluid
- $\boldsymbol{A}$: electromagnetic 4-potential : $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$

Proof: $\frac{d}{d\tau} \oint_{\mathcal{C}(\tau)} \boldsymbol{A} = \frac{d}{d\tau} \int_{\mathcal{S}(\tau)} \underbrace{\mathbf{d}\boldsymbol{A}}_{\boldsymbol{F}} = \frac{d}{d\tau} \int_{\mathcal{S}(\tau)} \boldsymbol{F} = \int_{\mathcal{S}(\tau)} \underbrace{\mathcal{L}_{\vec{u}}\, \boldsymbol{F}}_{0} = 0$

Non-relativistic limit: magnetic flux freezing : $\int_{\mathcal{S}} \vec{b} \cdot d\vec{S} = \mathrm{const}$ (Alfvén's theorem)

Magnetic induction equation (1/2)

We have obsviously $\boldsymbol{F} \cdot \vec{\boldsymbol{b}} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., \vec{\boldsymbol{b}}) = 0$

In addition, $\mathcal{L}_{\vec{\boldsymbol{b}}}\, \boldsymbol{F} = \vec{\boldsymbol{b}} \cdot \underbrace{\mathrm{d}\boldsymbol{F}}_{0} + \mathrm{d}(\underbrace{\vec{\boldsymbol{b}} \cdot \boldsymbol{F}}_{0}) = 0$

Hence

$$\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{b}} = 0} \text{ and } \boxed{\mathcal{L}_{\vec{\boldsymbol{b}}}\, \boldsymbol{F} = 0}$$

similarly to

$$\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0} \text{ and } \boxed{\mathcal{L}_{\vec{\boldsymbol{u}}}\, \boldsymbol{F} = 0}$$

Magnetic induction equation (2/2)

From $\mathcal{L}_{\vec{u}}\, \boldsymbol{F} = 0$ and $\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .)$, we get

$$(\mathcal{L}_{\vec{u}}\, \boldsymbol{\epsilon})(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .) + \boldsymbol{\epsilon}(\underbrace{\mathcal{L}_{\vec{u}}\, \vec{\boldsymbol{u}}}_{0}, \vec{\boldsymbol{b}}, ., .) + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \mathcal{L}_{\vec{u}}\, \vec{\boldsymbol{b}}, ., .) = 0$$

Now $\mathcal{L}_{\vec{u}}\, \boldsymbol{\epsilon} = (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{u}})\boldsymbol{\epsilon}$, hence $\boldsymbol{\epsilon}\left(\vec{\boldsymbol{u}},\ (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{u}})\vec{\boldsymbol{b}} + \mathcal{L}_{\vec{u}}\, \vec{\boldsymbol{b}}, ., .\right) = 0$

This implies

$$\mathcal{L}_{\vec{u}}\, \vec{\boldsymbol{b}} = \alpha\, \vec{\boldsymbol{u}} - (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{u}})\vec{\boldsymbol{b}} \qquad (1)$$

Magnetic induction equation (2/2)

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$$(\mathcal{L}_{\vec{u}}\, \boldsymbol{\epsilon})(\vec{u}, \vec{b}, ., .) + \boldsymbol{\epsilon}(\underbrace{\mathcal{L}_{\vec{u}}\, \vec{u}}_{0}, \vec{b}, ., .) + \boldsymbol{\epsilon}(\vec{u}, \mathcal{L}_{\vec{u}}\, \vec{b}, ., .) = 0$$

Now $\mathcal{L}_{\vec{u}}\, \boldsymbol{\epsilon} = (\boldsymbol{\nabla} \cdot \vec{u})\boldsymbol{\epsilon}$, hence $\boldsymbol{\epsilon}\left(\vec{u},\ (\boldsymbol{\nabla} \cdot \vec{u})\vec{b} + \mathcal{L}_{\vec{u}}\, \vec{b}, ., .\right) = 0$
This implies

$$\mathcal{L}_{\vec{u}}\, \vec{b} = \alpha\, \vec{u} - (\boldsymbol{\nabla} \cdot \vec{u})\vec{b} \qquad (1)$$

Similarly, the property $\mathcal{L}_{\vec{b}}\, \boldsymbol{F} = 0$ leads to $\boldsymbol{\epsilon}\left((\boldsymbol{\nabla} \cdot \vec{b})\vec{u} + \mathcal{L}_{\vec{b}}\, \vec{u},\ \vec{b}, ., .\right) = 0$
which implies

$$\mathcal{L}_{\vec{b}}\, \vec{u} = -\mathcal{L}_{\vec{u}}\, \vec{b} = -(\boldsymbol{\nabla} \cdot \vec{b})\vec{u} + \beta\, \vec{b} \qquad (2)$$

Comparison of (1) and (2) leads to

$$\boxed{\mathcal{L}_{\vec{u}}\, \vec{b} = (\boldsymbol{\nabla} \cdot \vec{b})\vec{u} - (\boldsymbol{\nabla} \cdot \vec{u})\vec{b}} \qquad (3)$$

Non-relativistic limit: $\dfrac{\partial \vec{b}}{\partial t} = \mathrm{curl}(\vec{v} \times \vec{b})$ (induction equation)

Some simple consequences

We have $\underline{u} \cdot \vec{b} = 0 \Longrightarrow \mathcal{L}_{\vec{u}}\, \underline{u} \cdot \vec{b} + \underline{u} \cdot \mathcal{L}_{\vec{u}}\, \vec{b} = 0$

Now $\mathcal{L}_{\vec{u}}\, \underline{u} = \underline{a}$ with $\vec{a} := \nabla_{\vec{u}} \vec{u}$ (fluid 4-acceleration)

and $\underline{u} \cdot \mathcal{L}_{\vec{u}}\, \vec{b} = (\nabla \cdot \vec{b}) \underbrace{\underline{u} \cdot \vec{u}}_{-1} - (\nabla \cdot \vec{u}) \underbrace{\underline{u} \cdot \vec{b}}_{0} = -\nabla \cdot \vec{b}$

Hence

$$\boxed{\nabla \cdot \vec{b} = \underline{a} \cdot \vec{b}}$$

Some simple consequences

We have $\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{b}} = 0 \Longrightarrow \mathcal{L}_{\vec{\boldsymbol{u}}}\, \underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{b}} + \underline{\boldsymbol{u}} \cdot \mathcal{L}_{\vec{\boldsymbol{u}}}\, \vec{\boldsymbol{b}} = 0$

Now $\mathcal{L}_{\vec{\boldsymbol{u}}}\, \underline{\boldsymbol{u}} = \underline{\boldsymbol{a}}$ with $\vec{\boldsymbol{a}} := \boldsymbol{\nabla}_{\vec{\boldsymbol{u}}} \vec{\boldsymbol{u}}$ (fluid 4-acceleration)

and $\underline{\boldsymbol{u}} \cdot \mathcal{L}_{\vec{\boldsymbol{u}}}\, \vec{\boldsymbol{b}} = (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{b}}) \underbrace{\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}}_{-1} - (\boldsymbol{\nabla} \cdot \vec{\boldsymbol{u}}) \underbrace{\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{b}}}_{0} = -\boldsymbol{\nabla} \cdot \vec{\boldsymbol{b}}$

Hence

$$\boxed{\boldsymbol{\nabla} \cdot \vec{\boldsymbol{b}} = \underline{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}}$$

If we invoke **baryon number conservation**

$$\boxed{\boldsymbol{\nabla} \cdot (n\, \vec{\boldsymbol{u}}) = 0} \iff \boldsymbol{\nabla} \cdot \vec{\boldsymbol{u}} = -\frac{1}{n} \mathcal{L}_{\vec{\boldsymbol{u}}}\, n$$

the magnetic induction equation (3) leads to a simple equation for the vector $\vec{\boldsymbol{b}}/n$:

$$\boxed{\mathcal{L}_{\vec{\boldsymbol{u}}} \left(\frac{\vec{\boldsymbol{b}}}{n} \right) = \left(\underline{\boldsymbol{a}} \cdot \frac{\vec{\boldsymbol{b}}}{n} \right) \vec{\boldsymbol{u}}}$$

Perfect fluid

From now on, we assume that the fluid is a perfect one: its energy-momentum tensor is

$$\boldsymbol{T}^{\rm fluid} = (\varepsilon + p)\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + p\boldsymbol{g}$$

Simple fluid model: all thermodynamical quantities depend on

- s: entropy density in the fluid frame,
- n: baryon number density in the fluid frame

Equation of state : $\varepsilon = \varepsilon(s, n) \Longrightarrow \begin{cases} T := \dfrac{\partial \varepsilon}{\partial s} & \text{temperature} \\ \mu := \dfrac{\partial \varepsilon}{\partial n} & \text{baryon chemical potential} \end{cases}$

First law of thermodynamics $\Longrightarrow p = -\varepsilon + Ts + \mu n$

$\Longrightarrow$ enthalpy per baryon : $\boxed{h = \dfrac{\varepsilon + p}{n} = \mu + TS}$, with $S := \dfrac{s}{n}$ (entropy per baryon)

Conservation of energy-momentum

Conservation law for the total energy-momentum:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{T}^{\rm fluid}+\boldsymbol{T}^{\rm em})=0 \qquad (4)$$

- from Maxwell equations, $\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm em}=-\boldsymbol{F}\cdot\vec{\boldsymbol{j}}$
- using the baryon number conservation, $\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm fluid}$ can be decomposed in two parts:
  - along $\vec{\boldsymbol{u}}$: $\vec{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm fluid}=-nT\vec{\boldsymbol{u}}\cdot\mathbf{d}S$
  - orthogonal to $\vec{\boldsymbol{u}}$: $\boxed{\perp_{\boldsymbol{u}}\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm fluid}=n(\vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}})-T\mathbf{d}S)}$

[Synge 1937] [Lichnerowicz 1941] [Taub 1959] [Carter 1979]

Since $\vec{\boldsymbol{u}}\cdot\boldsymbol{F}\cdot\vec{\boldsymbol{j}}=0$, Eq. (4) is equivalent to the system

$$\boxed{\vec{\boldsymbol{u}}\cdot\mathbf{d}S=0} \qquad (5)$$

$$\boxed{\vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}})-T\mathbf{d}S=\frac{1}{n}\boldsymbol{F}\cdot\vec{\boldsymbol{j}}} \qquad (6)$$

Eq. (6) is the MHD-Euler equation in canonical form.

Example of application : Kelvin's theorem

$\mathcal{C}(\tau)$: closed contour dragged along by the fluid (proper time τ)

Fluid circulation around $\mathcal{C}(\tau)$: $C(\tau) := \oint_{\mathcal{C}(\tau)} h\underline{\boldsymbol{u}}$

Variation of the circulation as the contour is dragged by the fluid:

$$\frac{dC}{d\tau} = \frac{d}{d\tau}\oint_{\mathcal{C}(\tau)} h\underline{\boldsymbol{u}} = \oint_{\mathcal{C}(\tau)} \mathcal{L}_{\vec{\boldsymbol{u}}}\,(h\underline{\boldsymbol{u}}) = \oint_{\mathcal{C}(\tau)} \vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}}) + \oint_{\mathcal{C}(\tau)} \mathbf{d}(h\underbrace{\underline{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}}}_{-1})$$

where the last equality follows from Cartan's identity

Now, since $\mathcal{C}(\tau)$ is closed, $\oint_{\mathcal{C}(\tau)} \mathbf{d}h = 0$

Using the MHD-Euler equation (6), we thus get

$$\boxed{\frac{dC}{d\tau} = \oint_{\mathcal{C}(\tau)} \left(T\mathbf{d}S + \frac{1}{n}\boldsymbol{F}\cdot\vec{\boldsymbol{j}} \right)}$$

If $\boldsymbol{F}\cdot\vec{\boldsymbol{j}} = 0$ (force-free MHD) and $T = \mathrm{const}$ or $S = \mathrm{const}$ on $\mathcal{C}(\tau)$, then C is conserved (Kelvin's theorem)

Outline

Stationary and axisymmetric spacetimes

Assume that $(\mathscr{M}, \boldsymbol{g})$ is endowed with two symmetries:

1. **stationarity** : $\exists$ a group action of $(\mathbb{R}, +)$ on $\mathscr{M}$ such that
   - the orbits are timelike curves
   - $\boldsymbol{g}$ is invariant under the $(\mathbb{R}, +)$ action : if $\vec{\xi}$ is a generator of the group action,

$$\boxed{\mathcal{L}_{\vec{\xi}}\, \boldsymbol{g} = 0} \qquad (7)$$

2. **axisymmetry** : $\exists$ a group action of $\mathrm{SO}(2)$ on $\mathscr{M}$ such that
   - the set of fixed points is a 2-dimensional submanifold $\Delta \subset \mathscr{M}$ (called the *rotation axis*)
   - $\boldsymbol{g}$ is invariant under the $\mathrm{SO}(2)$ action : if $\vec{\chi}$ is a generator of the group action,

$$\boxed{\mathcal{L}_{\vec{\chi}}\, \boldsymbol{g} = 0} \qquad (8)$$

(7) and (8) are equivalent to *Killing equations*:

$$\boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0} \quad \text{and} \quad \boxed{\nabla_\alpha \chi_\beta + \nabla_\beta \chi_\alpha = 0}$$

Stationary and axisymmetric spacetimes

No generality is lost by considering that the stationary and axisymmetric actions commute [Carter 1970] :
$(\mathscr{M}, \boldsymbol{g})$ is invariant under the action of the Abelian group $(\mathbb{R},+) \times \mathrm{SO}(2)$, and not only under the actions of $(\mathbb{R},+)$ and $\mathrm{SO}(2)$ separately. It is equivalent to say that the Killing vectors commute:

$$[\vec{\xi}, \vec{\chi}] = 0$$

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$$[\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}] = 0$$

$\Longrightarrow$ $\exists$ coordinates $(x^\alpha) = (t, x^1, x^2, \varphi)$ on $\mathscr{M}$ such that $\vec{\boldsymbol{\xi}} = \dfrac{\partial}{\partial t}$ and $\vec{\boldsymbol{\chi}} = \dfrac{\partial}{\partial \varphi}$

Within them, $g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2)$

Adapted coordinates are not unique:
$$\begin{cases} t' &= t + F_0(x^1, x^2) \\ x'^1 &= F_1(x^1, x^2) \\ x'^2 &= F_2(x^1, x^2) \\ \varphi' &= \varphi + F_3(x^1, x^2) \end{cases}$$

Stationary and axisymmetric electromagnetic field

Assume that the electromagnetic field is both stationary and axisymmetric:

$$\boxed{\mathcal{L}_{\vec{\xi}}\, \boldsymbol{F} = 0} \quad \text{and} \quad \boxed{\mathcal{L}_{\vec{\chi}}\, \boldsymbol{F} = 0} \qquad (9)$$

Cartan identity and Maxwell eq. $\Longrightarrow \mathcal{L}_{\vec{\xi}}\, \boldsymbol{F} = \vec{\xi} \cdot \underbrace{\mathbf{d}\boldsymbol{F}}_{0} + \mathbf{d}(\vec{\xi} \cdot \boldsymbol{F}) = \mathbf{d}(\vec{\xi} \cdot \boldsymbol{F})$

Hence (9) is equivalent to

$$\mathbf{d}(\vec{\xi} \cdot \boldsymbol{F}) = 0 \quad \text{and} \quad \mathbf{d}(\vec{\chi} \cdot \boldsymbol{F}) = 0$$

Poincaré lemma $\Longrightarrow \exists$ locally two scalar fields Φ and Ψ such that

$$\boxed{\vec{\xi} \cdot \boldsymbol{F} = -\mathbf{d}\Phi} \quad \text{and} \quad \boxed{\vec{\chi} \cdot \boldsymbol{F} = -\mathbf{d}\Psi}$$

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Link with the 4-potential $\boldsymbol{A}$: one may use the gauge freedom on $\boldsymbol{A}$ to set

$$\Phi = \boldsymbol{A}\cdot\vec{\xi} = A_t \quad \text{and} \quad \Psi = \boldsymbol{A}\cdot\vec{\chi} = A_\varphi$$

Symmetries of the scalar potentials

From the definitions of Φ and Ψ:

- $\mathcal{L}_{\vec{\xi}}\,\Phi = \vec{\xi}\cdot\mathbf{d}\Phi = -\boldsymbol{F}(\vec{\xi},\vec{\xi}) = 0$
- $\mathcal{L}_{\vec{\chi}}\,\Psi = \vec{\chi}\cdot\mathbf{d}\Psi = -\boldsymbol{F}(\vec{\chi},\vec{\chi}) = 0$
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We have $\mathbf{d}[\boldsymbol{F}(\vec{\xi},\vec{\chi})] = \mathbf{d}[\vec{\xi}\cdot\mathbf{d}\Psi] = \mathcal{L}_{\vec{\xi}}\mathbf{d}\Psi - \vec{\xi}\cdot\underbrace{\mathbf{dd}\Psi}_{0} = \mathcal{L}_{\vec{\xi}}(\boldsymbol{F}\cdot\vec{\chi}) = 0$

Hence $\boldsymbol{F}(\vec{\xi},\vec{\chi}) = \mathrm{const}$

Assuming that $\boldsymbol{F}$ vanishes somewhere in $\mathscr{M}$ (for instance at spatial infinity), we conclude that

$$\boxed{\boldsymbol{F}(\vec{\xi},\vec{\chi}) = 0}$$

Then $\boxed{\mathcal{L}_{\vec{\xi}}\Phi = \mathcal{L}_{\vec{\chi}}\Phi = 0}$ and $\boxed{\mathcal{L}_{\vec{\xi}}\Psi = \mathcal{L}_{\vec{\chi}}\Psi = 0}$

i.e. the scalar potentials Φ and Ψ obey to the two spacetime symmetries

Most general stationary-axisymmetric electromagnetic field

$$\boxed{\boldsymbol{F} = \mathbf{d}\Phi \wedge \boldsymbol{\xi}^* + \mathbf{d}\Psi \wedge \boldsymbol{\chi}^* + \frac{I}{\sigma}\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)} \qquad (10)$$

$$\boxed{\star\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\nabla}\Phi, \vec{\xi^*}, ., .) + \boldsymbol{\epsilon}(\vec{\nabla}\Psi, \vec{\chi^*}, ., .) - \frac{I}{\sigma}\underline{\xi} \wedge \underline{\chi}} \qquad (11)$$

with

- $\boldsymbol{\xi}^* := \frac{1}{\sigma}\left(-X\,\underline{\xi} + W\underline{\chi}\right)$, $\quad \boldsymbol{\chi}^* := \frac{1}{\sigma}\left(W\,\underline{\xi} + V\underline{\chi}\right)$
- $V := -\underline{\xi} \cdot \vec{\xi}$, $\quad W := \underline{\xi} \cdot \vec{\chi}$, $\quad X := \underline{\chi} \cdot \vec{\chi}$
- $\sigma := VX + W^2$
- $\boxed{I := \star\boldsymbol{F}(\vec{\xi}, \vec{\chi})}$ ← the only non-trivial scalar, apart from $\boldsymbol{F}(\vec{\xi}, \vec{\chi})$, one can form from $\boldsymbol{F}$, $\vec{\xi}$ and $\vec{\chi}$

$(\boldsymbol{\xi}^*, \boldsymbol{\chi}^*)$ is the dual basis of $(\vec{\xi}, \vec{\chi})$ in the 2-plane $\boxed{\Pi := \mathrm{Vect}(\vec{\xi}, \vec{\chi})}$:

$$\boldsymbol{\xi}^* \cdot \vec{\xi} = 1, \quad \boldsymbol{\xi}^* \cdot \vec{\chi} = 0, \quad \boldsymbol{\chi}^* \cdot \vec{\xi} = 0, \quad \boldsymbol{\chi}^* \cdot \vec{\chi} = 1$$

$$\forall \vec{v} \in \Pi^{\perp}, \quad \boldsymbol{\xi}^* \cdot \vec{v} = 0 \quad \text{and} \quad \boldsymbol{\chi}^* \cdot \vec{v} = 0$$

Most general stationary-axisymmetric electromagnetic field

The proof

Consider the 2-form $\boldsymbol{H} := \boldsymbol{F} - \mathbf{d}\Phi \wedge \boldsymbol{\xi}^* - \mathbf{d}\Psi \wedge \boldsymbol{\chi}^*$

It satisfies

$$\boldsymbol{H}(\vec{\xi}, .) = \underbrace{\boldsymbol{F}(\vec{\xi}, .)}_{-\mathbf{d}\Phi} - \underbrace{(\vec{\xi} \cdot \mathbf{d}\Phi)}_{0}\boldsymbol{\xi}^* + \underbrace{(\boldsymbol{\xi}^* \cdot \vec{\xi})}_{1}\mathbf{d}\Phi - \underbrace{(\vec{\xi} \cdot \mathbf{d}\Psi)}_{0}\boldsymbol{\chi}^* + \underbrace{(\boldsymbol{\chi}^* \cdot \vec{\xi})}_{0}\mathbf{d}\Psi = 0$$

Similarly $\boldsymbol{H}(\vec{\chi}, .) = 0$. Hence $\boldsymbol{H}|_{\Pi} = 0$

On $\Pi^\perp$, $\boldsymbol{H}|_{\Pi^\perp}$ is a 2-form. Another 2-form on $\Pi^\perp$ is $\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)\Big|_{\Pi^\perp}$

Since $\dim \Pi^\perp = 2$ and $\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)\Big|_{\Pi^\perp} \neq 0$, $\exists$ a scalar field I such that $\boldsymbol{H}|_{\Pi^\perp} = \frac{I}{\sigma}\, \boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)\Big|_{\Pi^\perp}$. Because both $\boldsymbol{H}$ and $\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)$ vanish on Π, we can extend the equality to all space:

$$\boldsymbol{H} = \frac{I}{\sigma}\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)$$

Thus $\boldsymbol{F}$ has the form (10). Taking the Hodge dual gives the form (11) for $\star\boldsymbol{F}$, on which we readily check that $I = \star\boldsymbol{F}(\vec{\xi}, \vec{\chi})$, thereby completing the proof.

Example: Kerr-Newman electromagnetic field

Using Boyer-Lindquist coordinates (t, r, θ, φ), the electromagnetic field of the Kerr-Newman solution (charged rotating black hole) is

$$\boldsymbol{F} = \frac{\mu_0 Q}{4\pi(r^2 + a^2\cos^2\theta)^2} \Big\{ \left[(r^2 - a^2\cos^2\theta)\,\mathbf{d}r - a^2 r \sin 2\theta\,\mathbf{d}\theta \right] \wedge \mathbf{d}t + \left[a(a^2\cos^2\theta - r^2)\sin^2\theta\,\mathbf{d}r + ar(r^2+a^2)\sin 2\theta\,\mathbf{d}\theta \right] \wedge \mathbf{d}\varphi \Big\}$$

Q: total electric charge, $a := J/M$: reduced angular momentum

For Kerr-Newman, $\boldsymbol{\xi}^* = \mathbf{d}t$ and $\boldsymbol{\chi}^* = \mathbf{d}\varphi$; comparison with (10) leads to

$$\boxed{\Phi = -\frac{\mu_0 Q}{4\pi}\,\frac{r}{r^2 + a^2\cos^2\theta}}, \quad \boxed{\Psi = \frac{\mu_0 Q}{4\pi}\,\frac{ar\sin^2\theta}{r^2 + a^2\cos^2\theta}}, \quad \boxed{I = 0}$$

Non-rotating limit $(a = 0)$: Reissner-Nordström solution: $\Phi = -\frac{\mu_0}{4\pi}\frac{Q}{r}$, $\Psi = 0$

Maxwell equations

First Maxwell equation: $\mathbf{d}\boldsymbol{F} = 0$

It is automatically satisfied by the form (10) of $\boldsymbol{F}$

Second Maxwell equation: $\mathbf{d} \star \boldsymbol{F} = \mu_0 \star \underline{\boldsymbol{j}}$

It gives the electric 4-current:

$$\mu_0 \vec{\boldsymbol{j}} = a\,\vec{\boldsymbol{\xi}} + b\,\vec{\boldsymbol{\chi}} + \frac{1}{\sigma}\vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} I, .) \qquad (12)$$

with

- $a := \nabla_\mu \left(\frac{X}{\sigma} \nabla^\mu \Phi - \frac{W}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[-X \mathscr{C}(\vec{\boldsymbol{\xi}}) + W \mathscr{C}(\vec{\boldsymbol{\chi}}) \right]$
- $b := -\nabla_\mu \left(\frac{W}{\sigma} \nabla^\mu \Phi + \frac{V}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[W \mathscr{C}(\vec{\boldsymbol{\xi}}) + V \mathscr{C}(\vec{\boldsymbol{\chi}}) \right]$
- $\mathscr{C}(\vec{\boldsymbol{\xi}}) := \star(\underline{\boldsymbol{\xi}} \wedge \underline{\boldsymbol{\chi}} \wedge \mathbf{d}\underline{\boldsymbol{\xi}}) = \epsilon^{\mu\nu\rho\sigma} \xi_\mu \chi_\nu \nabla_\rho \xi_\sigma$ (circularity factor)
- $\mathscr{C}(\vec{\boldsymbol{\chi}}) := \star(\underline{\boldsymbol{\xi}} \wedge \underline{\boldsymbol{\chi}} \wedge \mathbf{d}\underline{\boldsymbol{\chi}}) = \epsilon^{\mu\nu\rho\sigma} \xi_\mu \chi_\nu \nabla_\rho \chi_\sigma$ (circularity factor)

Remark: $\vec{\boldsymbol{j}}$ has no meridional component (i.e. $\vec{\boldsymbol{j}} \in \Pi$) $\iff \mathbf{d}I = 0$

Simplification for circular spacetimes

Spacetime $(\mathscr{M}, \boldsymbol{g})$ is circular $\iff$ the planes $\Pi^{\perp}$ are integrable in 2-surfaces
$\iff$ $\mathscr{C}(\vec{\boldsymbol{\xi}}) = \mathscr{C}(\vec{\boldsymbol{\chi}}) = 0$

Generalized Papapetrou theorem [Papapetrou 1966] [Kundt & Trümper 1966] [Carter 1969] : a stationary and axisymmetric spacetime ruled by the Einstein equation is circular iff the total energy-momemtum tensor $\boldsymbol{T}$ obeys to

$$\xi^{\mu} T_{\mu}^{\ [\alpha} \xi^{\beta} \chi^{\gamma]} = 0$$
$$\chi^{\mu} T_{\mu}^{\ [\alpha} \xi^{\beta} \chi^{\gamma]} = 0$$

Examples:

- circular spacetimes: Kerr-Newman, rotating star, magnetized rotating star with either purely poloidal magnetic field or purely toroidal magnetic field
- non-circular spacetimes: rotating star with meridional flow, magnetized rotating star with mixed magnetic field

In what follows, we do **not** assume that $(\mathscr{M}, \boldsymbol{g})$ is circular

Outline

Perfect conductor hypothesis (1/2)

$$\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0}$$

with the fluid 4-velocity decomposed as

$$\boxed{\vec{\boldsymbol{u}} = \lambda(\vec{\boldsymbol{\xi}} + \Omega\vec{\boldsymbol{\chi}}) + \vec{\boldsymbol{w}}, \qquad \vec{\boldsymbol{w}} \in \Pi^{\perp}}$$

$\vec{\boldsymbol{w}}$ is the meridional flow

$$\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1 \iff \lambda = \sqrt{\frac{1 + \underline{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}}}{V - 2\Omega W - \Omega^2 X}}$$

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We have

$$\mathcal{L}_{\vec{\boldsymbol{u}}}\, \Phi = 0 \quad \text{and} \quad \mathcal{L}_{\vec{\boldsymbol{u}}}\, \Psi = 0, \qquad (13)$$

i.e. the scalar potentials Φ and Ψ are constant along the fluid lines.

Proof: $\mathcal{L}_{\vec{\boldsymbol{u}}}\, \Phi = \vec{\boldsymbol{u}} \cdot \mathbf{d}\Phi = -\boldsymbol{F}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{u}}) = 0$ by the perfect conductor property.

Corollary: since we had already $\mathcal{L}_{\vec{\boldsymbol{\xi}}}\, \Phi = \mathcal{L}_{\vec{\boldsymbol{\chi}}}\, \Phi = 0$ and $\mathcal{L}_{\vec{\boldsymbol{\xi}}}\, \Psi = \mathcal{L}_{\vec{\boldsymbol{\chi}}}\, \Psi = 0$, it follows from (13) that

$$\vec{\boldsymbol{w}} \cdot \mathbf{d}\Phi = 0 \quad \text{and} \quad \vec{\boldsymbol{w}} \cdot \mathbf{d}\Psi = 0$$

Perfect conductor hypothesis (2/2)

Expressing the condition $\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0$ with the general form of a stationary-axisymmetric electromagnetic field yields

$$\underbrace{(\boldsymbol{\xi}^* \cdot \vec{\boldsymbol{u}})}_{\lambda}\mathbf{d}\Phi - \underbrace{(\mathbf{d}\Phi \cdot \vec{\boldsymbol{u}})}_{0}\boldsymbol{\xi}^* + \underbrace{(\boldsymbol{\chi}^* \cdot \vec{\boldsymbol{u}})}_{\lambda\Omega}\mathbf{d}\Psi - \underbrace{(\mathbf{d}\Psi \cdot \vec{\boldsymbol{u}})}_{0}\boldsymbol{\chi}^* + \frac{I}{\sigma}\underbrace{\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, ., \vec{\boldsymbol{u}})}_{-\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .)} = 0$$

Hence

$$\boxed{\mathbf{d}\Phi = -\Omega\, \mathbf{d}\Psi + \frac{I}{\sigma\lambda}\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .)} \qquad (14)$$

Case $\mathbf{d}\Psi \neq 0$

$\mathbf{d}\Psi \neq 0 \Longrightarrow \dim \mathrm{Vect}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}}\Psi) = 3$

Consider the 1-form $\boldsymbol{q} := \boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}}\Psi, .)$. It obeys

$$\boldsymbol{q} \cdot \vec{\boldsymbol{\xi}} = 0, \qquad \boldsymbol{q} \cdot \vec{\boldsymbol{\chi}} = 0, \qquad \boldsymbol{q} \cdot \vec{\boldsymbol{\nabla}}\Psi = 0$$

Besides

$$\underline{\boldsymbol{w}} \cdot \vec{\boldsymbol{\xi}} = 0, \qquad \underline{\boldsymbol{w}} \cdot \vec{\boldsymbol{\chi}} = 0, \qquad \underline{\boldsymbol{w}} \cdot \vec{\boldsymbol{\nabla}}\Psi = 0$$

Hence the 1-forms $\boldsymbol{q}$ and $\underline{\boldsymbol{w}}$ must be proportional: $\exists$ a scalar field a such that

$$\boxed{\underline{\boldsymbol{w}} = a\, \boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}}\Psi, .)} \qquad (15)$$

A consequence of the above relation is

$$\boxed{\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .) = a\sigma\, \mathbf{d}\Psi}$$

$a = 0 \iff$ no meridional flow

Perfect conductor relation with $\mathbf{d}\Psi \neq 0$

Inserting $\epsilon(\vec{\xi}, \vec{\chi}, \vec{w}, .) = a\sigma\, \mathbf{d}\Psi$ into the perfect conductor relation (14) yields

$$\boxed{\mathbf{d}\Phi = -\omega\, \mathbf{d}\Psi} \qquad (16)$$

with

$$\boxed{\omega := \Omega - \frac{aI}{\lambda}}$$

(16) implies

$$\mathbf{d}\omega \wedge \mathbf{d}\Psi = 0$$

from which we deduce that ω is a function of Ψ:

$$\boxed{\omega = \omega(\Psi)}$$

Remark: for a pure rotating flow $(a = 0)$, $\omega = \Omega$

Expression of the electromagnetic field with $\mathrm{d}\Psi \neq 0$

$$\boldsymbol{F} = \mathbf{d}\Psi \wedge (\boldsymbol{\chi}^* - \omega \boldsymbol{\xi}^*) + \frac{I}{\sigma} \boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, ., .)$$

$$\star\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\boldsymbol{\nabla}}\Psi, \vec{\boldsymbol{\chi}}^* - \omega \vec{\boldsymbol{\xi}}^*, ., .) - \frac{I}{\sigma} \underline{\boldsymbol{\xi}} \wedge \underline{\boldsymbol{\chi}}$$

Conservation of baryon number

Taking the Lie derivative along $\vec{u}$ of the relation $\epsilon(\vec{\xi}, \vec{\chi}, \vec{u}, .) = a\sigma\, \mathbf{d}\Psi$ and using $\mathcal{L}_{\vec{u}}\, \epsilon = (\nabla \cdot \vec{u})\epsilon$ yields

$$\mathcal{L}_{\vec{u}}\,(a\sigma) - a\sigma \nabla \cdot \vec{u} = 0$$

Invoking the baryon number conservation

$$\boxed{\nabla \cdot \vec{u} = -\frac{1}{n}\mathcal{L}_{\vec{u}}\, n}$$

leads to

$$\boxed{\mathcal{L}_{\vec{u}}\, K = 0}$$

where

$$\boxed{K := an\sigma}$$

K is thus constant along the fluid lines.

Conservation of baryon number

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where

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K is thus constant along the fluid lines.

Moreoever, we have

$$\begin{array}{lll} \mathbf{d}K \cdot \vec{\xi} = 0, & \mathbf{d}K \cdot \vec{\chi} = 0, & \mathbf{d}K \cdot \vec{w} = 0 \\ \mathbf{d}\Psi \cdot \vec{\xi} = 0, & \mathbf{d}\Psi \cdot \vec{\chi} = 0, & \mathbf{d}\Psi \cdot \vec{w} = 0 \end{array}$$

Hence $\mathbf{d}K \propto \mathbf{d}\Psi$ and

$$\boxed{K = K(\Psi)}$$

Comparison with previous work

Bekenstein & Oron (1978)

[Bekenstein & Oron (1978)] have shown that the quantity

$$C := \frac{F_{31}}{\sqrt{-g} n u^2}$$

is conserved along the fluid lines.

We have $\boxed{C = \frac{1}{K}}$

Remark: for a purely rotational fluid motion ($\vec{\boldsymbol{w}} = 0 \iff a = 0 \iff K = 0$),

$$C \to \infty$$

Helical vector

Let us introduce $\boxed{\vec{\boldsymbol{k}} = \vec{\boldsymbol{\xi}} + \omega\, \vec{\boldsymbol{\chi}}}$

Since in general ω is not constant, $\vec{\boldsymbol{k}}$ is not a Killing vector. However

- $\boldsymbol{\nabla} \cdot \vec{\boldsymbol{k}} = 0$
- for any scalar field f that obeys to spacetime symmetries, $\mathcal{L}_{\vec{\boldsymbol{k}}}\, f = 0$
- $\mathcal{L}_{\vec{\boldsymbol{k}}}\, \vec{\boldsymbol{u}} = 0$

All these properties are readily verified.

Moreover, $\vec{\boldsymbol{k}} \cdot \boldsymbol{F} = \underbrace{\vec{\boldsymbol{\xi}} \cdot \boldsymbol{F}}_{-\mathbf{d}\Phi} + \omega\, \underbrace{\vec{\boldsymbol{\chi}} \cdot \boldsymbol{F}}_{-\mathbf{d}\Psi} = 0$:

$$\boxed{\vec{\boldsymbol{k}} \cdot \boldsymbol{F} = 0}$$

A conserved quantity from the MHD-Euler equation

From now on, we make use of the MHD-Euler equation (6):

$$\vec{\boldsymbol{u}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}}) - T\mathbf{d}S = \frac{1}{n}\boldsymbol{F} \cdot \vec{\boldsymbol{j}}$$

Let us apply this equality between 1-forms to the helical vector $\vec{\boldsymbol{k}}$:

$$\vec{\boldsymbol{u}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}}) \cdot \vec{\boldsymbol{k}} - T\vec{\boldsymbol{k}} \cdot \mathbf{d}S = \frac{1}{n}\boldsymbol{F}(\vec{\boldsymbol{k}}, \vec{\boldsymbol{j}})$$

Now, from previously listed properties of $\vec{\boldsymbol{k}}$, $\vec{\boldsymbol{k}} \cdot \mathbf{d}S = 0$ and $\boldsymbol{F}(\vec{\boldsymbol{k}}, \vec{\boldsymbol{j}}) = 0$.

Hence there remains

$$\vec{\boldsymbol{k}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}}) \cdot \vec{\boldsymbol{u}} = 0 \qquad (17)$$

A conserved quantity from the MHD-Euler equation

Besides, via Cartan's identity,

$$\vec{\boldsymbol{k}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}}) = \mathcal{L}_{\vec{\boldsymbol{k}}}\,(h\underline{\boldsymbol{u}}) - \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{k}}) = \underbrace{\mathcal{L}_{\vec{\boldsymbol{\xi}}}\,(h\underline{\boldsymbol{u}})}_{0} + \omega \underbrace{\mathcal{L}_{\vec{\boldsymbol{\chi}}}\,(h\underline{\boldsymbol{u}})}_{0} + (h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\chi}})\mathbf{d}\omega - \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{k}})$$

Hence Eq. (17) becomes

$$(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\chi}}) \underbrace{\vec{\boldsymbol{u}} \cdot \mathbf{d}\omega}_{0} - \vec{\boldsymbol{u}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{k}}) = 0$$

Thus we conclude

$$\boxed{\mathcal{L}_{\vec{\boldsymbol{u}}}\, D = 0}$$

where

$$\boxed{D := h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{k}}}$$

Another conserved quantity from the MHD-Euler equation

Restart previous computation with $\vec{\xi}$ instead of $\vec{k}$:

MHD-Euler equation $\Longrightarrow \vec{u} \cdot \mathbf{d}(h\underline{u}) \cdot \vec{\xi} - T \underbrace{\vec{\xi} \cdot \mathbf{d}S}_{0} = \frac{1}{n} \underbrace{F(\vec{\xi}, \vec{j})}_{-\mathbf{d}\Phi \cdot \vec{j}}$

Since $\mathbf{d}\Phi = -\omega\, \mathbf{d}\Psi$, we get

$$\vec{u} \cdot \mathbf{d}(h\underline{u}) \cdot \vec{\xi} = \frac{\omega}{n} \vec{j} \cdot \mathbf{d}\Psi$$

Cartan ident. $\Longrightarrow \vec{\xi} \cdot \mathbf{d}(h\underline{u}) = \underbrace{\mathcal{L}_{\vec{\xi}}\, (h\underline{u})}_{0} - \mathbf{d}(h\underline{u} \cdot \vec{\xi}) \Longrightarrow \mathbf{d}(h\underline{u}) \cdot \vec{\xi} = \mathbf{d}(h\underline{u} \cdot \vec{\xi})$

Hence

$$\boxed{\mathcal{L}_{\vec{u}}\, (h\underline{u} \cdot \vec{\xi}) = \frac{\omega}{n} \vec{j} \cdot \mathbf{d}\Psi} \qquad (18)$$

There remains to evaluate the term $\vec{j} \cdot \mathbf{d}\Psi$

Another conserved quantity from the MHD-Euler equation

From the expression (12) for $\vec{j}$ along with the properties $\vec{\xi} \cdot \mathbf{d}\Psi = 0$ and $\vec{\chi} \cdot \mathbf{d}\Psi = 0$, we get

$$\vec{j} \cdot \mathbf{d}\Psi = \frac{1}{\mu_0 \sigma} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla} \Psi) = -\frac{1}{\mu_0 \sigma} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} \Psi, \vec{\nabla} I) \tag{19}$$

Another conserved quantity from the MHD-Euler equation

From the expression (12) for $\vec{j}$ along with the properties $\vec{\xi} \cdot \mathbf{d}\Psi = 0$ and $\vec{\chi} \cdot \mathbf{d}\Psi = 0$, we get

$$\vec{j} \cdot \mathbf{d}\Psi = \frac{1}{\mu_0 \sigma} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla}\Psi) = -\frac{1}{\mu_0 \sigma} \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla}\Psi, \vec{\nabla} I) \qquad (19)$$

Two cases must be considered:

(i) $a = 0$ $(\vec{w} = 0)$:

$\vec{u} = \lambda(\vec{\xi} + \Omega\vec{\chi}) \Longrightarrow \mathcal{L}_{\vec{u}} \, (h\underline{u} \cdot \vec{\xi}) = 0$.
Eqs. (18) and (19) then yield

$$\epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla}\Psi, \vec{\nabla} I) = 0$$

from which we deduce

$$\mathbf{d}I \propto \mathbf{d}\Psi$$

and

$$\boxed{I = I(\Psi)}$$

Another conserved quantity from the MHD-Euler equation

(ii) $a \neq 0$ ($\vec{w} \neq 0$): then Eq. (15) gives

$$\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla}\Psi, .) = \frac{1}{a}\,\underline{\boldsymbol{w}}$$

and we may write (19) as

$$\vec{j} \cdot \mathbf{d}\Psi = -\frac{1}{\mu_0 a \sigma}\,\underline{\boldsymbol{w}} \cdot \vec{\nabla} I = -\frac{1}{\mu_0 a \sigma}\,\vec{w} \cdot \mathbf{d}I = -\frac{1}{\mu_0 a \sigma}\,\vec{u} \cdot \mathbf{d}I = -\frac{1}{\mu_0 a \sigma}\,\mathcal{L}_{\vec{u}}\, I$$

Thus Eq. (18) becomes, using $K = an\sigma$,

$$\mathcal{L}_{\vec{u}}\,(h\underline{\boldsymbol{u}} \cdot \vec{\xi}) = -\frac{\omega}{\mu_0 K}\,\mathcal{L}_{\vec{u}}\, I$$

Since $\mathcal{L}_{\vec{u}}\,\omega = 0$ and $\mathcal{L}_{\vec{u}}\,K = 0$, we obtain

$$\boxed{\mathcal{L}_{\vec{u}}\, E = 0},$$

with

$$\boxed{E := -h\underline{\boldsymbol{u}} \cdot \vec{\xi} - \frac{\omega I}{\mu_0 K}} \qquad (20)$$

Another conserved quantity from the MHD-Euler equation

Similarly, using $\vec{\chi}$ instead of $\vec{\xi}$, we arrive at

$$\mathcal{L}_{\vec{u}}\,(h\underline{u}\cdot\vec{\chi}) = \frac{1}{\mu_0 n\sigma}\,\epsilon(\vec{\xi},\vec{\chi},\vec{\nabla}\Psi,\vec{\nabla}I)$$

Again we have to distinguish two cases:

(i) $a = 0$ $(\vec{w} = 0)$: then $\mathcal{L}_{\vec{u}}\,(h\underline{u}\cdot\vec{\chi}) = 0$ and we recover $I = I(\Psi)$ as above

(ii) $a \neq 0$ $(\vec{w} \neq 0)$: we obtain then

$$\boxed{\mathcal{L}_{\vec{u}}\,L = 0},$$

with

$$\boxed{L := h\underline{u}\cdot\vec{\chi} - \frac{I}{\mu_0 K}} \qquad (21)$$

Remark: the conserved quantities D, E and L are not independent since

$$\boxed{D = -E + \omega L}$$

Summary

- **For purely rotational fluid motion** $(a = 0)$: any scalar quantity which obeys to the spacetime symmetries is conserved along the fluid lines
- **For a fluid motion with meridional components** $(a \neq 0)$: there exist four scalar quantities which are constant along the fluid lines:

$$\omega, \quad K, \quad E, \quad L$$

(D being a combination of ω, E and L)

If there is no electromagnetic field, $E = -h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}}$ and the constancy of E along the fluid lines is the relativistic Bernoulli theorem

Comparison with previous work
Bekenstein & Oron (1978)

The constancy of ω, K, D, E and L along the fluid lines has been shown first by [Bekenstein & Oron (1978)]

Bekenstein & Oron have provided coordinate-dependent definitions of ω and K, namely

$$\omega := -\frac{F_{01}}{F_{31}} \qquad \text{and} \qquad K^{-1} := \frac{F_{31}}{\sqrt{-g}nu^2}$$

Besides, they have obtained expressions for E and L slightly more complicated than (20) and (21), namely

$$E = -\left(h + \frac{|b|^2}{\mu_0 n}\right) \underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}} - \frac{1}{\mu_0 K} (\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{k}}) \, (\underline{\boldsymbol{b}} \cdot \vec{\boldsymbol{\xi}})$$

$$L = \left(h + \frac{|b|^2}{\mu_0 n}\right) \underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\chi}} + \frac{1}{\mu_0 K} (\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{k}}) \, (\underline{\boldsymbol{b}} \cdot \vec{\boldsymbol{\chi}})$$

It can be shown that these expressions are equivalent to (20) and (21)

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