3+1 formalism in general relativity

Eric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
CNRS / Observatoire de Paris / Université Paris Diderot
F-92190 Meudon, France

eric.gourgoulhon@obspm.fr

http://www.luth.obspm.fr/~luthier/gourgoulhon/

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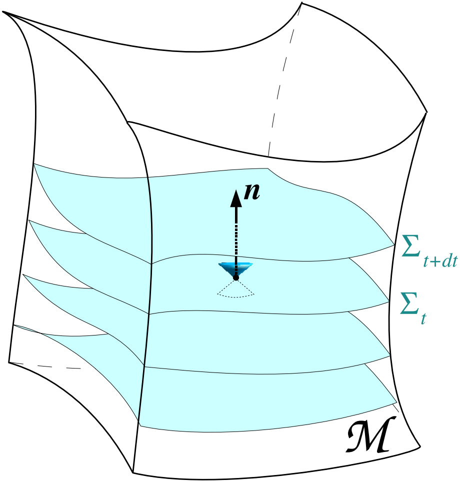
Plan

1. The 3+1 foliation of spacetime
2. 3+1 decomposition of Einstein equation
3. The Cauchy problem
4. Conformal decomposition

Outline

Framework: globally hyperbolic spacetimes



4-dimensional spacetime $(\mathscr{M}, \boldsymbol{g})$:

- $\mathscr{M}$: 4-dimensional smooth manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$: $\mathrm{sign}(\boldsymbol{g}) = (-, +, +, +)$

$(\mathscr{M}, \boldsymbol{g})$ is assumed to be **time orientable**: the light cones of $\boldsymbol{g}$ can be divided continuously over $\mathscr{M}$ in two sets (past and future)

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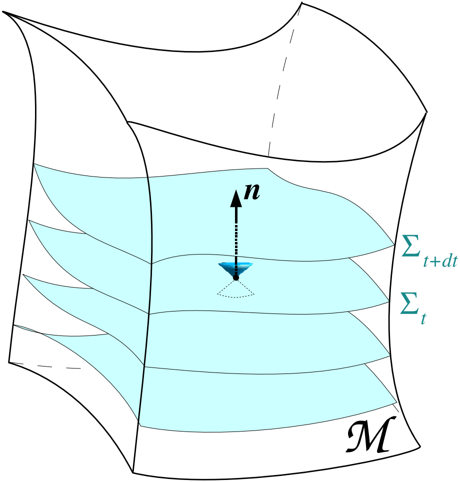

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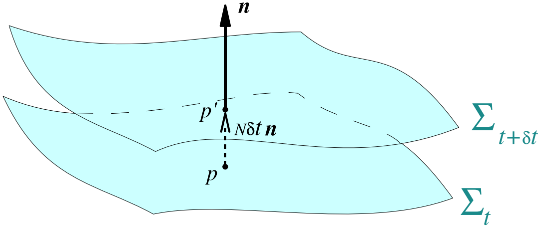
The spacetime $(\mathscr{M}, \boldsymbol{g})$ is assumed to be **globally hyperbolic**: $\exists$ a **foliation** (or **slicing**) of the spacetime manifold $\mathscr{M}$ by a family of spacelike hypersurfaces Σ_t :

$$\mathscr{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$$

hypersurface = submanifold of $\mathscr{M}$ of dimension 3

Unit normal vector and lapse function



$\boldsymbol{n}$: unit normal vector to Σ_t

Σ_t spacelike $\iff$ $\boldsymbol{n}$ timelike

$\boldsymbol{n} \cdot \boldsymbol{n} := \boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n}) = -1$

$\boldsymbol{n}$ chosen to be future directed

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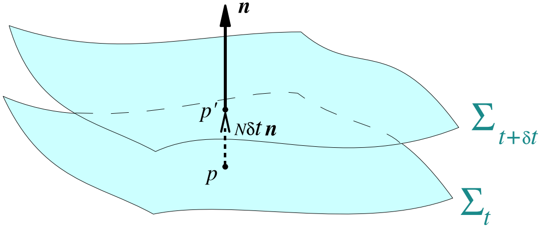

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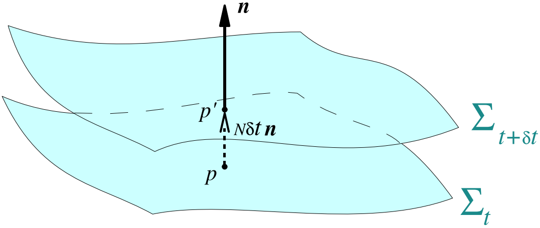
The 1-form $\underline{\boldsymbol{n}}$ associated with $\boldsymbol{n}$ is proportional to the gradient of t:

$\underline{\boldsymbol{n}} = -N\mathbf{d}t$ $\qquad (n_\alpha = -N\nabla_\alpha t)$

N: **lapse function** ; $N > 0$

Elapse proper time between p and p': $\delta\tau = N\delta t$

Unit normal vector and lapse function



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Normal evolution vector : $\boldsymbol{m} := N\boldsymbol{n}$

$\langle \mathbf{d}t, \boldsymbol{m}\rangle = 1 \Rightarrow \boldsymbol{m}$ Lie drags the hypersurfaces Σ_t

Induced metric (first fundamental form)

The **induced metric** or **first fundamental form** on Σ_t is the bilinear form $\boldsymbol{\gamma}$ defined by

$$\forall(\boldsymbol{u},\boldsymbol{v}) \in \mathcal{T}_p(\Sigma_t) \times \mathcal{T}_p(\Sigma_t), \quad \boldsymbol{\gamma}(\boldsymbol{u},\boldsymbol{v}) := \boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})$$

Σ_t spacelike $\iff$ $\boldsymbol{\gamma}$ positive definite (Riemannian metric)

$\boldsymbol{D}$: Levi-Civita connection associated with $\boldsymbol{\gamma}$: $\boldsymbol{D}\boldsymbol{\gamma} = 0$
$\boldsymbol{\mathcal{R}}$: Riemann tensor of $\boldsymbol{D}$:

$$\forall \boldsymbol{v} \in \mathcal{T}(\Sigma_t),\ (D_i D_j - D_j D_i) v^k = \mathcal{R}^k{}_{lij}\, v^l$$

$\boldsymbol{R}$: Ricci tensor of $\boldsymbol{D}$: $R_{ij} := R^k{}_{ikj}$
R : scalar curvature (or **Gaussian curvature**) of $(\Sigma, \boldsymbol{\gamma})$: $R := \gamma^{ij} R_{ij}$

Orthogonal projector

Since γ is not degenerate we have the orthogonal decomposition:

$$\mathcal{T}_p(\mathscr{M}) = \mathcal{T}_p(\Sigma_t) \oplus \mathrm{Vect}(\boldsymbol{n})$$

The associated **orthogonal projector onto** Σ_t is

$$\vec{\gamma} : \begin{array}{ccc} \mathcal{T}_p(\mathscr{M}) & \longrightarrow & \mathcal{T}_p(\Sigma) \\ \boldsymbol{v} & \longmapsto & \boldsymbol{v} + (\boldsymbol{n} \cdot \boldsymbol{v})\, \boldsymbol{n} \end{array}$$

In particular, $\vec{\gamma}(\boldsymbol{n}) = 0$ and $\forall \boldsymbol{v} \in \mathcal{T}_p(\Sigma_t),\ \vec{\gamma}(\boldsymbol{v}) = \boldsymbol{v}$
Components: $\gamma^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + n^{\alpha} n_{\beta}$

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"Extended" induced metric :

$$\forall (\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}_p(\mathscr{M}) \times \mathcal{T}_p(\mathscr{M}), \quad \boldsymbol{\gamma}(\boldsymbol{u}, \boldsymbol{v}) := \boldsymbol{\gamma}(\vec{\gamma}(\boldsymbol{u}), \vec{\gamma}(\boldsymbol{v}))$$

$$\boxed{\boldsymbol{\gamma} = \boldsymbol{g} + \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}} \qquad (\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha\, n_\beta)$$

(hence the notation $\vec{\gamma}$ for the orthogonal projector)

Extrinsic curvature (second fundamental form)

The **extrinsic curvature** (or **second fundamental form**) of Σ_t is the bilinear form defined by

$$\begin{array}{rccc} \boldsymbol{K}: & \mathcal{T}_p(\Sigma_t)\times\mathcal{T}_p(\Sigma_t) & \longrightarrow & \mathbb{R} \\ & (\boldsymbol{u},\boldsymbol{v}) & \longmapsto & -\boldsymbol{u}\cdot\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{n} \end{array}$$

It measures the "bending" of Σ_t in $(\mathscr{M},\boldsymbol{g})$ by evaluating the change of direction of the normal vector $\boldsymbol{n}$ as one moves on Σ_t

Weingarten property: $\boldsymbol{K}$ is symmetric: $\boldsymbol{K}(\boldsymbol{u},\boldsymbol{v})=\boldsymbol{K}(\boldsymbol{v},\boldsymbol{u})$

Trace: $K := \mathrm{tr}_{\boldsymbol{\gamma}}\boldsymbol{K} = \gamma^{ij}K_{ij}$ = (3 times) the **mean curvature** of Σ_t

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$$\Longrightarrow \boxed{\boldsymbol{\nabla}\underline{\boldsymbol{n}} = -\boldsymbol{K} - \boldsymbol{D}\ln N\otimes\underline{\boldsymbol{n}}} \qquad (\nabla_\beta n_\alpha = -K_{\alpha\beta} - D_\alpha \ln N\, n_\beta)$$

$$\boxed{K = -\boldsymbol{\nabla}\cdot\boldsymbol{n}}$$

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$$\boxed{K = -\boldsymbol{\nabla}\cdot\boldsymbol{n}}$$

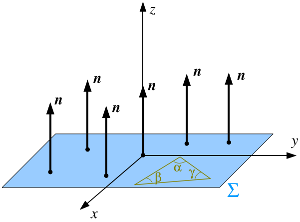
Σ_t being part of a foliation, an alternative expression of $\boldsymbol{K}$ is available:

$$\boxed{\boldsymbol{K} = -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\,\boldsymbol{\gamma}}$$

Intrinsic and extrinsic curvatures
Examples in the Euclidean space

- intrinsic curvature: Riemann tensor $\boldsymbol{\mathcal{R}}$
- extrinsic curvature: second fundamental form $\boldsymbol{K}$

plane



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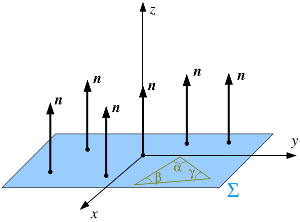
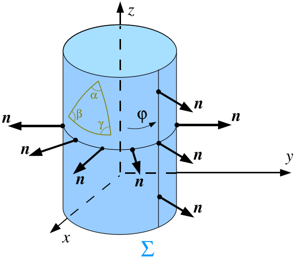


$\mathcal{R} = 0$
$\boldsymbol{K} = 0$

cylinder



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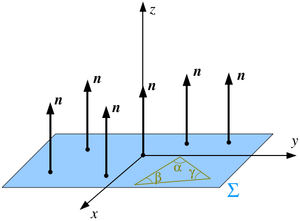

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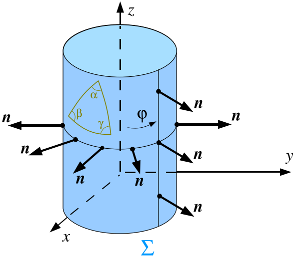

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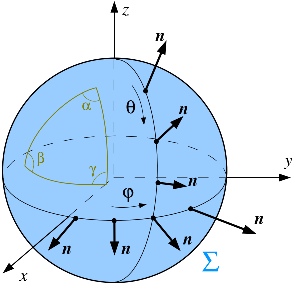
sphere



$\mathcal{R} \neq 0$
$\boldsymbol{K} \neq 0$

Link between the ∇ and D connections

For any tensor field $\boldsymbol{T}$ tangent to Σ_t:

$$D_\rho T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q} = \gamma^{\alpha_1}{}_{\mu_1}\cdots\gamma^{\alpha_p}{}_{\mu_p}\,\gamma^{\nu_1}{}_{\beta_1}\cdots\gamma^{\nu_q}{}_{\beta_q}\,\gamma^{\sigma}{}_{\rho}\,\nabla_\sigma T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q}$$

For two vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$ tangent to Σ_t, $\boxed{\boldsymbol{D_u v} = \boldsymbol{\nabla_u v} + \boldsymbol{K}(\boldsymbol{u},\boldsymbol{v})\,\boldsymbol{n}}$

3+1 decomposition of the Riemann tensor

- Gauss equation: $\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\gamma^{\gamma}{}_{\rho}\gamma^{\sigma}{}_{\delta}\,{}^4\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \mathcal{R}^{\gamma}{}_{\delta\alpha\beta} + K^{\gamma}{}_{\alpha}K_{\delta\beta} - K^{\gamma}{}_{\beta}K_{\alpha\delta}$

  contracted version :

  $\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\,{}^4R_{\mu\nu} + \gamma_{\alpha\mu}n^{\nu}\gamma^{\rho}{}_{\beta}n^{\sigma}\,{}^4\mathcal{R}^{\mu}{}_{\nu\rho\sigma} = R_{\alpha\beta} + KK_{\alpha\beta} - K_{\alpha\mu}K^{\mu}{}_{\beta}$

  trace: ${}^4R + 2\,{}^4R_{\mu\nu}n^{\mu}n^{\nu} = R + K^2 - K_{ij}K^{ij}$ *(Theorema Egregium)*

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  contracted version :

  $\gamma^\mu{}_\alpha \gamma^\nu{}_\beta \, {}^4R_{\mu\nu} + \gamma_{\alpha\mu} n^\nu \gamma^\rho{}_\beta n^\sigma \, {}^4\mathcal{R}^\mu{}_{\nu\rho\sigma} = R_{\alpha\beta} + K K_{\alpha\beta} - K_{\alpha\mu} K^\mu{}_\beta$

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- Codazzi equation: $\gamma^\gamma{}_\rho \, n^\sigma \, \gamma^\mu{}_\alpha \gamma^\nu{}_\beta \, {}^4\mathcal{R}^\rho{}_{\sigma\mu\nu} = D_\beta K^\gamma{}_\alpha - D_\alpha K^\gamma{}_\beta$

  contracted version : $\gamma^\mu{}_\alpha n^\nu \, {}^4R_{\mu\nu} = D_\alpha K - D_\mu K^\mu{}_\alpha$

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- Codazzi equation: $\gamma^{\gamma}{}_{\rho}\,n^{\sigma}\,\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\,{}^4\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = D_{\beta}K^{\gamma}{}_{\alpha} - D_{\alpha}K^{\gamma}{}_{\beta}$

  contracted version : $\gamma^{\mu}{}_{\alpha}n^{\nu}\,{}^4R_{\mu\nu} = D_{\alpha}K - D_{\mu}K^{\mu}{}_{\alpha}$

- Ricci equation: $\gamma_{\alpha\mu}\,n^{\rho}\gamma^{\nu}{}_{\beta}\,n^{\sigma}\,{}^4\mathcal{R}^{\mu}{}_{\rho\nu\sigma} = \frac{1}{N}\mathcal{L}_{\boldsymbol{m}}\,K_{\alpha\beta} + \frac{1}{N}D_{\alpha}D_{\beta}N + K_{\alpha\mu}K^{\mu}{}_{\beta}$

  combined with the contracted Gauss equation :

  $\gamma^{\mu}{}_{\alpha}\gamma^{\nu}{}_{\beta}\,{}^4R_{\mu\nu} = -\frac{1}{N}\mathcal{L}_{\boldsymbol{m}}\,K_{\alpha\beta} - \frac{1}{N}D_{\alpha}D_{\beta}N + R_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}{}_{\beta}$

Outline

Einstein equation

The spacetime $(\mathscr{M}, \boldsymbol{g})$ obeys Einstein equation

$$^4\boldsymbol{R} - \frac{1}{2}{}^4R\,\boldsymbol{g} = 8\pi\boldsymbol{T}$$

where $\boldsymbol{T}$ is the matter stress-energy tensor

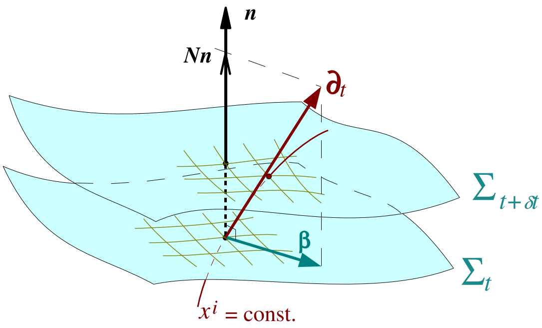
3+1 decomposition of the stress-energy tensor

$\mathcal{E}$: Eulerian observer = observer of 4-velocity $\boldsymbol{n}$

- $E := \boldsymbol{T}(\boldsymbol{n}, \boldsymbol{n})$: **matter energy density** as measured by $\mathcal{E}$
- $\boldsymbol{p} := -\boldsymbol{T}(\boldsymbol{n}, \vec{\gamma}(.))$: **matter momentum density** as measured by $\mathcal{E}$
- $\boldsymbol{S} := \boldsymbol{T}(\vec{\gamma}(.), \vec{\gamma}(.))$: **matter stress tensor** as measured by $\mathcal{E}$

$$\boxed{\boldsymbol{T} = \boldsymbol{S} + \underline{\boldsymbol{n}} \otimes \boldsymbol{p} + \boldsymbol{p} \otimes \underline{\boldsymbol{n}} + E\, \underline{\boldsymbol{n}} \otimes \underline{\boldsymbol{n}}}$$

Spatial coordinates and shift vector



$(x^i) = (x^1, x^2, x^3)$ coordinates on Σ_t

(x^i) vary smoothly between neighbouring hypersurfaces ⇒ $(x^\alpha) = (t, x^1, x^2, x^3)$ well behaved coordinate system on $\mathscr{M}$

Associated natural basis :

$$\boldsymbol{\partial}_t := \frac{\partial}{\partial t}$$

$$\boldsymbol{\partial}_i := \frac{\partial}{\partial x^i}, \quad i \in \{1, 2, 3\}$$

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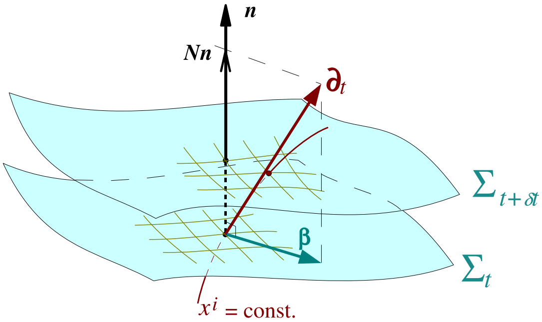

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$\langle \mathbf{d}t, \boldsymbol{\partial}_t \rangle = 1 \Rightarrow \boldsymbol{\partial}_t$ Lie-drags the hypersurfaces Σ_t, as $\boldsymbol{m} := N\boldsymbol{n}$ does. The difference between $\boldsymbol{\partial}_t$ and $\boldsymbol{m}$ is called the **shift vector** and is denoted $\boldsymbol{\beta}$:

$$\boxed{\boldsymbol{\partial}_t =: \boldsymbol{m} + \boldsymbol{\beta}}$$

Notice: $\boldsymbol{\beta}$ is tangent to Σ_t: $\boldsymbol{n} \cdot \boldsymbol{\beta} = 0$

Metric tensor in terms of lapse and shift

Components of β w.r.t. (x^i): $\boldsymbol{\beta} =: \beta^i \, \boldsymbol{\partial}_i$ and $\underline{\boldsymbol{\beta}} =: \beta_i \, \mathbf{d}x^i$

Components of $\boldsymbol{n}$ w.r.t. (x^α):

$n^\alpha = \left(\frac{1}{N}, -\frac{\beta^1}{N}, -\frac{\beta^2}{N}, -\frac{\beta^3}{N} \right)$ and $n_\alpha = (-N, 0, 0, 0)$

Components of $\boldsymbol{g}$ w.r.t. (x^α):

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

or equivalently $g_{\mu\nu} \, dx^\mu \, dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$

Components of the inverse metric:

$$g^{\alpha\beta} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & \frac{\beta^j}{N^2} \\ \frac{\beta^i}{N^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{N^2} \end{pmatrix}$$

Relation between the determinants : $\sqrt{-g} = N \sqrt{\gamma}$

3+1 Einstein system

Thanks to the Gauss, Codazzi and Ricci equations reminder, the Einstein equation is equivalent to the system

- $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right) \gamma_{ij} = -2NK_{ij}$ kinematical relation $\boldsymbol{K} = -\frac{1}{2}\boldsymbol{\mathcal{L}_n}\,\boldsymbol{\gamma}$
- $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right) K_{ij} = -D_i D_j N + N\Big\{ R_{ij} + K K_{ij} - 2K_{ik}K^k_{\ \ j}$ $+4\pi\left[(S-E)\gamma_{ij} - 2S_{ij}\right]\Big\}$ dynamical part of Einstein equation
- $R + K^2 - K_{ij}K^{ij} = 16\pi E$ Hamiltonian constraint
- $D_j K^j_{\ \ i} - D_i K = 8\pi p_i$ momentum constraint

The full PDE system

Supplementary equations:

$$D_i D_j N = \frac{\partial^2 N}{\partial x^i \partial x^j} - \Gamma^k{}_{ij} \frac{\partial N}{\partial x^k}$$

$$D_j K^j{}_i = \frac{\partial K^j{}_i}{\partial x^j} + \Gamma^j{}_{jk} K^k{}_i - \Gamma^k{}_{ji} K^j{}_k$$

$$D_i K = \frac{\partial K}{\partial x^i}$$

$$\mathcal{L}_{\boldsymbol{\beta}}\, \gamma_{ij} = \frac{\partial \beta_i}{\partial x^j} + \frac{\partial \beta_j}{\partial x^i} - 2\Gamma^k{}_{ij} \beta_k$$

$$\mathcal{L}_{\boldsymbol{\beta}}\, K_{ij} = \beta^k \frac{\partial K_{ij}}{\partial x^k} + K_{kj} \frac{\partial \beta^k}{\partial x^i} + K_{ik} \frac{\partial \beta^k}{\partial x^j}$$

$$R_{ij} = \frac{\partial \Gamma^k{}_{ij}}{\partial x^k} - \frac{\partial \Gamma^k{}_{ik}}{\partial x^j} + \Gamma^k{}_{ij} \Gamma^l{}_{kl} - \Gamma^l{}_{ik} \Gamma^k{}_{lj}$$

$$R = \gamma^{ij} R_{ij}$$

$$\Gamma^k{}_{ij} = \frac{1}{2} \gamma^{kl} \left(\frac{\partial \gamma_{lj}}{\partial x^i} + \frac{\partial \gamma_{il}}{\partial x^j} - \frac{\partial \gamma_{ij}}{\partial x^l} \right)$$

History of 3+1 formalism

- G. Darmois (1927): 3+1 Einstein equations in terms of (γ_{ij}, K_{ij}) with $\alpha = 1$ and $\boldsymbol{\beta} = 0$ (Gaussian normal coordinates)
- A. Lichnerowicz (1939) : $\alpha \neq 1$ and $\boldsymbol{\beta} = 0$ (normal coordinates)
- Y. Choquet-Bruhat (1948) : $\alpha \neq 1$ and $\boldsymbol{\beta} \neq 0$ (general coordinates)
- R. Arnowitt, S. Deser & C.W. Misner (1962) : *Hamiltonian formulation* of GR based on a 3+1 decomposition in terms of (γ_{ij}, π^{ij})
  NB: spatial projection of *Einstein tensor* instead of *Ricci tensor* in previous works
- J. Wheeler (1964) : coined the terms *lapse* and *shift*
- J.W. York (1979) : modern 3+1 decomposition based on spatial projection of *Ricci tensor*

Outline

GR as a 3-dimensional dynamical system

3+1 Einstein system $\Longrightarrow$ Einstein equation = time evolution of tensor fields $(\gamma, \boldsymbol{K})$ on a single 3-dimensional manifold Σ
(Wheeler's *geometrodynamics* (1964))

No time derivative of N nor $\boldsymbol{\beta}$: lapse and shift are not dynamical variables
(best seen on the ADM Hamiltonian formulation)
This reflects the coordinate freedom of GR reminder :

$$\text{choice of foliation } (\Sigma_t)_{t\in\mathbb{R}} \iff \text{choice of lapse function } N$$
$$\text{choice of spatial coordinates } (x^i) \iff \text{choice of shift vector } \boldsymbol{\beta}$$

Constraints

The dynamical system has two **constraints**:

- $R + K^2 - K_{ij}K^{ij} = 16\pi E$ Hamiltonian constraint
- $D_j K^j{}_i - D_i K = 8\pi p_i$ momentum constraint

Similar to $\boldsymbol{D} \cdot \boldsymbol{B} = 0$ and $\boldsymbol{D} \cdot \boldsymbol{E} = \rho/\epsilon_0$ in Maxwell equations for the electromagnetic field

Cauchy problem

The first two equations of the 3+1 Einstein system reminder can be put in the form of a **Cauchy problem:**

$$\frac{\partial^2 \gamma_{ij}}{\partial t^2} = F_{ij}\left(\gamma_{kl}, \frac{\partial \gamma_{kl}}{\partial x^m}, \frac{\partial \gamma_{kl}}{\partial t}, \frac{\partial^2 \gamma_{kl}}{\partial x^m \partial x^n}\right) \quad (1)$$

Cauchy problem: given initial data at $t = 0$: γ_{ij} and $\frac{\partial \gamma_{ij}}{\partial t}$, find a solution for $t > 0$

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Cauchy problem: given initial data at $t = 0$: γ_{ij} and $\frac{\partial \gamma_{ij}}{\partial t}$, find a solution for $t > 0$

But this Cauchy problem is subject to the constraints

- $R + K^2 - K_{ij}K^{ij} = 16\pi E$ Hamiltonian constraint
- $D_j K^j{}_i - D_i K = 8\pi p_i$ momentum constraint

Preservation of the constraints

Thanks to the Bianchi identities, it can be shown that if the constraints are satisfied at $t = 0$, they are preserved by the evolution system (1)

Existence and uniqueness of solutions

Question:

Given a set $(\Sigma_0, \gamma, \boldsymbol{K}, E, \boldsymbol{p})$, where Σ_0 is a three-dimensional manifold, γ a Riemannian metric on Σ_0, $\boldsymbol{K}$ a symmetric bilinear form field on Σ_0, E a scalar field on Σ_0 and $\boldsymbol{p}$ a 1-form field on Σ_0, which obeys the constraint equations, does there exist a spacetime $(\mathscr{M}, \boldsymbol{g}, \boldsymbol{T})$ such that $(\boldsymbol{g}, \boldsymbol{T})$ fulfills the Einstein equation and Σ_0 can be embedded as an hypersurface of $\mathscr{M}$ with induced metric γ and extrinsic curvature $\boldsymbol{K}$?

Answer:

- the solution exists and is unique in a vicinity of Σ_0 for **analytic** initial data (Cauchy-Kovalevskaya theorem) (Darmois 1927, Lichnerowicz 1939)
- the solution exists and is unique in a vicinity of Σ_0 for **generic** (i.e. smooth) initial data (Choquet-Bruhat 1952)
- there exists a unique maximal solution (Choquet-Bruhat & Geroch 1969)

Outline

Conformal metric

Introduce on Σ_t a metric $\tilde{\gamma}$ conformally related to the induced metric γ:

$$\boxed{\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}}$$

Ψ : **conformal factor**

Inverse metric:

$$\gamma^{ij} = \Psi^{-4} \, \tilde{\gamma}^{ij}$$

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Motivations:

- the gravitational field degrees of freedom are carried by conformal equivalence classes (York 1971)

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Motivations:

- the gravitational field degrees of freedom are carried by conformal equivalence classes (York 1971)
- the conformal decomposition is of great help for preparing initial data as solution of the constraint equations

Conformal connection

$\tilde{\gamma}$ Riemannian metric on Σ_t: it has a unique Levi-Civita connection associated to it: $\tilde{\boldsymbol{D}}\tilde{\boldsymbol{\gamma}} = 0$

Christoffel symbols: $\tilde{\Gamma}^k{}_{ij} = \frac{1}{2}\tilde{\gamma}^{kl}\left(\frac{\partial\tilde{\gamma}_{lj}}{\partial x^i} + \frac{\partial\tilde{\gamma}_{il}}{\partial x^j} - \frac{\partial\tilde{\gamma}_{ij}}{\partial x^l}\right)$

Relation between the two connections:

$$D_k T^{i_1\ldots i_p}{}_{j_1\ldots j_q} = \tilde{D}_k T^{i_1\ldots i_p}{}_{j_1\ldots j_q} + \sum_{r=1}^{p} C^{i_r}{}_{kl}\, T^{i_1\ldots l\ldots i_p}{}_{j_1\ldots j_q} - \sum_{r=1}^{q} C^{l}{}_{kj_r}\, T^{i_1\ldots i_p}{}_{j_1\ldots l\ldots j_q}$$

with $C^k{}_{ij} := \Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij}$

One finds

$$\boxed{C^k{}_{ij} = 2\left(\delta^k{}_i\tilde{D}_j \ln\Psi + \delta^k{}_j\tilde{D}_i \ln\Psi - \tilde{D}^k \ln\Psi\,\tilde{\gamma}_{ij}\right)}$$

Application: divergence relation : $D_i v^i = \Psi^{-6}\tilde{D}_i\left(\Psi^6 v^i\right)$

Conformal decomposition of the Ricci tensor

From the Ricci identity:

$$R_{ij} = \tilde{R}_{ij} + \tilde{D}_k C^k{}_{ij} - \tilde{D}_i C^k{}_{kj} + C^k{}_{ij} C^l{}_{lk} - C^k{}_{il} C^l{}_{kj}$$

In the present case this formula reduces to

$$R_{ij} = \tilde{R}_{ij} - 2\tilde{D}_i \tilde{D}_j \ln \Psi - 2\tilde{D}_k \tilde{D}^k \ln \Psi \, \tilde{\gamma}_{ij} + 4\tilde{D}_i \ln \Psi \, \tilde{D}_j \ln \Psi - 4\tilde{D}_k \ln \Psi \, \tilde{D}^k \ln \Psi \, \tilde{\gamma}_{ij}$$

Scalar curvature :

$$\boxed{R = \Psi^{-4} \tilde{R} - 8\Psi^{-5} \tilde{D}_i \tilde{D}^i \Psi}$$

where $R := \gamma^{ij} R_{ij}$ and $\tilde{R} := \tilde{\gamma}^{ij} \tilde{R}_{ij}$

Conformal decomposition of the extrinsic curvature

- First step: traceless decomposition:

$$K^{ij} =: A^{ij} + \frac{1}{3} K \gamma^{ij}$$

with $\gamma_{ij} A^{ij} = 0$

- Second step: conformal decomposition of the traceless part:

$$A^{ij} = \Psi^{\alpha} \tilde{A}^{ij}$$

with α to be determined

"Time evolution" scaling $\alpha = -4$

Time evolution of the 3-metric reminder: $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right) \gamma^{ij} = 2NK^{ij}$

- trace part : $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right) \ln \Psi = \frac{1}{6}\left(\tilde{D}_i \beta^i - NK - \frac{\partial}{\partial t} \ln \tilde{\gamma}\right)$
- traceless part : $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right) \tilde{\gamma}^{ij} = 2N\Psi^4 A^{ij} + \frac{2}{3}\tilde{D}_k \beta^k \, \tilde{\gamma}^{ij}$

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- traceless part : $\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right) \tilde{\gamma}^{ij} = 2N\Psi^4 A^{ij} + \frac{2}{3}\tilde{D}_k \beta^k \, \tilde{\gamma}^{ij}$

This suggests to introduce

$$\boxed{\tilde{A}^{ij} := \Psi^4 A^{ij}} \qquad \text{(Nakamura 1994)}$$

$\Longrightarrow$ momentum constraint becomes

$$\tilde{D}_j \tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j \ln \Psi - \frac{2}{3}\tilde{D}^i K = 8\pi \Psi^4 p^i$$

"Momentum-constraint" scaling $\alpha = -10$

Momentum constraint: $D_j K^{ij} - D^i K = 8\pi p^i$

Now $D_j K^{ij} = D_j A^{ij} + \frac{1}{3} D^i K$ and

$$\begin{aligned} D_j A^{ij} &= \tilde{D}_j A^{ij} + C^i{}_{jk} A^{kj} + C^j{}_{jk} A^{ik} \\ &= \tilde{D}_j A^{ij} + 2\left(\delta^i{}_j \tilde{D}_k \ln\Psi + \delta^i{}_k \tilde{D}_j \ln\Psi - \tilde{D}^i \ln\Psi \, \tilde{\gamma}_{jk}\right) A^{kj} + 6\tilde{D}_k \ln\Psi \, A^{ik} \\ &= \tilde{D}_j A^{ij} + 10 A^{ij} \tilde{D}_j \ln\Psi - 2\tilde{D}^i \ln\Psi \underbrace{\tilde{\gamma}_{jk} A^{jk}}_{=0} \end{aligned}$$

Hence $D_j A^{ij} = \Psi^{-10} \tilde{D}_j \left(\Psi^{10} A^{ij}\right)$

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Hence $D_j A^{ij} = \Psi^{-10} \tilde{D}_j \left(\Psi^{10} A^{ij}\right)$

This suggests to introduce

$$\boxed{\hat{A}^{ij} := \Psi^{10} A^{ij}} \qquad \text{(Lichnerowicz 1944)}$$

$\Longrightarrow$ momentum constraint becomes

$$\boxed{\tilde{D}_j \hat{A}^{ij} - \frac{2}{3} \Psi^6 \tilde{D}^i K = 8\pi \Psi^{10} p^i}$$

Hamiltonian constraint as the Lichnerowicz equation

Hamiltonian constraint: $R + K^2 - K_{ij} K^{ij} = 16\pi E$

Now reminder $R = \Psi^{-4} \tilde{R} - 8 \Psi^{-5} \tilde{D}_i \tilde{D}^i \Psi$ and $K_{ij} K^{ij} = \Psi^{-12} \hat{A}_{ij} \hat{A}^{ij} + \frac{K^2}{3}$

so that

$$\tilde{D}_i \tilde{D}^i \Psi - \frac{1}{8} \tilde{R} \Psi + \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \, \Psi^{-7} + \left(2\pi E - \frac{1}{12} K^2 \right) \Psi^5 = 0$$

This is **Lichnerowicz equation** (or **Lichnerowicz-York** equation).

Summary: conformal 3+1 Einstein system

Version $\alpha = -4$ (Shibata & Nakamura 1995):

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\Psi = \frac{\Psi}{6}\left(\tilde{D}_i\beta^i - NK - \frac{\partial}{\partial t}\ln\tilde{\gamma}\right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\tilde{\gamma}^{ij} = 2N\tilde{A}^{ij} + \frac{2}{3}\tilde{D}_k\beta^k\,\tilde{\gamma}^{ij}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)K = -\Psi^{-4}\left(\tilde{D}_i\tilde{D}^iN + 2\tilde{D}_i\ln\Psi\,\tilde{D}^iN\right) + N\left[4\pi(E+S) + \tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{3}\right]$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\tilde{A}^{ij} = \Psi^{-4}\left[N\left(\tilde{R}^{ij} - 2\tilde{D}^i\tilde{D}^j\ln\Psi\right) - \tilde{D}^i\tilde{D}^jN\right] + \cdots$$

$$\begin{cases} \tilde{D}_i\tilde{D}^i\Psi - \frac{1}{8}\tilde{R}\Psi + \left(\frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\Psi^5 = 0 \\ \tilde{D}_j\tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j\ln\Psi - \frac{2}{3}\tilde{D}^iK = 8\pi\Psi^4p^i \end{cases}$$

Summary: conformal 3+1 Einstein system

Version $\alpha = -10$:

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\Psi = \frac{\Psi}{6}\left(\tilde{D}_i\beta^i - NK - \frac{\partial}{\partial t}\ln\tilde{\gamma}\right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\tilde{\gamma}^{ij} = 2N\tilde{A}^{ij} + \frac{2}{3}\tilde{D}_k\beta^k\,\tilde{\gamma}^{ij}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)K = -\Psi^{-4}\left(\tilde{D}_i\tilde{D}^iN + 2\tilde{D}_i\ln\Psi\,\tilde{D}^iN\right) + N\left[4\pi(E+S) + \tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{3}\right]$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}}\right)\tilde{A}^{ij} = \Psi^{-4}\left[N\left(\tilde{R}^{ij} - 2\tilde{D}^i\tilde{D}^j\ln\Psi\right) - \tilde{D}^i\tilde{D}^jN\right] + \cdots$$

$$\begin{cases} \tilde{D}_i\tilde{D}^i\Psi - \frac{1}{8}\tilde{R}\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\,\Psi^{-7} + \left(2\pi E - \frac{1}{12}K^2\right)\Psi^5 = 0 \\ \tilde{D}_j\hat{A}^{ij} - \frac{2}{3}\Psi^6\tilde{D}^iK = 8\pi\Psi^{10}p^i \end{cases}$$