General relativity computations with SageManifolds

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Outline

1. Computer differential geometry and tensor calculus
2. The SageManifolds project
3. Let us practice!
4. Other examples
5. Conclusion and perspectives
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Introduction

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- In 1965, J.G. Fletcher developed the GEOM program, to compute the Riemann tensor of a given metric.
- In 1969, during his PhD under Pirani supervision, Ray d’Inverno wrote ALAM (Atlas Lisp Algebraic Manipulator) and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, Applications of SHEEP (1994)].
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Since then, many softwares for tensor calculus have been developed...
Free packages for tensor computer algebra in Mathematica, developed by José Martín-García et al. [http://www.xact.es/]

The xAct system

- **Spinors**
  - "Spinor calculus in GR"
  - A. García-Parrado and J.M. Martín-García.

- **Invar**
  - "Riemann tensor Invariants"
  - J.M. Martín-García, R. Portugal and D. Yllanes.

- **SymManipulator**
  - "Symmetrized tensor expressions"
  - T. Bäckdahl.

- **xCoba**
  - "Component tensor algebra"
  - J.M. Martín-García and D. Yllanes.

- **xTensor**
  - Abstract tensor algebra

- **xPerm**
  - Permutation Group theory

- **Harmonics**
  - "Tensor spherical harmonics"
  - D. Brizuela, J.M. Martín-García and G. Mena Marugán.

- **xPert**
  - "Perturbation theory"
  - D. Brizuela, J.M. Martín-García and G. Mena Marugán.

- **xPrint**
  - "Graphical front-end"
  - A. Stecchina.

- **MathLink**

- **xCore**
  - *Mathematica* tools

- **xperm.c**
  - C-language module

[Garía-Parrado Gómez-Lobo & Martín-García, Comp. Phys. Comm. 183, 2214 (2012)]
Software for differential geometry

Packages for general purpose computer algebra systems:
- **xAct** free package for Mathematica  [J.-M. Martin-Garcia]
- **Ricci** free package for Mathematica  [J. L. Lee]
- **MathTensor** package for Mathematica  [S. M. Christensen & L. Parker]
- **GRTensor** package for Maple  [P. Musgrave, D. Pollney & K. Lake]
- **DifferentialGeometry** included in Maple  [I. M. Anderson & E. S. Cheb-Terrab]
- **Atlas 2** for Maple and Mathematica
- ...

Standalone applications:
- **SHEEP, Classi, STensor**, based on Lisp, developed in 1970’s and 1980’s (free)  
- **Cadabra** field theory (free)  [K. Peeters]
- **SnapPy** topology and geometry of 3-manifolds, based on Python (free)  [M. Culler, N. M. Dunfield & J. R. Weeks]
- ...

cf. the complete list at [http://www.xact.es/links.html](http://www.xact.es/links.html)
SageMath in a few words

- **SageMath** (*nickname: Sage*) is a **free open-source** mathematics software system.
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and provides a **uniform interface** to them
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**The mission**

*Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.*
Some advantages of SageMath

SageMath is free

Freedom means

1. everybody can use it, by downloading the software from http://sagemath.org
2. everybody can examine the source code and improve it
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SageMath is based on Python

- no need to learn any specific syntax to use it
- easy access for students
- Python is a very powerful object oriented language, with a neat syntax
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SageMath is developing and spreading fast

...sustained by an enthousiast community of developers
As an object-oriented language, Python (and hence SageMath) makes use of the following **postfix notation** (same in C++, Java, etc.):

\[
\text{result} = \text{object.function}(\text{arguments})
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In a procedural language, this would be written as

\[
\text{result} = \text{function}(\text{object,arguments})
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`Eric Gourgoulhon (LUTH)`

GR computations with SageManifolds

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result = function(object,arguments)
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**Examples**

1. `riem = g.riemann()`
2. `lie_t_v = t.lie_der(v)`

NB: no argument in example 1
SageMath relies on a Parent / Element scheme: each object \( x \) on which some calculus is performed has a “parent”, which is another SageMath object \( X \) representing the set to which \( x \) belongs. The calculus rules on \( x \) are determined by the algebraic structure of \( X \). Conversion rules prior to an operation, e.g. \( x + y \) with \( x \) and \( y \) having different parents, are defined at the level of the parents.
SageMath approach to computer mathematics

SageMath relies on a **Parent / Element** scheme: each object $x$ on which some calculus is performed has a “parent”, which is another SageMath object $X$ representing the set to which $x$ belongs. The calculus rules on $x$ are determined by the *algebraic structure* of $X$.

*Conversion rules* prior to an operation, e.g. $x + y$ with $x$ and $y$ having different parents, are defined at the level of the parents.

**Example**

```python
sage: x = 4 ; x.parent()
Integer Ring
sage: y = 4/3 ; y.parent()
Rational Field
sage: s = x + y ; s.parent()
Rational Field
sage: y.parent().has_coerce_map_from(x.parent())
True
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This approach is similar to that of Magma and is different from that of Mathematica, in which everything is a tree of symbols.
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The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

Implement real smooth manifolds of arbitrary dimension in Sage and tensor calculus on them

In particular:

- one should be able to introduce an arbitrary number of coordinate charts on a given manifold, with the relevant transition maps
- tensor fields must be manipulated as such and not through their components with respect to a specific (possibly coordinate) vector frame
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Concretely, the project amounts to creating new Python classes, such as Manifold, Chart, TensorField or Metric, within SageMath’s Parent/Element framework.
Given a (topological) manifold $M$ of dimension $n \geq 1$, a **coordinate chart** is a homeomorphism $\varphi : U \to V$, where $U$ is an open subset of $M$ and $V$ is an open subset of $\mathbb{R}^n$. 
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In general, more than one chart is required to cover the entire manifold:

Examples:

- at least 2 charts are necessary to cover the $n$-dimensional sphere $S^n$ ($n \geq 1$) and the torus $\mathbb{T}^2$
- at least 3 charts are necessary to cover the real projective plane $\mathbb{RP}^2$
The SageManifolds project

Implementing coordinate charts

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In SageManifolds, an arbitrary number of charts can be introduced.

To fully specify the manifold, one shall also provide the transition maps on overlapping chart domains (SageManifolds class `CoordChange`).
A scalar field on manifold $M$ is a smooth mapping

$$f : \quad U \subset M \quad \longrightarrow \quad \mathbb{R}$$

$$p \quad \longmapsto \quad f(p)$$

where $U$ is an open subset of $M$. 
Implementing scalar fields

A scalar field on manifold $M$ is a smooth mapping

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A scalar field maps points, not coordinates, to real numbers

$\longrightarrow$ an object $f$ in the ScalarField class has different coordinate representations in different charts defined on $U$. 
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$\Rightarrow$ an object $f$ in the ScalarField class has different coordinate representations in different charts defined on $U$.

The various coordinate representations $F, \hat{F}, \ldots$ of $f$ are stored as a Python dictionary whose keys are the charts $C, \hat{C}, \ldots$:

$$f._\text{express} = \{C : F, \hat{C} : \hat{F}, \ldots\}$$

with

$$f(p) = F(x^1, \ldots, x^n) = \hat{F}(\hat{x}^1, \ldots, \hat{x}^n) = \ldots$$

point in chart $C$ \quad coord. of $p$ in chart $\hat{C}$
The scalar field algebra

Given an open subset $U \subset M$, the set $C^\infty(U)$ of scalar fields defined on $U$ has naturally the structure of a **commutative algebra over** $\mathbb{R}$:

1. it is clearly a vector space over $\mathbb{R}$
2. it is endowed with a commutative ring structure by pointwise multiplication:

   $$\forall f, g \in C^\infty(U), \quad \forall p \in U, \quad (f \cdot g)(p) := f(p)g(p)$$

The algebra $C^\infty(U)$ is implemented in SageManifolds via the class ScalarFieldAlgebra.
Classes for scalar fields

- **UniqueRepresentation**
- **Parent**
- **CommutativeAlgebraElement**

**ScalarFieldAlgebra**
- **ring**: SR
- **element**: ScalarField

**category**: CommutativeAlgebras(SR)

**ScalarField**
- **parent**: ScalarFieldAlgebra

Native **Sage** class

**SageManifolds** class
(differential part)
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Various ways to install/access SageMath

- **Install on your computer:**
  3 options:
  - **compile from source (Linux, MacOS X):**
    
git clone git://github.com/sagemath/sage.git
cd sage
MAKE='make -j8' make
  - install a compiled binary version (Linux, MacOS X)
  - run in virtual machine (Windows)

- **Sage Debian Live USB key:**
  http://sagedebianlive.metelu.net/
comes along with SageMath (boosted with octave, scilab), GeoGebra, LaTex, gimp, vlc, LibreOffice,...

- **SageMathCloud:**
  https://cloud.sagemath.com/

- **SageMathCell:**
  Single cell mode: http://sagecell.sagemath.org/
Let us practice!

Various ways to run SageMath

- **Console mode:**
  run the command `sage`

- **Standard Sage Notebook:**
  run the command `sage -n`
  \[\Rightarrow\] worksheet file format: `sws`

- **Jupyter Notebook\(^1\):**
  run the command `sage -n jupyter`
  \[\Rightarrow\] worksheet file format: `ipynb`

- **SageMathCloud:**
  in your browser, open [https://cloud.sagemath.com/](https://cloud.sagemath.com/)
  \[\Rightarrow\] worksheet file format: `sagews, ipynb`

---

\(^1\)the future standard notebook
Let us practice!

A full example: deriving and solving the TOV equations

Let us use SageManifolds to (i) derive the Tolman-Oppenheimer-Volkoff (TOV) equations from the Einstein equation and (ii) to solve the TOV system to get some numerical models of relativistic stars

See the worksheet at
http://nbviewer.jupyter.org/github/egourgoulhon/NewCompStarSchool/blob/master/WorkSheets/TOV.ipynb

The source is stored at GitHub, from which it can be downloaded:
https://github.com/egourgoulhon/NewCompStarSchool

A copy of the worksheet is also publicly available on the SageMathCloud (click on the icon “Files”):
https://cloud.sagemath.com/projects/8f20b8d0-aac0-4454-95d5-dc929a02e1e5/files/TOV.ipynb
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The 2-sphere example

Stereographic coordinates on the 2-sphere

Two charts:
- $X_1: \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- $X_2: \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

See the worksheet at http://sagemanifolds.obspm.fr/examples/html/SM_sphere_S2.html
The 2-sphere example

Vector frame associated with the stereographic coordinates \((x, y)\) from the North pole

\[
\begin{align*}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{align*}
\]

← picture obtained via function

\texttt{VectorField.plot()}
Charts on Schwarzschild spacetime
The Carter-Penrose diagram

Two charts of standard Schwarzschild-Droste coordinates \((t, r, \theta, \varphi)\) plotted in terms of compactified coordinates \((\tilde{T}, \tilde{X}, \theta, \varphi)\); see the worksheet at http://luth.obspm.fr/~luthier/gourgoulhon/bh16/sage.html
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Conclusion and perspectives

- **SageManifolds** is a work in progress
  - ~ 64,000 lines of Python code up to now (including comments and doctests)
- A preliminary version (v0.9) is freely available (GPL) at [http://sagemanifolds.obspm.fr/](http://sagemanifolds.obspm.fr/)
Already present (v0.9):

- maps between manifolds, pullback operator
- submanifolds, pushforward operator
- curves in manifolds
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- all monoterm tensor symmetries
- exterior calculus (wedge product, exterior derivative, Hodge duality)
- Lie derivatives of tensor fields
- affine connections, curvature, torsion
- pseudo-Riemannian metrics, Weyl tensor
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations, via the Python library multiprocessing
Current status

- **Not implemented yet (but should be soon):**
  - extrinsic geometry of pseudo-Riemannian submanifolds
  - computation of geodesics (numerical integration via SageMath/GSL or Gyoto)
  - integrals on submanifolds
Conclusion and perspectives

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- **Future prospects:**
  - add more graphical outputs
  - add more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
  - connection with numerical relativity: using SageMath to explore numerically-generated spacetimes
Integration into SageMath

SageManifolds is aimed to be fully integrated into SageMath

- The **algebraic part** (tensors on free modules of finite rank) has been submitted to SageMath Trac as ticket #15916 and got a positive review → integrated in SageMath 6.6
- The **differential part** has been split in various tickets for submission to SageMath Trac (cf. the metaticket #18528); 4 tickets have been already accepted and integrated in SageMath 7.3
- Until complete integration, the full SageManifold has to be downloaded from http://sagemonifolds.obspm.fr/
- SageManifolds v0.9 is installed in the SageMathCloud → open a free account and use it online: https://cloud.sagemath.com/

Want to join the project or simply to stay tuned?

visit http://sagemonifolds.obspm.fr/
(download page, documentation, example worksheets, mailing list)