Vectorial Boson star

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Contains

1) The Vectorial Field in a Minkoskian space

2) The vectorial field in a psudo-Riemann space

3) Solution of the vectorial equation of motion in spherical gravitation andd geometry

In this section we consider a 4 Minkoskian manifold with Cartesian metric

$$g_{ik} = \delta_{ik}, \quad g_{\alpha\, i} = 0, \quad g_{00} = -1 \tag{1}$$

(geek indexes run from 0 to 4, Latin indexes run from 1 to 3).

Following N.N. Bogolioubov and D.V. Chirkov *Introduction a' la theorie des champs* Ed Dunod (1960) the Lagrangian $\mathcal{L}$ is not uniquely defined. [1])

We require that the momentum energy tensor $T^{\alpha\,\beta}$ satisfies 2 conditions:

1) The tensor is symmetric,

2) The time component $T^{0\,0}$ must be negative defined.

[1] The Lagrangian is supposed to be a quadratic form of the field variables A_α and of its derivatives. In the case of a scalar field, the Lagrangian is uniquely defined.

The simplest Lagrangian $\mathcal{L}_1$ that satisfies the condition 1) is

$$\mathcal{L}_1 = -\frac{1}{2} g^{\alpha\beta} g^{\gamma\eta} \partial_\alpha A_\gamma \partial_\beta A_\eta + \frac{1}{2}\mu^2 A_\alpha A^\alpha \qquad (2)$$

where μ is the mass of the field. [2] By applying the Noether theorem to the action

$$\mathcal{A}_1 = \int \mathcal{L}_1 \, d^4x \qquad (3)$$

we obtain the equation of the motion

$$-\frac{\partial^2 A_\alpha}{\partial x_0^2} + \triangle A_\alpha - \mu^2 A_\alpha = 0 \qquad (4)$$

an the energy tensor $T_{\alpha\beta}$. This tensor is symmetric but T_{00} is not definite negative. By imposing

$$\partial_\alpha \, A^\alpha = 0 \qquad (5)$$

the T_{00} component becomes definite negative.
Note that because of the Eq.(5) only 3 components of the vector field A_α are independent, in agreement with the fact that a massive vectorial field has 3 degrees of freedom: 2 transverse polarisations and one longitudinal. By taking the 4-divergence of the Eq.(4) we have

$$\Box \, \nabla_\alpha A^\alpha - \mu^2 \nabla_\alpha A^\alpha = 0 \qquad (6)$$

That means that if $\nabla_\alpha \, A^\alpha = 0$ vanishes with its time derivative at the time t_0 will vanish at all time.

The Lagrangian $\mathcal{L}_2$

[2] $\mathcal{L}_\infty$ is the Lagrangian used in Q.E.D.

An other Lagrangian can be used to obtain the field equation:

$$\mathcal{L}_2 = -\frac{1}{4} H_{\alpha\,\beta} H^{\alpha\,\beta} + \frac{\mu^2}{2} A_\alpha A^\alpha \tag{7}$$

where

$$H_{\alpha\,\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{8}$$

Note that $\mathcal{L}_2$ differs from $\mathcal{L}_1$ by the term

$$\mathcal{L}_2 - \mathcal{L}_1 = \frac{1}{2}\, g^{\alpha\,\beta} g^{\eta\delta}\, \partial_\eta A_\alpha\, \partial_\beta A_\eta = \frac{1}{2} g^{\alpha\beta}\, g^{\eta\,\delta} \left[\partial_\eta \left(A_\alpha \partial_\beta A_\delta\right) - A_\alpha \partial^2_{\beta\,\eta} A_\delta\right] \tag{9}$$

If the condition given by the Eq.(5) is fulfilled, the Lagrangians $\mathcal{L}_\in$ and $\mathcal{L}_\infty$ differ by a divergence.

The equation of the motion obtained from the Lagrangian $\mathcal{L}_\in$ reads

$$-\frac{\partial^2 A^\alpha}{\partial x_0^2} + \triangle A^\alpha - \partial_\eta \partial_\alpha A^\eta - \mu^2 A^\alpha = 0 \tag{10}$$

The motion equation Eq.(4) and Eq.(10) are equivalent if the condition given by the Eq.(5) is fulfilled. By taking the 4-divergence of the Eq.(10) we obtain

$$\partial_\alpha\, A^\alpha \equiv 0$$

Note the difference between the results obtained with the Lagrangian $\mathcal{L}_\infty$ and $\mathcal{L}_\in$: In the first case the free divergence condition is imposed, in the second case, this condition is automatically fulfilled. [3]

The energy-momentum tensor $\Theta_{\alpha\beta}$ obtained from the La-

[3] In Q.E.D., i.e. when the mass μ vanishes and when the variables A_α are operators, problems are found in computing the commutators of the operators A_α in using the Lagrangian $\mathcal{L}_2$. It turns out that the Q.E.D. cannot be obtained from a massive field by making $\mu \to 0$

grangian reads

$$\Theta_{\alpha\,\beta} = -2\,g^{\eta\,\delta}\,H_{\alpha\eta}H_{\beta\,\delta}+\frac{1}{2}g_{\alpha\,\beta}\,H_{\delta\,\eta}H^{\delta\eta}+2\mu^2 A_\alpha A_\beta-\mu^2 g_{\alpha\beta}\,A_\delta A^\delta \quad (11)$$

The above energy momentum tensor is symmetric more ever the time component T_{00} is definite positive.
The two energy tensors $T_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ differ by a divergence, consequently the global quantities (Energy, 4-momentum,spin) are identical.
Problems arise when the energy momentum tensor is required, for example in studding the vectorial bosons stars: which tensor must be used ? The tensor $T^{\alpha\beta}$ or $\Theta^{\alpha\beta}$? or a linear combination of the two tensors ? I what it follows, we shall show that the presence of the curvature terms of a pseudo- Riemann manifold eliminates the above degeneracy.

Vectorial field in curved space time

The field equations and the energy tensor in curved space can be obtained by minimising the action $\mathcal{A}$

$$\mathcal{A} = \int \hat{\mathcal{L}}\sqrt{-g}\,d^4\,x \quad (12)$$

where $-g$ is the determinant of the metric $g_{\alpha\beta}$ The equivalent $\hat{\mathcal{L}}_1$ or $\hat{\mathcal{L}}_2$ Lagrangian can be obtained by replacing the operator ∂_α by the co-variant derivative ∇_α
We have for $\hat{L}_1$

$$\hat{\mathcal{L}}_1 = -\frac{1}{2}g^{\alpha\beta}\nabla_\alpha A_\eta \nabla_\beta A^\eta + \frac{1}{2}\mu^2 A_\eta A^\eta \quad (13)$$

The same one for $\hat{\mathcal{L}}_2$

The equation of motion

The equation of motion can be obtained in the usual way. Starting from the Lagrangian $\hat{\mathcal{L}}_1$ we obtain

$$\nabla_\beta \nabla^\beta A^\alpha = \mu^2 A^\alpha \tag{14}$$

The above equation looks similar to Eq.(4) but this similitude is misleading, in fact if we take the 4-divergence of both sides of the Eq.(14 and taking into account that the operators $\nabla_\alpha \nabla\beta$ do not commute we have

$$\nabla_\alpha \nabla_\beta A^\beta - \nabla_\beta \left(R^{\delta\,\beta} A_\delta \right) - \mu^2 \nabla_\alpha A^\alpha = 0 \tag{15}$$

where

$$R_{\alpha\beta} = R^{\delta}_{\ \alpha\,\delta\,\beta} \tag{16}$$

We see, that contrary the minkoskian case, $\nabla_\alpha A^\alpha = 0$ is not conserved during the motion. Finally is worth to note that the operator $\nabla_\alpha \nabla^\alpha$ appearing in the Eq.(14) contains second order derivatives of the metric, consequently the vector field is coupled with it self via the the Einstein equation.

The situation is different is different when the equation of the motion are obtained by using the Lagrangian $\hat{\mathcal{L}}_2$ The Equation of the motion reads

$$\nabla_\alpha \nabla^\alpha A_\beta - \nabla_\alpha \nabla_\beta A^\alpha - \mu^2 A_\beta = 0 \tag{17}$$

or equivalently

$$\nabla_\alpha \nabla^\alpha A_\beta - \nabla_\beta \nabla_\alpha A^\alpha - R^{\ \alpha}_{\beta}\, A_\alpha - \mu^2 A_\beta = 0 \tag{18}$$

The above differential operator is the *de Rhamm operator*

Note that this operator is different from the one given by the Eq.(14):in fact the Lagrangian $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}$ do not differ by a divergence.

This operator has the following properties:

1) It no contains second derivatives of the metric tensor $g_{\alpha\,\beta}$

We can verify this claim by performing a long and tedious explicit computation of the above operator. A faster and smarter demonstration consists by noting that antisymmetric tensor $H_{\alpha}\,\beta$ has the following properties (If the torsion vanishes) Eq.(8)

$$H_{\alpha\,\beta} = \nabla\alpha A_{\beta} - \nabla_{\beta} A_{\alpha} \equiv \partial_{\alpha}\, A_{\beta} - \partial_{\beta}\, A_{\alpha} \tag{19}$$

consequently no second order derivatives appear in the operator given by the Eq.**??**

If we consider a locally Cartesian free falling frame in a point $x_0^a lpha$

$$g_{\alpha\,\beta} = \delta_{\alpha\,\beta} + C_{\alpha\,\beta\eta\,\delta}\,(x^{\eta} - x_0^{\eta})\left(x^{\delta} - x_0^{\delta}\right) \tag{20}$$

where $C_{\alpha\beta\,\eta\delta}$ are constant quantities and we make the eikonal approximation for the equation of motion, the obtain the equation of motion of a classical point particle following a geodesic.

2) By taking the 4-divergence of both sides of the Eq.(18) we obtain

$$\mu^2\,\nabla_{\alpha} A^{\alpha} \equiv 0 \tag{21}$$

Conclusion: The tensor $\Theta^{\alpha\beta}$ and the equation of motion

Eq.(18) seems fulfil all the required properties.

Solution of the equation of motion in a Minkokian space in spherical coordinates and spherical components.

We look for a solution of the an harmonic solution of the Eq.(4) i.e. depending on the factor $\exp i\omega\, t$ (Note: In a flat space, the Eq.(4) and Eq.(10) are equivalent.) We look for a solution that for $r \to \infty$ is of the form $\exp i(= \kappa r - \omega\, t$ The Eq. of motion can be written as

$$\triangle A_\alpha = \left(-\omega^2 + \mu^2\right) A_\alpha \tag{22}$$

we introduce the spherical normed components

$$A_r = A_1, \quad A_\theta = \frac{1}{r} A_2, \quad A_\phi = \frac{A_3}{r \sin\theta} \tag{23}$$

and no vanishing components of the metric $g_{\alpha\beta}$

$$g_{0\,0} = -1, \quad g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin\theta^2 \tag{24}$$

It turns be worth to introduce the following angular potential

$$A_\theta = \partial_\theta \eta - \frac{1}{\sin\theta}\,\partial_\phi \chi, \quad A_\phi = \frac{1}{\sin\theta}\,\partial_\phi \eta + \partial_\theta\, \chi \tag{25}$$

we shall explicit the operator for each component We perform an expansion in spherical harmonics $P_l^m(\theta)$. We have for the component $(A_r)_{lm}$

$$\left[\frac{d^2}{dr^2} + \frac{4}{r}\frac{d}{dr} + \frac{1}{r^2}\left(2 - l\,(l+1)\right)\right](A_r)_{lm} - \frac{2}{r}(div\vec{A})_{lm} = \left(-\omega^2 + \mu^2\right)(A_r)lm \tag{26}$$

where $div\vec{A}$ is the 3-space divergence of the 3 vector A^i

$$\left(div\vec{A}\right)_{lm} = \left[\frac{d}{dr} + \frac{2}{r}\right](A_r)_{lm} - l(l+1)\frac{\eta_{lm}}{r} \tag{27}$$

The equation for the potential χ reads

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right]\chi_m = \left(-\omega^2 + \mu^2\right)\chi_{lm} \tag{28}$$

Finally by using the 4-divergence equation $\nabla_\alpha A^\alpha = 0$ we obtain the equation for A^0

$$i\omega A^0_{lm} = (div\vec{A})_{lm} \tag{29}$$

(See Eq.(27) The problem has 3 degrees of freedom, consequently it exist 3 independent elementary solutions. We start with the Eq.(28, for simplicity we consider the case $m = 0$ An elementary solution is

$$\chi_{l0}(r,\theta,t) = z_l(kr)\,P^0_l(\theta)\,\exp -i\omega\,t \tag{30}$$

where $z_l(\kappa r)$ is any kind of Bessel function. with the dispersion law

$$\kappa^2 = \omega^2 - \mu^2 \tag{31}$$

if we take $z_l(kr) = h^1_l(kr)$ the solution pour $r \to \infty$ behaves

$$\chi_{l0} = \frac{P^0_l(\theta)}{\kappa r}\exp i(\kappa r - \omega\,t \tag{32}$$

i.e. an out going wave. If $z_l = h^2-(\kappa r)$ we have an ingoing wave. **Note**: The function $h^1_l(\kappa r)$ can be easily computed. In fact we have

$$4h^1_0(\kappa r) = \frac{\exp i\,\kappa r}{\kappa r}, \quad h^1_1(\kappa r) = \left[\frac{1}{\kappa r} + \frac{i}{(\kappa r)^2}\right]\exp i\kappa r \tag{33}$$

The function h^1_l and their derivatives for $l > 1$ can be computed easily by recurrence.
By using the Eq.25), we an compute A_θ and A_ϕ no trivial solutions exists only for $l > 0$ ($P^0_0(\theta) = 1$). For $m = 0$

only A_ϕ does not vanish. This particular solution,that we shall call, for the analogy with the electromagnetism *Transverse magnetic* is

$$A_0^{TM} = 0; \;\; A_r^{TM} = 0, \;\; A_\theta^{TM} = 0, \;\; A_\phi^{TM} = h_l^1(\kappa r)\, \partial_\theta\, P_l^0(\theta) \;\exp\left(i(\kappa r - \omega t)\right) \tag{34}$$

An other solution can be found by requiring $div\vec{A} = 0$
We have $A_0^{TE} = 0$ and

$$(A_r^{TE})_{=}\frac{h_l^1(\kappa r)}{kr} P_l^0(\theta) \exp\left(i(kr - \omega t)\right) \tag{35}$$

by using the Eq,(27) the potential η is computed.
This solution is transverse asymptotically when $r \to \infty$
When the mass of the $\mu = 0$ we find the solutions of the electromagnetism. If the mass $\mu \neq 0$ a longitudinal solution exists We take

$$A_{lm}^L = h_l^1(kr) P^0 l(\theta) \exp\left(i(kr - \omega t)\right), \;\; \eta = 0, \;\; \chi = 0 \tag{36}$$

By introducing this solution in the Eq.(26 and by using the Eq.(29)we obtain

$$div\vec{A}^L = \left[\frac{d}{dr} + \frac{1}{r}\right] A_{lm}^L = i\omega A_{lm}^0 \tag{37}$$

This is a pure longitudinal wave.
Note: This solution exists also for $l = 0$

Analogy with a electromagnetic wave i a plasma

An electromagnetic in a plasma has a dispersion law identical to the one given by the Eq.(31) where μ is replaced by the plasma frequency ω_p, (no propagation for

$\omega \leq \omega_p$) but the longitudinal modes are different: In the massif vectorial field, the longitudinal mode has the same dispersion law and the same phase velocity that the transverse one. I a wave in the plasma the longitudinal mod is a *Lngmuir wave* the propagation velocity of which depends on the plasma properties in a different way than the transverse one.