Scale-Relativistic Cosmology*

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Abstract

The principle of relativity, when it is applied to scale transformations, leads to the suggestion of a generalization of fundamental dilations laws. These new special-scale-relativistic resolution transformations involve log-Lorentz factors and lead to the occurrence of a minimal and of a maximal length-scale in nature, which are invariant under dilations. The minimal length-scale, that replaces the zero from the viewpoint of its physical properties, is identified with the Planck-length l_P, and the maximal scale, that replaces infinity, is identified with the cosmic scale $\mathbb{L} = \Lambda^{-1/2}$, where Λ is the cosmological constant.

The new interpretation of the Planck scale has several implications for the structure and history of the early Universe: we consider the questions of the origin, of the status of physical laws at very early times, of the horizon / causality problem and of fluctuations at recombination epoch.

The new interpretation of the cosmic scale has consequences for our knowledge of the present universe, concerning in particular Mach's principle, the large number coincidence, the problem of the vacuum energy density, the nature and the value of the cosmological constant. The value (theroretically predicted ten years ago) of the scaled cosmological constant $\Omega_\Lambda = 0.75 \pm 0.15$ is now supported by several different experiments (Hubble diagram of Supernovae, Boomerang measurements, gravitational lensing by clusters of galaxies).

The scale-relativity framework also allows one to suggest a general solution to the missing mass problem, and to make theoretical predictions of fundamental energy scales, thanks to the interpretation of new structures in scale space: fractal /classical transitions as Compton lengths, mass-coupling relations and critical value $4\pi^2$ of inverse couplings. Among them, we find a structure at $3.27 \pm 0.26 \times 10^{20}$ eV, which agrees closely with the observed highest energy cosmic rays at $3.2 \pm 0.9 \times 10^{20}$ eV, and another at 5.3×10^{-3} eV, which corresponds to the typical neutrino mass needed to explain solar neutrino oscillations.

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1 Introduction

The theory of scale-relativity is founded on a generalization of the standard description of space-time, and, as such, it has obvious cosmological consequences. In this approach, the principle of relativity is applied to scale transformations of space-time resolutions (i.e., of observing scales). One relaxes the hypothesis of derivability of coordinates, so that, in such an extended framework, space-time is not only curved, but also fractal.

The theory of motion relativity, which started with Galileo [1], amounts to describe space-time in terms of a twice-differentiable manifold. Its successive steps were marked by parallel generalizations of the principle of relativity and of the geometry of space and time: Galilean relativity and Newton's Euclidean space; Poincaré-Einstein special relativity [2, 3] and Minkowski's still absolute space-time; finally Einstein's generalized motion relativity and Riemannian relative manifolds [4].

In the scale-relativity construction, we work in a continuous manifold that may be differentiable or not-differentiable depending on the scale range considered. We follow the same kind of steps [5, 6, 7] for the description of the space of scales as in the historical developments of the space of positions (and instants); however, new elements of description must also be added due to the motion-scale interaction [8, 9], which lead to a re-interpretation of gauge transformations.

Hence a Galilean scale relativity is first constructed [10, 6], in which standard scale invariant laws (with constant fractal dimensions) are obtained as a scale equivalent of inertial motion. This scale symmetry is also spontaneously broken at some transition scale which gives a geometric meaning to fundamental mass scales. Moreover the laws of displacement that include in their description these underlying fractal structures can be given a quantum form [6, 9, 11]. In particular, the fondamental law of dynamics is integrated in this framework in terms of a Schrödinger equation. This result supports the earlier suggestions that a fractal geometry of space-time could be the source of quantum behavior [12, 13, 10, 14, 15]. We have presented in the joint paper [16] some cosmological consequences for structure formation and for the "missing mass" problem of the use of this approach in the macroscopic gravitational case.

The second step consists of introducing a special scale-relativity in which the laws of scale contraction and dilation have the structure of the Lorentz group [5]. The main result of this generalization is the reinterpretation of (i) the Planck length-scale $l_P = (\hbar G/c^3)^{1/2}$ as a minimal, impassable scale, invariant under dilations, that replaces the zero point from the view point of its physical properties; (ii) the cosmic scale $\mathbb{L} = \Lambda^{-1/2}$, which gives the genuine physical meaning of the cosmological constant Λ, as a maximal, impassable scale, invariant under dilations, that replaces infinity from the view point of its physical properties. The number $\mathbb{C} = \ln(\mathbb{L}/l_P) = 139.8$ plays for scale transformations the same part as played by the light velocity for motion transformation. Such log-Lorentzian scale laws has also been applied, with a different interpretation of the variables, to the general problem of finite size scale invariance and its application to turbulence (see [17] and references therein).

The third step accounts for the coupling between motion and scale transformations. This means, in other words, to look for non-linear scale transformations in which the scales of resolution become themselves functions of the coordinates. Such a development therefore comes under general scale-relativity. It allows one to suggest a new interpretation of gauge invariance, of the nature of gauge fields and therefore of the electric charge itself [8, 9, 18]. In this framework universal relations can be constructed between mass scales and coupling constants, which are supported by the experimental data.

The present contribution is devoted to the study of the cosmological consequences of these proposals. Strictly they should be studied in the framework of a complete theory which would be both motion-general-relativistic and scale-general-relativistic. Such a general theory is still in construction, so that only hints about its expected conclusions can be up to now given. The new interpretation of the Planck scale has several implications for the structure and history of the early Universe: we consider the questions of the origin, of the status of physical laws at very early times, of the horizon / causality problem and of fluctuations at recombination epoch. The new interpretation of the cosmic scale changes our view of the present universe, concerning in particular Mach's principle, the large number coincidence, the problem of the vacuum energy density, the power spectrum / correlation function of present large scale structures of the Universe, the nature and value of the cosmological constant, etc... We recall in particular that we have theroretically predicted ten years ago a value of $\Omega_\Lambda = 0.7$ (for a Hubble constant $H_0 = 70$ km/s.Mpc), which is now supported by several different experiments (Hubble diagram of Supernovae, Boomerang measurements, gravitational lensing by clusters of galaxies). Finally we make proposals concerning the very high energy cosmic rays at 3.2×10^{20} eV and the neutrino mass, and we show that the various new cosmological scale-relativistic effects allow several independent and converging numerical estimates of the cosmological constant.

2 Theoretical background

2.1 Galilean scale relativity

2.1.1 Scale-invariant fractal laws

Simple fractal scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws. Consider a non-differentiable coordinate $\mathcal{L}$. The fundamental theorem that links non-differentiability to fractality [9] implies that $\mathcal{L}$ is an explicit function $\mathcal{L}(\varepsilon)$ of the resolution interval ε. We have recalled in the joint paper [16] how one can recover scale-invariant laws with constant fractal dimension as solutions of a first order scale differential equation:

$$\frac{d \ln \mathcal{L}}{d \ln(\lambda/\varepsilon)} = \delta, \tag{1}$$

where δ is a constant. The solution is a fractal, power-law dependence:

$$\mathcal{L} = \mathcal{L}_0 (\lambda/\varepsilon)^\delta, \tag{2}$$

where the scale dimension $\delta = D - D_T$, D being the fractal dimension and D_T the topological dimension. The Galilean structure of the group of scale transformation that corresponds to this law can be checked in a straightforward manner from the fact that it transforms in a scale transformation $\varepsilon \to \varepsilon'$ as

$$\ln \frac{\mathcal{L}(\varepsilon')}{\mathcal{L}_0} = \ln \frac{\mathcal{L}(\varepsilon)}{\mathcal{L}_0} + \delta(\varepsilon) \ln \frac{\varepsilon}{\varepsilon'} \quad ; \quad \delta(\varepsilon') = \delta(\varepsilon). \tag{3}$$

This transformation has exactly the structure of the Galileo group (but here for scales instead of motion), as confirmed by the law of composition of dilations $\varepsilon \to \varepsilon' \to \varepsilon''$, which writes:

$$\ln \rho'' = \ln \rho + \ln \rho', \tag{4}$$

with $\rho = \varepsilon'/\varepsilon$, $\rho' = \varepsilon''/\varepsilon'$ and $\rho'' = \varepsilon''/\varepsilon$.

2.1.2 Breaking of the scale symmetry

More generally, one can write a first order equation where the scale variation of $\mathcal{L}$ depends on $\mathcal{L}$ only, i.e. $d\mathcal{L}/d\ln\varepsilon = \beta(\mathcal{L})$. The function $\beta(\mathcal{L})$ is a priori unknown, but we may consider as a first step a perturbative approach and take its Taylor expansion. We obtain the equation:

$$\frac{d\mathcal{L}}{d\ln\varepsilon} = a + b\mathcal{L} + \ldots \tag{5}$$

This equation is solved in terms of a standard power law of power $\delta = -b$, broken at some relative scale λ (which is a constant of integration):

$$\mathcal{L} = \mathcal{L}_0 \left[1 + \left(\frac{\lambda}{\varepsilon}\right)^{\delta}\right]. \tag{6}$$

Depending on the sign of δ, this solution represents either a small-scale fractal behavior (in which the scale variable is a resolution), broken at larger scales, or a large-scale fractal behavior (in which the scale variable ε would now represent a changing window for a fixed resolution λ), broken at smaller scales.

This transition scale λ between the differentiable and non-differentiable regimes acquires a very fundamental meaning in the theory. Indeed, it can be identified with the conservative quantity that is built from the scale symmetry (see [6] chap. 6.9). As will be specified in more detail hereafter, it therefore provides a geometric meaning to the concept of mass (inertial mass at small scales and gravitational mass at large scales). Conversely, theoretical predictions of mass scales can be performed thanks to this new definition of mass.

2.2 Special scale-relativity

It is well known that the Galileo group of motion is only a degeneration of the more general Lorentz group of motion. The same is true for scale laws, with an additional subtlety due to the above scale symmetry breaking.

Indeed, if one looks for the general linear laws of scale that come under the principle of scale relativity, one demonstrates that, once they are expressed in logarithm form (this allowing to jump from a multiplicative group to an additive group), they have the structure of the Lorentz group [5]. Therefore, in the framework of special scale relativity, we substitute to the Galilean law of composition of dilations $\ln(\varepsilon'/\lambda) = \ln\rho + \ln(\varepsilon/\lambda)$ the more general log-Lorentzian law:

$$\ln \varrho" = \frac{\ln \varrho + \ln \varrho'}{1 + \ln \varrho \ln \varrho' / \mathbb{C}^2}. \tag{7}$$

Under this form the scale relativity symmetry remains unbroken. Such a law corresponds, at the present epoch, only to the null mass limit. As we shall see hereafter, it is expected to apply in a universal way during the very first instants of the universe. This law assumes that, at very high energy, no static scale and no space or time unit can be defined, so that only pure contractions and dilations have physical meaning. In Eq. 7, there appears a universal purely numerical constant $\mathbb{C} = \ln \mathbb{K}$. As we recall in what follows, $\mathbb{K}$ can be identified with the ratio of the cosmic length $\mathbb{L} = \Lambda^{-1/2}$ over the Planck length l_P, where Λ is the cosmological constant. The recently measured value of the cosmological constant (see what follows) agrees with its theoretically predicted value, that corresponds to a value of $\mathbb{K} = 5.3 \times 10^{60}$ [6, 9].

Now, as we have seen in the previous section, the scale symmetry is broken in microphysics by the mass of elementary particles, i.e., by the emergence of their Compton-de Broglie length:

$$\lambda_c = \frac{\hbar}{mc} \quad ; \quad \lambda_{dB} = \frac{\hbar}{mv} \tag{8}$$

and for extended objects by the thermal de Broglie length:

$$\lambda_{\rm th} = \frac{\hbar}{m \langle v^2 \rangle^{1/2}}. \tag{9}$$

The transition scale in Eq. 6 can indeed be identified with the Einstein-de Broglie scale [6, 9, 19]. This scale becomes in rest frame $\tau = \hbar/mc^2$, and it is therefore equivalent to the mass of the particle, up to fundamental constants.

In the cosmological realm, the scale symmetry is broken toward the small scales by the emergence of static structures (galaxies, groups, cluster cores) of typical size:

$$\lambda_g \approx \frac{1}{3} \frac{GM}{\langle v^2 \rangle}. \tag{10}$$

The effect of these two symmetry breakings is to separate the scale space into three domains, a quantum (scale-dependent) domain at small scales, a classical (scale-independent) domain at intermediate scales and a cosmological (scale-dependent) domain toward the large scales [6, 9], in which explicit scale-dependence and quantum-like laws take place again (see e.g. the joint paper [16]).

The consequence is that, in the two scale-dependent domains, these transition scales λ can be taken as references for scale ratios, so that one do not deal any longer with dimensionless ratios, but with a new law involving dimensioned space and time intervals [5]. The new law of composition of dilations writes, instead of $\varepsilon' = \rho \times \varepsilon$:

$$\ln \frac{\varepsilon'}{\lambda} = \frac{\ln \rho + \ln(\varepsilon/\lambda)}{1 + \ln \rho \ln(\varepsilon/\lambda) / \ln^2(L/\lambda)}. \tag{11}$$

This means that we combine Galilean scale-relativity with Lorentzian scale-relativity on two different ranges of scales separated by the transition scale λ. While the dilation ratio from λ to ε is given by the Galilean ratio λ/ε (the same being true for λ to ε'), this is no longer the case of the dilation ratio ρ from ε to ε': namely, $\varepsilon'/\varepsilon \neq \rho$ in the above formula. In the Lorentzian law of composition of dilations, a dimensioned scale L appears, such that $\mathbb{C} = \ln(L/\lambda)$, which, as can be directly checked in the above expression, is invariant under dilations [5, 6]. It can be identified [5, 6]:

* toward the small scales with the Planck length-scale $L = l_P = (\hbar G/c^3)^{1/2}$,

* and toward the large scales with the cosmic scale $L = \mathbb{L} = \Lambda^{-1/2}$, where Λ is the cosmological constant.

The law of transformation of a fractal coordinate $\mathcal{L}$ in a scale change of the reference system by a dilation ρ is now given, in the asymptotic scale-dependent domain, by:

$$\ln(\mathcal{L}'/\mathcal{L}_0) = \frac{\ln(\mathcal{L}/\mathcal{L}_0) + \delta \ln \varrho}{\sqrt{1 - \ln^2 \varrho/\mathbb{C}^2}}. \tag{12}$$

Moreover, the scale dimension δ, which is a constant in the standard fractal case (here identified with Galilean scale-relativity) becomes itself a variable: we have called "djinn" this variable scale dimension. More precisely, the couple it makes with the fractal coordinate $[\delta, \ln(\mathcal{L}/\lambda)]$ becomes a scale vector. In other words, the variable scale-dimension is now identified with a fifth dimension of the fractal space-time (in analogy with the passage to the four-dimensional Minkowskian space-time of special relativity of motion). In a scale transformation by a factor ρ this fifth dimension transforms as:

$$\delta(\varepsilon') = \frac{\delta(\varepsilon) + \ln \rho \ln(\mathcal{L}/\mathcal{L}_0)/\mathbb{C}^2}{\sqrt{1 - \ln^2 \rho/\mathbb{C}^2}}, \tag{13}$$

where $\mathbb{C} = \ln(L/\lambda)$, and where λ still denotes the fractal-nonfractal transition scale. It is apparent in these expressions that possible scale ratios in nature are now limited by the ratio L/λ, in the same way as velocities are limited by the invariant velocity c. Indeed, in the limit $\ln \rho \rightarrow \mathbb{C}$, the scale-dimension tends to infinity, in agreement with El Naschie's result [14, 15] that the dimension of the fractal space-time is ultimately infinite. This limit corresponds to the Planck length-time scale toward very small scales and to the cosmic scale $\mathbb{L}$ toward very large scales. Therefore in the special scale-relativity framework, these two scales can neither been reached nor crossed (i.e. nothing actually exists beyond them), and they replace from the viewpoint of their physical properties respectively the zero and the infinite.

2.3 Toward general scale-relativity: five-dimensional fractal space-time

The full meaning of these limiting scales is readily understood in the 5-dimensional theory, i.e. by working in the five-dimensional "space-time-djinn". Once again the analogy with motion relativity is enlightning. We already know from the Poincaré-Einstein work that the very nature of the velocity of light is an horizon, a vanishing point of perspective resulting of a projection effect from 4-dimensional space-time to 3-dimensional space (as confirmed by the fact that the 4-velocities tend to infinity when $v \to c$). In this view, velocities in space are nothing but rotations in space-time.

The same is true in the scale-relativity theory, through the addition of the fifth extra dimension or "djinn" that plays for scale transformations of resolutions the same role as played by time for motion laws. One should however be cautious with the fact that this fifth dimension is not combined with the four standard classical space-time dimensions. As indeed recalled in the joint paper [16] and in the previous section, scale differential equations yield solutions (Eq. 6) in which the position variable becomes the sum of a classical derivable variable and of a fractal, non-derivable fluctuation, i.e., $dX^\mu = dx^\mu + d\xi^\mu$. While the dx^μ remain four-dimensional and submitted to a local Minkowski metric, the $d\xi^\mu$ are combined in an Euclidean way in Galilean scale relativity (as errors do), then become submitted to a locally Minkowkian 5-dimensional metric in special and in general scale-relativity:

$$d\sigma^2 = G_{\alpha\beta}\, d\xi^\alpha d\xi^\beta, \tag{14}$$

with α and $\beta = 0$ to 4 and $d\xi^0 = d\delta$. In coordinate systems where $G_{0\mu} = 0$ (with $\mu = 1$ to 4), it may be written under the form:

$$d\sigma^2 = G_{00}\, d\delta^2 - \tilde{G}_{\mu\nu}\, d\ln \xi^\mu d\ln \xi^\nu. \tag{15}$$

When crossing the transition scale in the scale space, one jumps from a regime dominated by the four classical variables dx^μ to a new regime dominated by the five fractal variables $d\xi^\alpha$, made of the four fractal fluctuations and of the djinn (the variable fractal dimension).

Note that the idea of a non-compactified fifth dimension has also been recently suggested in a different context [20] and its cosmological implications are currently actively studied. However this concept remains different from the scale-relativity one, since it deals with a fourth space-like dimension. Moreover this extra dimension is postulated arbitrarily, and therefore it still needs a direct detection or an indirect one through its possible effects, while the introduction of the djinn relies on the already established existence of the four resolutions for the four space-time coordinates, and on the description of their dilations and contractions as rotations in the 5-dimensional space-time-djinn.

2.4 Scale-motion coupling and gauge transformations

2.4.1 Gauge invariance

The theory of scale relativity also allows one to get new insights about the physical meaning of gauge invariance [8, 9]. In the scale laws recalled hereabove, only scale transformations at a given point were considered. But we may also wonder about what happens to the structures in scale-space of a scale-dependent object such as an electron or another charged particle, when it is displaced. Consider anyone of these structures, lying at some (relative) resolution ε (such that $\varepsilon < \lambda$, where λ is the Compton length of the particle) for a given position of the particle. In a displacement, the relativity of scales implies that the resolution at which this given structure appears in the new position will a priori be different from the initial one. In other words, $\varepsilon = \varepsilon(x, t)$ is now a function of the space-time coordinates, and we expect the occurrence of dilations of resolutions induced by translations, so that we are led to introduce a covariant derivative:

$$e \frac{D\varepsilon}{\varepsilon} = e \frac{d\varepsilon}{\varepsilon} - A_\mu dx^\mu, \tag{16}$$

where a four-vector A_μ must be introduced since dx^μ is itself a four-vector and $d \ln \varepsilon$ a scalar (in the case of a global dilation, as considered here as a first step).

However, if one wants such a "field" A_μ to be physical, it should be defined whatever the initial scale from which we started. Starting from another scale $\varepsilon' = \rho\varepsilon$, we get the same expression as in Eq.(16), but with A_μ replaced by A'_μ. Therefore, we obtain the relation:

$$A'_\mu = A_\mu + e\, \partial_\mu \ln \rho, \tag{17}$$

which depends on the relative "state of scale", $\bar{V} = \ln \rho = \ln(\varepsilon/\varepsilon')$, that is now a function of the coordinates.

One may therefore identify A_μ with an electromagnetic potential, and Eq.(17) with the property of gauge invariance, which therefore finds an explanation in the scale-relativity framework [8, 9]. This approach shares some common features with Weyl's theory of gauge invariance [21]. However, as we now recall, the fact to give a physical and geometric meaning to the basic variables (namely, the internal resolutions) on which the scale transformations are applied allows one to go one step further and to demonstrates the quantization of the electric charge.

2.4.2 Universal relation between mass-scales and couplings

Applying a gauge transformation to the electromagnetic field implies to change also the wave function of the electron, that becomes:

$$\psi' = \psi\, e^{i4\pi\alpha \ln \rho} \tag{18}$$

where α is a coupling constant.

While in Galilean scale relativity, the scale ratio ρ is unlimited, in the more general framework of special scale relativity it is limited by the Planck-scale/Compton-scale ratio. This limitation implies the quantization of charge, following a general mass-charge relation [9]:

$$\alpha \times \ln\left(\frac{m_P}{m}\right) = \frac{k}{2}, \tag{19}$$

where k is integer. Let us recall how such a relation compares with experimental data in the case of the fine structure constant [8, 9]. In applying it to the electron, one should account for the electroweak theory, according to which the electromagnetic coupling is only 3/8 of its high energy value (plus radiative corrections). We get:

$$\frac{8}{3}\, \alpha_{\text{em}} \ln\left(\frac{m_P}{m_e}\right) = 1 \tag{20}$$

where $\alpha_{\text{em}} = 1/137.036$ is the low energy fine structure constant and m_e is the electron mass. This relation is supported by the experimental data with a relative precision of

2×10^{-3}, becoming 10^{-4} when accounting for threshhold effects [9] . It can be interpreted as stating that all or most of the electron mass is of electroweak origin, and can be used to theoretically predict the value of the electron mass in terms of its charge (see Fig. 1).

This approach can be generalized to more general, non-abelian gauge theories, since, as recalled hereabove, we can define four different and independent dilations along the four space-time resolutions instead of only one global dilation. The above U(1) field is then expected to be embedded into a larger field, in agreement with the electroweak and grand unification theories, and the charge e to be one element of a more complicated, "vectorial" charge (see [22, 23, 24] for first attempts in this direction, including a tentative theoretical prediction of the Higgs mass).

2.4.3 Critical value $4\pi^2$ of inverse couplings

Let us recall our simple argument [9] according to which the natural "bare" value of any coupling constant is given by the optimized value $\alpha_b = 1/4\pi^2$.

Assume that there exists a Coulomb-like force between two bodies. In the classical theory, this force is attributed to the fact that the bodies carry "charges" Q_1 and Q_2, and it writes:

$$F = \frac{Q_1\, Q_2}{4\pi\, r^2}. \tag{21}$$

Now the microscopic quantum theory, since the work of Einstein [25], interprets this process in terms of quanta. Namely, the two macroscopic bodies are themselves made of elementary substructures that carry quanta of charges e, so that $Q_1 = Z_1\, e$ and $Q_2 = Z_2\, e$. The force between two quanta of charge now writes:

$$F = \frac{e^2}{4\pi\, r^2}. \tag{22}$$

It can be re-expressed in terms of a coupling constant α under the quantum form:

$$F = \frac{\alpha\, \hbar c}{r^2}, \tag{23}$$

so that one makes the identification $e^2 = 4\pi\alpha\hbar c$.

The second quantization theory states that the force itself is carried by elementary particles, namely, by the continous exchange of zero mass bosons between the two charges. Indeed the existence of a mass for the intermediate particle would introduce a vanishing Yukawa term $\exp(-r/\lambda_c)$ in the expression of the force. The macroscopic, classical force therefore results from the average effect of the boson exchanges:

$$F = \langle \frac{\delta p}{\delta t} \rangle, \tag{24}$$

i.e., it is given by the average variation of momentum over the time interval δt. Now, each charge quanta emits bosons isotropically, while only those which are absorbed by the other quanta participate in the interaction. The 3-D nature of space therefore implies a geometric factor $1/4\pi$. We obtain:

$$F = \frac{1}{4\pi}\, \frac{\delta p}{\delta t}. \tag{25}$$

Now the exchanged momentum comes under a Heisenberg inequality. It will therefore be optimized by the "saturating", minimal value of this inequality, which is of universal meaning (this was the basic starting point of the construction of the scale-relativity theory, see [10, 6]). However, one should be cautious about the precise Heisenberg relation to be used here. We indeed do not look here for orders of magnitude, but for exact statements.

Fortunately a general method making use of the concept of information entropy has been devised by Finkel [26] for constructing any exact Heisenberg relation between any couple of variables. This method has been used in [7] to compare in a precise way the fractal description of inequalities and the quantum mechanical one. Here we look for the Heisenberg relation that relates the distance interval r and the momentum difference δp. While the usual Heisenberg relation for variances is saturated for the standard value $\hbar/2$, one finds in the case of intervals:

$$\delta p \times r \geq \frac{\hbar}{\pi}. \tag{26}$$

The limit value is precisely reached by the harmonic oscillator vacuum state.

The last step of the demonstration makes use of the fact that the intermediate bosons are of null mass, so that they move at the velocity of light c, and therefore $\delta t = r/c$. We finally find for the optimized amplitude of the force

$$F = \frac{1}{4\pi^2} \frac{\hbar c}{r^2}. \tag{27}$$

The comparison with Eq. 23 yields the announced result [9]:

$$\alpha_b = \frac{1}{4\pi^2}. \tag{28}$$

Let us give another simple geometric and dimensional argument also leading to the result $\alpha^{-1} = 4\pi^2$. In the framework of a possible future unified theory of the presently known fields, the Newton and Coulomb forces would become manifestations of various aspects of the same geometry. Now one of the essential geometric relation issued from Einstein's and Newton's theory of gravitation is Kepler's third law in a $1/r$ central potential, that writes:

$$(GM)^{-1}\, a^3\, T^{-2} = \frac{1}{4\pi^2}, \tag{29}$$

where M is the mass of the central body, a the semi-major axis and T the period. The $1/4\pi^2$ factor directly comes from the specific relation involved here between space and time: namely, for a circular orbit, the distance a is covered in a time $T/2\pi$. Coulomb's law has the same structure as Newton's law, with GMm replaced by e^2. But the dimensionality of the squared charge is precisely $[\mathrm{M\ L^3\ T^{-2}}]$, while there is a fundamental duality ($M \leftrightarrow 1/GM$) between the inertial mass and the active gravitational mass (see [10, 27] and what follows). We suggest that, in a geometric interpretation of electromagnetism such as that suggested in the scale-relativity theory, there is the same kind of effect linked to orbit closure in the definition of charge as in Kepler's third law, so that the dimensionality relation should be written $[\mathrm{M\ L^3\ (T/2\pi)^{-2}}]$. The expected natural critical value 1 for this combination is therefore translated in a critical value $1/4\pi^2$ for the coupling constant.

This result may seem, at first sight, to be in contradiction with the experimental values of coupling constants. However, the reason why this value is not directly seen in the experimental measurement of coupling constants is twofold:

(i) The couplings are actually explicitly scale-dependent ("running" with energy-scale), because of radiative corrections for the effects of fermion-antifermion pairs. Therefore the $4\pi^2$ value of couplings is expected to be reached only at some particular, critical scales, to be determined.

(ii) In the standard model, the bare couplings (at infinite energy) remain either (logarithmically) divergent or vanishing. On the contrary, in the special scale-relativistic framework [5], the divergence problem of mass and charge left unsolved by the renormalization theory is solved, so that one becomes able to look at the bare values.

When accounting for this running character of couplings, the results obtained (see Fig. 1) support the systematic occurrence of the critical value $4\pi^2$ for inverse couplings [5, 6, 9] at several specific and fundamental scales. We shall see in what follows that it also allows one to predict new very high energy structures: for example, the solution of the equation $\alpha_2^{-1}(E) = 4\pi^2$ is $E = 3.27 \pm 0.26 \times 10^{20}$ eV).

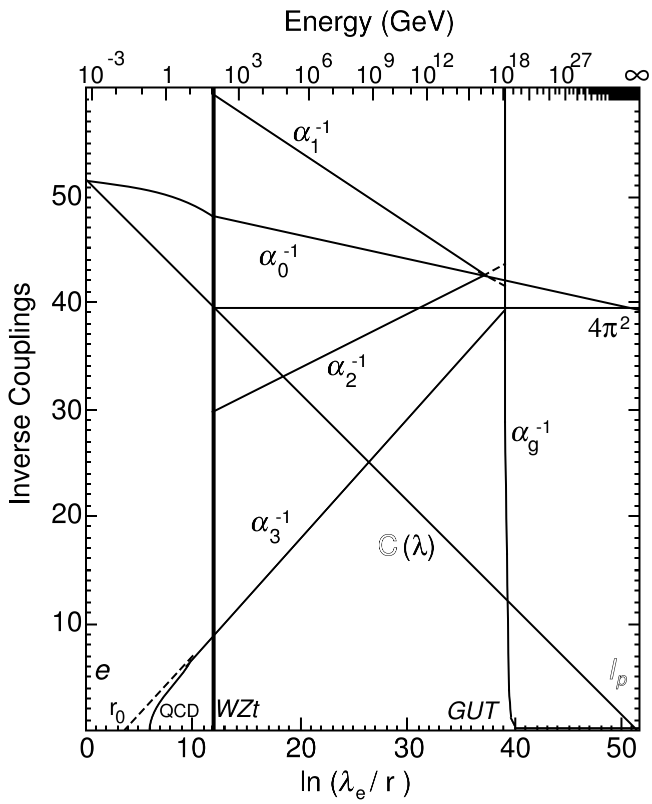

Figure 1: Variation with scale of the inverse couplings of the fundamental interactions U(1), SU(2) and SU(3) in the scale-relativistic minimal standard model. This scale-relativistic diagram (mass-scale versus inverse coupling constants) shows well-defined structures and symmetries that are accounted for by the mass / charge relations (see text).

2.4.4 Consequence for space dimensionality

The above way of recovering the expression for the Coulomb force provides us with another result, namely, a possible demonstration that the topological dimension of space is expected to be $D_T^S = 3$ at large scales. Indeed, the classical distance dependence of an infinite range force is $r^{1-D_T^S}$, while we have obtained here a r^{-2} dependence only from the Heisenberg relation. More generally, a generalized Heisenberg relation can be demonstrated for any fractal dimension D of the geodesics of the fractal space-time [6], that writes:

$$\delta p \times r^{D-1} \geq (\hbar/2)\, \lambda_c^{D-2}, \tag{30}$$

where λ_c is the Compton length. We recover the standard Heisenberg relation for the critival value $D = 2$ (see e.g. [9]), which is the universal Feynman value of the fractal dimension (that corresponds to standard quantum mechanics).

We therefore obtain a general relation between the topological dimension of space and the fractal dimension of geodesics:

$$D_T^S = D + 1. \tag{31}$$

The special scale-relativistic transformation in which the effective value of D becomes a variable does not contradict this result, since the relevant fractal dimension in this case is the proper (or scale-covariant) dimension, which is nothing but the 5-dimensional invariant itself (which remains $\sigma = 2$).

Moreover, while from the fractal dimension $D = 2$ of quantum trajectories at low energy we recover a dimensionality of space $D_T^S = 3$, the expected increase of the fractal dimension at high energies in the special scale relativity framework leads to the possibility of extra topological dimensions, which are a necessary ingredient of the superstrings approach.

2.5 Theoretical prediction of mass scales

Before considering the cosmological implications of the scale-relativity approach, let us sum up the various new ways by which a theoretical prediction of mass-energy scales can be performed (see Fig. 1).

• Transitions from fractal (non-derivable) to standard (derivable) regime are identified with fundamental mass scales: this allows one to rely the cosmological constant to elementary particle masses and to suggest a value for a neutrino mass.

• Critical value $4\pi^2$ for inverse couplings: the electromagnetic coupling at infinite energy and the strong coupling at Planck energy satisfy the relation $\alpha^{-1}(E) = 4\pi^2$. A new prediction can be reached by applying it to the SU(2) weak coupling.

• Universal mass scale - coupling (charge) relations: such relations, that are demonstrated from our reinterpretation of gauge invariance, apply to the mass and charge of the electron, to the electroweak scale and the critical coupling $1/4\pi^2$, and it allows one to predict new energy scales when applied to the SU(3) and SU(2) couplings (see Figure 1). Moreover, the gravitational Schrödinger equation (see [16] and the next section) being also gauge invariant, it can be applied to an attempt of theoretical prediction of the new gravitational coupling constant α_g.

3 Cosmological implications of scale relativity

3.1 The "missing mass" problem

Before applying the scale-relativity principle to the study of the global geometry of the Universe, let us briefly come back on our proposed solution to one of the most important problem of today's cosmology, that of the so-called "missing mass" or "dark matter". This question has already been considered in the joint paper [16], in particular as concerns the flat rotation curves of spiral galaxies, but we shall give here a more general argument. It suggests that the various dynamical and lensing effects that are tentatively interpreted in the standard approach as necessitating the existence of large amounts of unseen matter can be readily explained by the fractality of space.

Recall that, starting from the three simplest new properties of a non-differentible manifold (as compared with a Riemannian manifold) namely: (i) infinity of geodesics; (ii) decomposition of each elementary displacement in terms of the sum of a classical variable and a fractal variable of fractal dimension 2; (iii) two-valuedness of the velocity vector due to irreversibility in the reflexion $dt \leftrightarrow -dt$, the geodesics equation in a curved and fractal space(-time) can be integrated in the form of a Schrödinger equation [6, 11] that writes at the Newtonian limit:

$$\mathcal{D}^2 \bigtriangleup \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{1}{2}\varphi\, \psi, \tag{32}$$

where φ is the Newtonian potential, which is a solution of the Poisson equation:

$$\bigtriangleup\varphi = 4\pi G\rho. \tag{33}$$

From our description of the motion in terms of an infinite family of geodesics, the meaning of $P = \psi\psi^{\dagger}$ is imposed as giving the probability density of the particle positions, in agreement with Born's postulate. Indeed, separating the real and imaginary parts of the Schrödinger equation and writing it in terms of P and the classical velocity V, we get respectively a generalized Euler-Newton equation and a continuity equation [28]:

$$(\frac{\partial}{\partial t} + V \cdot \nabla)V = -\nabla(\varphi + q), \tag{34}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0, \tag{35}$$

This system of equations is equivalent to the classical one used in the standard approach of gravitational structure formation, except for the appearance of an extra potential term q that writes:

$$q = -2\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{36}$$

This potential is a manifestation of the fractality of space, in the same way as the Newtonian potential is a manifestation of space-time curvature. We suggest that its existence explains the various dynamical effects presently attributed to unseen, dark matter. Indeed, let us come back to the Schrödinger form of these equations. Two extreme situations (and any intermediate between them) can be considered:

(i) The particles fill the probability density distribution, so that $\rho \propto P$. In this case the system of equations is a coupled Schrödinger-Poisson (Hartree) system, of the kind used to describe superconductivity (by this way our conclusions meet those of Agop et al.[29], who attempt to describe the effect of a Cantorian-fractal space-time in terms of superconducting properties of matter). This case corresponds to a self-gravitating body such as a cluster of galaxies. Now Markowich et al. [30] have recently demonstrated the general existence and non-linear stability of steady states of the Schrödinger-Poisson system, with conserved total energy.

(ii) There are only very few test-particles, so that from the view-point of matter density, we deal with the vacuum. This case corresponds to the outer regions of spiral galaxies (in the absence of dark matter as assumed here). Therefore φ is a solution of $\Delta\varphi = 0$, i.e. $\varphi = -G\,\Sigma_i(M_i/r_i)$. The Schrödinger equation with such a potential has also general stationary solutions.

Therefore in both cases, we can write a time-independent Schrödinger equation that takes the simplified form:

$$2\mathcal{D}^2 \bigtriangleup \psi + (\mathcal{E} - \varphi)\psi = 0, \tag{37}$$

where $\mathcal{E} = E/m$ and φ is the steady-state solution for the potential. In the gravitational macroscopic case considered here, this equation is subjected to the principle of equivalence (contrarily to the standard microscopic quantum mechanics, where $\mathcal{D} = \hbar/2m$, and therefore it does not depend on the inertial mass m of the bodies the distribution of which it describes. Note that in [16] the equations are written in terms of the potential energy $Q = m\,q$: our use here of the potential instead of the potential energy allows to manifest the vanishing of the inertial mass from these equations. For this steady-state solution, one finds the relation:

$$\mathcal{E} = \varphi + q. \tag{38}$$

Since $\mathcal{E} = E/m \propto v^2$, this relation expresses the main results of the scale-relativity-Schrödinger approach to gravitational structuration: namely, (i) the expected (and now observationally supported [16]) universal quantization of velocities, i.e. the theoretical prediction that in various gravitational systems the probability distribution of the velocities will not be in general flat, but instead will have a tendency to show structures at values independent of the particular system [6] Chap. 7.2 [32, 33, 34, 35, 36, 28, 31]; (ii) the observed values of the velocity (e.g., rotation curves of galaxies, velocity dispersion in clusters of galaxies, ...) is determined, not only by Newton's potential, but also by the new potential q.

Our proposal is, therefore, that this potential, of non-Poissonian behavior (Eq. 36) and determined only by usual matter, is at the origin of the various dynamical and lensing effects usually attributed to unseen additional mass. For example, in the Kepler problem (that applies in the outskirts of spiral galaxies), the additional potential writes $q = \mathcal{E} + GM/r = -(GM/r_0)(1 - r_0/r)$, and we directly recover the result obtained in [16] for the fundamental level, but now whatever the state.

Another proposal concerns the new fundamental gravitational coupling constant α_g [28]. Contrarily to what happens in the classical theory, the equation of motion (Eq. 32) can be shown to be gauge invariant. If the potential $\phi = m\varphi$ is replaced by $\phi + GMm\,\partial\chi(t)/c\partial t$, where the factor GMm ensures a correct dimensionality, then Eq. (32)

remains invariant provided ψ is replaced by $\psi\, e^{-i\alpha_g \chi}$, with α_g related to $\mathcal{D}$ by:

$$\alpha_g \times 2\mathcal{D} = \frac{GM}{c}, \tag{39}$$

which establishes the relation between $\mathcal{D}$ and $\alpha_g = w_0/c$ (see the joint paper [16]).

Finally, in similarity with the electromagnetic case, we can interpret the arbitrary gauge function χ, up to some numerical constant, as the logarithm of a scale factor lnρ in resolution space. In the special scale-relativity framework, such a scale factor is limited by the ratio of the maximal cosmic scale over the Planck scale, i.e. $\ln\rho < \mathbb{C}_U$. This limitation of χ in the phase of the wave function ψ implies a quantization of its conjugate quantity α_g, following the relation (for three independent scale transformations on the three space resolutions):

$$\mu \times 3\alpha_g \mathbb{C}_U = \frac{k}{2}, \tag{40}$$

where k is integer and the numerical constant μ remains to be determined. It may once again involve powers of π factors. It is remarkable in this respect that the relation:

$$\frac{3}{2}\pi^2\, \alpha_g \mathbb{C}_U = 1 \tag{41}$$

yields a value $\alpha_g^{-1} = 2070.10 \pm 0.15$, i.e. $w_0 = 144.82 \pm 0.01$ km/s, which is in good agreement with it present observational determinations $w_0 = 144.7 \pm 0.5$ km/s. Reversely, from such a relation, if it was confirmed, a precise measurement of w_0 would provide one with a new way of determining the cosmological constant [28].

3.2 Primeval density fluctuations

Another fundamental consequence of the quantum gravitational theory concerns the question of density fluctuations at the recombination epoch. In the standard approach to the problem of structure formation, no structure can be formed from a strictly constant matter density. The standard model of formation therefore assumes that the present structures originate from the gravitational growth of early fluctuations that existed at the recombination epoch. These structures have actually been observed through their temperature signature in the Cosmic Microwave Background Radiation field, though at a low level of $\delta T/T \approx 2 \times 10^{-5}$.

However, in the standard approach these fluctuations should themselves be understood in terms of still earlier fluctuations. But the strong isotropization that prevails during the prerecombination phase poses a difficult problem. The standard solution to this problem consists of making the postulate that the Universe would have known an inflation phase (i.e. exponential expansion) of unknown origin, which would have multiplied quantum fluctuations by an enormous factor.

We can suggest another solution to this problem in the scale-relativity framework, without any need of an inflationary phase. Indeed, as described in more detail in the joint paper and in references therein [16] and as recalled in the previous section, the fractality of space-time involves a transformation the geodesics equations in a Schrödinger-like equation. As a direct consequence, there is a tendency to form structures at any epoch:

these structures are described by probability density distributions given by the square of the modulus of the probability amplitudes which are solutions of this gravitational Schrödinger equation [6, 11]. The classical approach, because of its deterministic and differentiable nature, predicts structures at a given epoch which are the result of an evolution from previous existing structures: these structures of the past are taken as initial conditions (in position and velocity). The new quantum-like approach is organized in a fundamentally different way. The loss of determinism of individual trajectories is compensated by a determinism of structures. At each epoch, stationary or steady-state solutions can be found in correspondence with the shape of the potential and the limiting and matching conditions. These structures do also evolve (as given by the time-dependent Poisson-Schrödinger system), in correspondence with the evolution of the environment.

Therefore one expects the occurence of quantum fluctuations (structures) at the decoupling epoch, but according to the macroscopic quantum theory based on $\mathcal{D} = GM/2\alpha_g c$ which applies at this epoch, instead of the microscopic quantum theory for which $\mathcal{D} = \hbar/2m$. No inflation is needed in the new framework to obtain a scale invariant, quantum-like, spectrum at $z = 1000$.

3.3 Dimensionless physics

Some consequences of this new interpretation of the Planck-length-time scale have been considered elsewhere [5, 9], concerning in particular the unification of fundamental fields. Let us briefly discuss here its consequence as concerns the status of units. We already know that special motion-relativity has changed the status of space and time units. Indeed, the very existence of space-time implies to use the same units for length and time intervals. This is achieved since 1985, the unit of length being derived from the unit of time, and c fixed. Therefore, the genuine nature of velocities consists of dimensionless pure numbers always smaller than one.

A new step can be made in this direction using special scale relativity. Indeed, in its framework, every length (or time) interval is written in terms of its ratio over the Planck length (time)-scale. The Planck length and time scales thus appear as natural units of length and time intervals, whose genuine nature is found to be pure, dimensionless numbers larger than one. In the end, this implies that space and time units do not really exist, since, in the same way as the limitation of 3-velocity is a pure effect of projection from 4-space-time to 3-space, the Planck limit is the simple result of projection from 5-dimensional to 4 dimensional space. More generally, if one replaces the three fundamental constants G, $\hbar$ and c by their expressions in terms of the Planck-time, the Planck-length and the Planck-mass in any equation of physics, all quantities appear in these equations in terms of their ratio over the corresponding Planck units, which, ultimately, vanish from physics. Let us give a simple example.

The gravitational force is written since Newton as:

$$F = \frac{-GMm}{r^2}. \tag{42}$$

A first step toward dimensionless physics consists of writing it in terms of quantum units, i.e., $\hbar c$, as currently done for the electric force. One obtains:

$$F = -\hbar c \, \frac{(M/m_P)(m/m_P)}{r^2}. \tag{43}$$

But by introducing a Planck unit of force according to its dimensionality, i.e., $f_P = m_P \, l_P \, t_P^{-2} = c^4 G^{-1}$, one can write Newton's force under the form:

$$\frac{F}{f_P} = -\frac{(M/m_P)(m/m_P)}{(r/l_P)^2}. \tag{44}$$

The (Galilean) scale relativity of mass and length becomes fully apparent in this expression. Each of the ratios is now a pure number ($f = F/f_P$, $\mu = M/m_P$, $\mu' = m/m_P$, $\rho = r/l_P$ and the Newtonian force has taken a dimensionless, constantless form:

$$f = -\frac{\mu \, \mu'}{\rho^2}. \tag{45}$$

Such an operation is possible on any physical expression, and it takes sense through the interpretation of the Planck-scale as a scale invariant under dilations, making of the Planck-length and Planck-time the natural units of length and of time.

3.4 Mach principle and the nature of mass

Concerning mass, let us be more specific about the status of the Planck mass. We have seen in the above expression that it appears as a natural mass unit simply by writing the Newton force in quantum units. But a more complete answer may be reached by an analysis of another important problem of physics, which is what Einstein called the Mach principle.

As reminded in [6, 39], there are actually three levels of Mach principle.

(i) Mach I: The first level amounts to determine how inertial systems are defined. Enstein's general relativity solves the problem thanks to the equivalence principle: inertial systems are defined locally as systems which are in free fall in the gravitational field (i.e. that follow geodesic motion in the curved space-time).

(ii) Mach II: The second level expresses the hope that the amplitude of "inertial forces" (i.e., of forces that appear precisely when motion is no longer inertial) could be determined from the gravitational field. As analysed by Sciama [38], the implementation of this Mach II principle makes sense only by considering inertial forces as a gravitational force of induction, i.e. as a very manifestation of the general relativity of motion (in analogy with the understanding by Einstein and Poincaré of the Faraday electromagnetic induction as a special relativity effect of change of the reference system). Sciama has shown that a simple mean to express the Mach II principle is to write that, if this principle is true, any body should be in free motion in its own proper reference frame. Therefore the total energy of a body, which is the sum of its own energy and of its energy of interaction with all the other bodies in the Universe, should be zero in its rest frame:

$$mc^2 + \sum_i \frac{-GmM_i}{r_i} = 0. \tag{46}$$

The sum of the (M_i/r_i)'s is convergent in finite Universes (as spherical models) and in Universes having a horizon. In particular, (as demonstrated in [6] Chap. 7.1 and recalled below)) the spatially flat models (which are now favored by cosmological tests) are Machian. In the convergent cases, this sum defines a mass scale M_U and a length-scale R_U for the Universe. Setting

$$\sum_i \frac{M_i}{r_i} = 2\frac{M_U}{R_U}, \tag{47}$$

the Mach II principle is finally expressed in terms of a Schwarzschild-like condition for the whole Universe [37, 38, 39]:

$$\frac{2G}{c^2}\frac{M_U}{R_U} = 1. \tag{48}$$

Note that the inertial mass of the body has disappeared from this equation. This means that this view of Mach's principle agrees with the weak principle of equivalence, and predicts no anisotropy of inertia. This is different from Weinberg's conception [40], who considers that the absence of observed anisotropy of inertia (now measured better than $\Delta m/m < 10^{-24}$) favors the equivalence principle against Mach's principle. Now there is indeed a direct and fundamental opposition between Mach's principle and the strong equivalence principle, since one expects an anisotropy of active gravitational mass, even though this effect remains too small to be presently observable.

Recall also that it is precisely the discovery by Einstein [37] that the implementation of the Mach II principle involved a relation M/R =constant for the whole Universe that lead him to introduce the cosmological constant in his field equations. We shall come back on this point in the forthcoming section about the cosmological constant. Einstein finally concluded that the Mach principle was implemented by the constraint that the Universe be described by a finite spehrical model. However, infinite models can also be Machian: in particular, this is the case of the Einstein-de Sitter model [6]. Indeed, the integral should be made up to the horizon $R_U = c/H$, and the mass inside this horizon is $M_U = (\frac{4}{3}\pi\rho(c/H)^3$, so that the Schwarzschild relation becomes:

$$\frac{2G}{c^2}\frac{M_U}{R_U} = \frac{8\pi G\rho}{3H^2} = 1. \tag{49}$$

which is exactly the space flatness $\Omega_M = 1$ condition. This remains true with a cosmological constant, since the Schwarzschild relation reads in this case $2GM/c^2R + \Lambda R^2/3 = 1$, which is exactly the flatness condition $\Omega_{\rm tot} = \Omega_M + \Omega_\Lambda = (8\pi G\rho + \Lambda c^2)/3H^2 = 1$.

Let us now go on with the question of the nature of masses. The general relation Eq. 48 between lengths and masses has given the hope [41] that, ultimately, the concept of mass disappear from physics and be replaced by a relative combination of the interdistances between all bodies in the Universe. This hope was apparently deceived.

But maybe we are looking very far for things that already lie under our eyes. Indeed, the gravitational mass is already, since the emergence of Einstein's general relativity, fully equivalent to the Schwarzschild radius. Let us define a Schwarzschild length $l_S = GM/c^2$ equal to half the Schwarzschild radius. Introducing the Planck scales in the spirit previously described, the relation between the active mass and this length writes:

$$\frac{M}{m_P} = \frac{l_S}{l_P}. \tag{50}$$

In we now look more closely at the various formulas where the active gravitational mass appears, we discover that it can be replaced everywhere by the Schwarzschild radius without any loss of meaning nor of generality. Actually, we discover an even more profound fact, namely, that the length carries the fundamental meaning and that the mass is only a derived concept. For example, the g_{00} term of Schwarzschild metrics is naturally integrated as $g_{00} = 1 - 2\, l_S/r$, and l_S is only subsequently interpreted as giving the active gravitational mass. Even in Newton's theory, the very nature of the central mass in the Kepler problem is that it determines the natural length unit of the system: hence Kepler's third law between the semi-major axes a and the period T of planetary orbits writes:

$$\left(\frac{a}{l_S}\right)^3 = \left(\frac{cT}{2\pi l_S}\right)^2. \tag{51}$$

This is also apparent from the fact that the product GM for bodies in the solar system (and therefore their Schwarzschild length) is known with a much higher precision than G and M separately.

Finally, such an analysis leads one to write Newton's force Eq. 42 under another equivalent and highly significant dimensionless form:

$$\frac{F}{f_P} = -\frac{r_1}{r} \times \frac{r_2}{r}, \tag{52}$$

where $r_1 = GM/c^2$ and $r_2 = Gm/c^2$. Such a writing is fundamentally scale-relativistic, since lengths appear here only through ratios.

Let us now consider the inertial mass. Since the introduction of matter waves by de Broglie, an inertial mass is also fully equivalent to a length, namely, to the Compton length λ_c associated to a particle of mass m:

$$\lambda_c = \frac{\hbar}{mc}. \tag{53}$$

The situation is exactly similar to the gravitational, macroscopic case: namely, it is easy to verify that in any formula in which the inertial mass of a body appears, it can be replaced by its Compton length. For example, the energy of the fundamental level of the hydrogen atom writes, in dimensionless terms:

$$\frac{E}{E_P} = -8\pi^2\alpha^2 \frac{m_e}{m_P} = -8\pi^2\alpha^2 \frac{l_P}{\lambda_e}, \tag{54}$$

where $E_P = m_P\, c^2$ is the Planck energy. Therefore we suggest that, like the gravitational mass, the inertial mass is but an intermediate concept which is ultimately fated to vanish from physics. Such a suggestion is supported by the scale-relativistic explanation of the nature of the Compton length of a particle. Recall indeed that we identify "particles" with the geodesics family of a non-differentiable fractal space-time, and that the Compton length is a geometric characterization of these geodesics. Namely, it gives the scale of transition from the fractal regime to the scale -independent, classical regime in the scale space [6, 9, 11, 19]. Therefore mass ratios can be replaced everywhere by length ratios, so that the Planck mass itself is destined to disappear from physics as a fundamental concept:

only the Planck-length-time scale remains, representing a fundamental "1" playing for scales the same part as played by the light velocity for motion.

Finally, the equality of the inertial and gravitational mass for a given body (in accordance with the strong equivalence principle) is translated in terms of an inverse relation between the Compton length and the Schwarzschild length:

$$\left(\frac{l_P}{\lambda_c} = \frac{m_I}{m_P}\right) \quad = \quad \left(\frac{M_A}{m_P} = \frac{l_S}{l_P}\right). \tag{55}$$

They are but symmetric elements in the scale space, i.e., $\ln(l_S) = -\ln(\lambda_c)$ in Planck (dimensionless) units. Moreover, both of them are solutions of the search for conservative scale quantities, obtained from applying Noether's theorem in the framework of scale-relativistic mechanics [5, 6].

(iii) Mach III: the third level of Mach's principle considers that there are fundamental relations between elementary particle scales and cosmic scales, and it is therefore related to the Eddington-Dirac large number hypothesis (see Ref. liwos Chap. 7.1). We shall be more specific about it in the forthcoming Section on the cosmological constant and the vacuum energy density.

3.5 Consequences for the Early Universe

The history of the early Universe is nowadays described in correspondence with our knowledge of high energy particle physics. This is due to the fact that, when going toward the past, the primeval Universe encounters conditions of ever increasing density, pressure and temperature, so that it is described in terms of a gas of elementary particles of increasing energy. This means that the various progress that are made in our understanding of elementary particles (transition scales and values of the coupling constants) have immediatly their counterpart in our view of the primeval universe. We shall not in what follows attempt to give a complete new description of the scale-relativistic history of the early Universe, which would be too long for the present contribution, but only point out some features by which it differs from the standard view.

3.5.1 Scale-relativistic theoretical predictions of masses and couplings

In the new framework, theoretical predictions of some of the free parameters of the standard model become possible. We have presented and checked such predictions in previous works [5, 6, 9]. But in the recent years, there has been an improvement of several experimental measurements [42], so that it may now be interesting to check them again. The new values for, respectively, top quark mass, Higgs boson mass, W and Z boson masses, strong coupling constant at Z scale, fine structure constant at Z scale, and $\sin^2\theta$ of weak mixing angle at Z scale in the modified minimal substraction scheme (where it is defined through the SU(2) charge g and the U(1) charge g') are:

$m_t = 174.3 \pm 5.1\,\mathrm{GeV/c^2}$; $m_H = 108 - 154\,\mathrm{GeV/c^2}$
$m_W = 80.42 \pm 0.04\,\mathrm{GeV/c^2}$; $m_Z = 91.1872 \pm 0.0021\mathrm{GeV/c^2}$
$\alpha_3(m_Z) = 0.118 \pm 0.002$; $\alpha^{-1}(m_Z) = 128.92 \pm 0.03$
$\hat{s}_Z^2 = g'^2/(g^2 + g'^2) = 0.23117 \pm 0.00016.$

Fine structure constant. In [9], we derived a prediction to 1/1000 of the low energy fine structure constant based on its renormalization group evolution from its bare (infinite energy) value, which is expected in our framework to be given by $\alpha_\infty^{-1} = 32\pi^2/3 = 105.276$.

The experimental and theoretical improvements since the writing of this paper now allow us to improve this calculation. We start by defining a formal QED inverse coupling $\alpha_0^{-1} = \frac{5}{8}\alpha_1^{-1} + \frac{3}{8}\alpha_2^{-1} = \frac{3}{8}\alpha^{-1}$, where α_1 and α_2 are respectively the U(1) and SU(2) running couplings and α is the running fine structure constant. Due to the fact that 3 among the 4 gauge bosons acquire mass through the Higgs mechanisms, the QED coupling jumps from α_0 to $\alpha = \frac{3}{8}\alpha_0$ at the Z scale.

While in the standard model α_0 is logarithmically divergent, it becomes convergent in the special scale-relativistic framework, its predicted infinite energy value being $4\pi^2$ (see the demonstration hereabove and Fig. 1). Therefore we expect $\alpha_\infty^{-1} = 32\pi^2/3$.

The difference between the infinite energy and Z value was computed using the solutions to the renormalization group equation for the running coupling. The prediction at the Z value for 1 Higgs doublet is (see more detail in [9, 22]):

$$\alpha^{-1}(m_Z) = \frac{32\pi^2}{3} + \frac{22\pi}{3} + \frac{6}{\pi^2} = 128.922\,. \tag{56}$$

Now the difference between the Z value and the low energy value of α has been recently greatly improved [43, 42]: $\Delta\alpha^{-1} = 8.12 \pm 0.03$. We therefore deduce a prediction for the low energy fine structure constant from its scale-relativistic bare value:

$$\alpha^{-1} = 137.04 \pm 0.03, \tag{57}$$

which compares well with the experimental value $\alpha^{-1} = 137.036$. Note that the error comes here from the uncertainties in the coupling variation, not from the new theoretical prediction, which is exact.

Therefore the comparison between experimental data and theoretical prediction becomes clearer if one compute the bare value of α_0^{-1} from the experimental low energy value and its renormalization group evolution. One finds $\alpha_0^{-1}(\infty) = 39.477 \pm 0.011$, which agrees within uncertainties with our exact prediction $4\pi^2 = 39.478...$. The improvement of precision since our first attempts ten years ago [5], which yielded 40.0 ± 0.5, is remarkable.

Strong coupling. From the conjecture that the strong inverse coupling reaches the critical value $4\pi^2$ at unification scale, i.e. at $m_P/2\pi$ in the special scale-relativistic modified standard model (see Fig. 1), we obtained a predicted value $\alpha_3(m_Z) = 0.116 \pm 0.0005$ from the solution to the renormalization group equation of the running coupling [5, 9] (the uncertainty 0.0002 quoted in [9] was underestimated). This expectation remains in agreement (within about one σ) with the recently improved experimental value 0.118 ± 0.002 [42].

In this case also it may be clearer to make the comparison in the reverse way. One starts from the experimental value $\alpha_3(m_Z) = 0.118 \pm 0.002$, then one runs it to the Planck energy scale in the special-scale-relativity framework (see [9]). One obtains $\alpha_3^{-1}(m_P) = 39.33 \pm 0.14$, within 1σ of $4\pi^2 = 39.48$.

SU(2) coupling and coupling ratios. In [9], we also attempted to apply the mass-charge relation to the SU(2) coupling α_2. We found that the relation

$$3\,\alpha_{2Z}\,\mathbb{C}_Z = 4 \tag{58}$$

was precisely achieved at the Z scale. However the factor 3 was not accounted for in that work. The solution to this problem relies on the generalization of scale (i.e. gauge) transformations to dilations which are no longer global: one can instead consider different and independent dilations on the internal resolutions corresponding to the various coordinates. The group SU(2) corresponds to rotations in a 3-dimensional scale-space. Therefore the phase term in a fermion field writes:

$$\alpha_2\,\ln(\frac{\varepsilon_x}{\lambda}) + \alpha_2\,\ln(\frac{\varepsilon_y}{\lambda}) + \alpha_2\,\ln(\frac{\varepsilon_z}{\lambda}) < 3\,\alpha_2\,\ln(\frac{l_P}{\lambda}), \tag{59}$$

since the same coupling applies to the three variables, and since all three resolutions are limited toward small scales by the Planck scale. From Eq. (58) we expect a value $\alpha_{2Z}^{-1} = 29.8169 \pm 0.0002$. The present precise experimental value is:

$$\alpha_{2Z}^{-1} = \alpha_Z^{-1} \times \hat{s}_Z^2 = 29.802 \pm 0.027, \tag{60}$$

which lies within 1σ of the theoretical prediction.

More generally, the universality of mass-coupling relations for any gauge field implies that coupling ratios at Z scale should be integers. Present values ($\alpha_{1Z}^{-1} = 59.471 \pm 0.026$; $\alpha_{2Z}^{-1} = 29.802 \pm 0.027$; $\alpha_{3Z}^{-1} = 8.47 \pm 0.14$) [42] support this prediction within a 2σ confidence interval, since one obtains:

$$\left(\frac{\alpha_3}{\alpha_1}\right)_Z = 7.02 \pm 0.12 \quad ; \quad \left(\frac{\alpha_2}{\alpha_1}\right)_Z = 1.996 \pm 0.002. \tag{61}$$

Vacuum expectation value of the Higgs field. As recalled hereabove, there are fundamental arguments for introducing a bare inverse coupling at infinite energy (i.e., in special scale relativity, at Planck length-scale) given by the critical value $4\pi^2$. Moreover, our re-interpretation of gauge invariance as scale-invariance on space-time resolution led us to construct general relations between couplings and scale ratios. Therefore one expects the emergence of a new fundamental scale given by:

$$\ln\left(\frac{\lambda}{l_P}\right) = 4\pi^2, \tag{62}$$

where l_P is the Planck length-scale. This relation provides a solution to the hierarchy problem, according to which there is a misunderstood factor $\approx 10^{17}$ between the electroweak scale and the Planck scale (expected to be the full unification scale). Indeed the length-scale λ defined above is $e^{4\pi^2} = 1.397 \times 10^{17}$ larger than the Planck scale. As a first approximation, we can apply this relation to mass ratios. This gives a mass scale of 87.39 GeV, intermediate between the Z and W masses. However, mass-scales and length-scales are no longer directly inverse in the scale-relativity framework. There is a "log-Lorentz" factor between them (when they are referred to low energies). Namely, by taking as reference the electron Compton scale, the new mass-scale is more precisely given by:

$$\ln\left(\frac{m}{m_e}\right) = \frac{\ln(\lambda_e/\lambda)}{\sqrt{1-\ln^2(\lambda_e/\lambda)/\mathbb{C}_e^2}}. \tag{63}$$

With the currently accepted value of the gravitational constant (for which the error is now thought to be 12 times larger than previously given, see [42]), we obtain for the fundamental constant $\mathbb{C}_e = \ln(\lambda_e/l_P) = 51.52797(70)$. Then the new theoretically predicted mass scale is

$$m_v = 123.23 \pm 0.09\,\mathrm{GeV/c^2}. \tag{64}$$

Such a mass scale seems to be closely linked with the vacuum expectation value v of the Higgs field, since one finds from the experimental data $v/\sqrt{2} = m_W/g = 123.11 \pm 0.03\,\mathrm{GeV}$ (where g is the SU(2) weak charge). Now some work remains to be done to really understand why the new mass-scale should have precisely this interpretation.

3.5.2 New formulation of the question of the origin

The new meaning attributed to the Planck-length and the Planck-time drastically changes the standard view about the primeval singularity. The first new physical law of cosmological importance is the disappearance of the zero instant from meaningful physical concepts. The evolution of the Universe does not begin any more at the instant $t = 0$ (i.e. $\log(t/t_0) = -\infty$ for any $t_0 > 0$), but instead at the Planck time $t_P = l_P/c$ (see [6] Chap.7.1). However this new structure should not be misinterpreted: in the new theory, the Planck scale owns all the properties of the previous zero instant. This means that temperature, redshift, energy, density and all the quantities Q which were previously diverging as t^{-k} are now diverging toward the past when t tends to t_P as

$$\log\frac{Q}{Q_0} = \frac{k\log(t_0/t)}{\sqrt{1-\frac{\log^2(t_0/t)}{\log^2(t_0/t_P)}}} \tag{65}$$

The scale factor of expansion of the Universe is also submitted to a similar new constraint, since it can no longer become smaller than the Planck length l_P. Therefore the Universe starts asymptotically from the Planck scale. This solved the problem of the singularity, which did not make part of the model in the standard general relativistic approach. For example, in the hyperbolic Robertson-Walker solution with trivial topology, the Universe in infinite at any instant $t \neq 0$, as small as it could be, while the spatial part of the metric vanishes at $t = 0$ since $R(0) = 0$. This kind of problem is no longer posed in the scale-relativistic framework.

3.5.3 Very early dimensionless phase of the Universe

Actually, even such a description in terms of a limiting Planck scale remains misleading. Indeed, we have insisted from the very beginning on the fact that only scale ratios do have a physical meaning, so that any description in terms of a dimensioned scale assumes that a static, invariant scale can be used as reference (unit). For example, the expansion of the Universe is meaningful at the present epoch only because there does exist small scales length-units that are not carried away by the expansion, in particular the Bohr radius which is linked to the line energies which serve to measure the redshift.

At the present epoch the static /non static transition scale is given by the typical radius of typical self-gravitating extragalactic objects such as galaxies and the centers of clusters of galaxies, i.e. $\lambda_g \approx GM/3\langle v^2 \rangle$ [6]. In the early Universe, static scales are given by scales such as the QCD confinement scale, the Compton scale of the electron, then the Bohr scale, but they are effective only at epoch later than the phase transition to which they correspond (i.e., quark-hadron transition, $e^+ - e^-$ pairs annihilation, recombination).

If one goes back still further in the past, one reaches the Planck energy scale. In the special scale-relativity framework, it is no longer equivalent to the Planck length-time scale (that now corresponds to infinite energy). Moreover, we have found [5] that the Planck mass scale becomes naturally the scale of full unification between gravitation and the U(1), SU(2) and SU(3) fields (see Fig. 1). This solves the main problem encountered by the minimal GUT theories based on the SU(5) group, namely, the life time of the proton whose theoretical expectation becomes in agreement with the experimental limits [5, 6, 9] (now the presence of the gravitational field implies that the SU(5) group can be the unifying group only on a small scale interval). The X bosons that carry the unified field have therefore a Planck mass.

This means that the GUT length-scale, which identifies with the Compton length of the X particles and which is predicted to be $10^{4\pi^2 (\log e)/\sqrt{2}} = 10^{12}$ times smaller than the Z scale, is the minimal possible static scale in nature. Before the GUT transition, physics can therefore be only pure dimensionless physics. The Planck-length and the cosmic length lose their meaning (more precisely, in terms of evolution toward the future, they have no meaning yet), and only their ratio $\mathbb{K} = 5.3 \times 10^{60}$ is meaningful. During this very early era, unbroken scale relativity holds, and all the physical laws should be expressed in terms of pure, dimensionless numbers. The dimensioned physics as we know it now becomes possible only after and thanks to the Planck energy symmetry breakdown.

3.5.4 Horizon / causality problem

One of the main results of scale relativity concerning the primeval Universe is its ability to solve the causality / horizon problem [6]. Let us recall the nature of this problem. When looking at two directions separated by a large angle, e.g. two opposite directions, we observe regions of the Universe which, for a large enough redshift, may have never been connected in the past. The problem is particularly strong concerning the microwave background radiation, due to its high isotropy ($\delta T/T \approx 2 \times 10^{-5}$) and its early origin ($z \approx 1000$): at least twenty such independent regions would be observed in the framework of standard cosmology.

Such causally disconnected regions should behave as completely independent universes, and it becomes very strange that no large fluctuation of the microwave background temperature is observed. The solution to this problem is usually searched for in the framework of inflationary cosmology. However one may remark that inflation is to some extent an ad hoc solution, in particular as concerns its cause (now unobservable scalar field , primordial black holes...), that must be postulated additionally to the presently known content of the Universe. Moreover it does not solve the problem in principle: in its framework the presently observed regions of the universe would have been causally connected in the past, but this does not remain true in the distant future.

The horizon / causality problem is simply solved in the special scale-relativity framework without needing an inflation phase, thanks to the new role played by the Planck lenth-time scale. It is identified with a limiting scale, invariant under dilations. This implies a causal connection of all points of the universe at the Planck epoch. This is due to the fact that, if one accounts for the scale-Lorentz factor, one finds that the light cones that rest on two arbitrary distant points flare when $t \to t_P$ and finally always cross themselves in their past (see Fig. 7.1 of Ref. [6] and Fig. 13 of [9]).

3.5.5 Ultra high energy cosmic rays

Ultra high energy cosmic rays observed beyond 10^{19} eV are still of unknown origin. The highest energy shower has been observed in 1991 by the Fly's Eye detector at $3.2 \pm 0.9 \times 10^{20}$ eV [44, 45]. The second highest energy event is at 2.1×10^{20} eV. Such detections, and more generally the highest energy spectrum of cosmic rays poses an important problem, because one expects a Greisen-Zatsepin-Kuzmin cutoff of the energy spectrum at $\approx 4 \times 10^{19}$ eV for travel distances larger than about 30 Mpc, due to pion photoproduction energy losses. Such a constraint strongly limits the possible astrophysical sources for such events, provided they are produced by accelerated known particles.

In order to circumvent this problem, it has been proposed that the highest energy cosmic rays originate in the decay of topological spacetime defects such as cosmic strings or vortons [46]. Such theories would predict a continuing cosmic ray flux all the way up the grand-unification mass scale (10^{23} eV in the minimal standard model, 10^{28} eV if the unification is at the Planck energy).

A similar proposal can be made in the scale-relativity framework, but with another mass scale for the primary particle. Indeed, we expect the ocurrence of new kinds of spacetime structures, linked in particular to the mass-charge relation and to the critical value $4\pi^2$ of inverse couplings. One of these structures is given by the equation (see Fig. 1):

$$\alpha_2^{-1}(E) = 4\pi^2, \tag{66}$$

where α_2 is the SU(2) weak coupling and E is the energy-scale to be solved for. The scale dependence of the α_2 running coupling is given by the solution to its renormalization group equation, that reads to first order (see e.g. [6] Chap. 6.2 and references therein):

$$\alpha_2^{-1}(r) = \alpha_2^{-1}(\lambda_Z) + \left(\frac{5}{3\pi} - \frac{N_H}{12\pi}\right) \ln\left(\frac{\lambda_Z}{r}\right), \tag{67}$$

where λ_Z is the Compton length of the Z boson, N_H is the number of Higgs doublets, and r a running length-scale. As demonstrated in [5, 6], this solution remains correct in the special scale-relativity framework, provided it is written in terms of length-scale. Conversely, while one can replace $\ln(\lambda_Z/r)$ by $\ln(m/m_Z)$ in the standard model, this is no longer the case in special scale-relativity, since a log-Lorentz factor is now involved in the mass-scale / length-scale transformation, namely,

$$\ln\left(\frac{r_Z}{r}\right) = \frac{\ln(m/m_Z)}{\sqrt{1+\ln^2(m/m_Z)/\ln^2(m_P/m_Z)}} \tag{68}$$

Now solving Eq. 66 for $E = mc^2$ with the precise value of $\alpha_2^{-1}(m_Z) = 29.802 \pm 0.027$ [42] yields:

$$E = 3.20 \pm 0.26 \times 10^{20}\text{eV}. \tag{69}$$

Including second order corrections in the renormalization group equation and accounting for the scale-relativistic correction on the fundamental constant $\mathbb{C}_Z = \ln(\lambda_Z/l_P) = 39.756$ (which differs by 1% from $\ln(m_P/m_Z) = 39.436$, [6]), one obtains an equivalent result,

$$E = 3.27 \pm 0.26 \times 10^{20}\text{eV}, \tag{70}$$

which agrees very closely with the maximal energy of cosmic rays observed at $3.2 \pm 0.9 \times 10^{20}$ eV (see Fig. 2). Due to the large value of this energy, the agreement would remain remarkable even if the experimental error revealed to be underestimated.

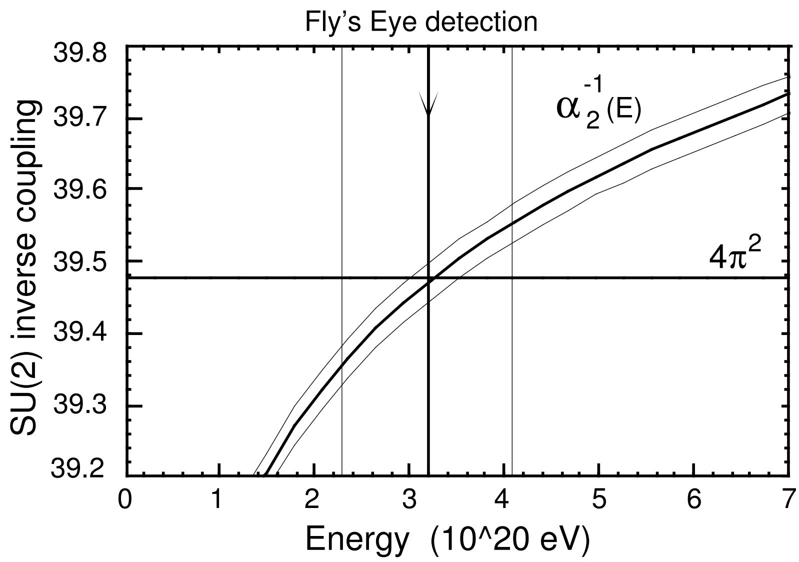

Figure 2: Theoretical prediction of a high energy weak field structure, which is solution of the equation $\alpha_2(E) = 1/4\pi^2$, where $\alpha_2(E)$ is the running SU(2) coupling. From precise recent determinations of the coupling constant at Z scale [42] $\alpha_{2Z} = 29.802 \pm 0.027$ and using the solution to the renormalization group equation for the running coupling, one obtains $E = 3.27 \pm 0.26 \times 10^{20}$ eV, to be compared with the Fly's Eye detection at $3.2 \pm 0.9 \times 10^{20}$ eV [45].

Such a result, provided it is confirmed by future observations, e.g. at the Pierre Auger observatory, allows one to put to the test the number of Higgs doublets and the scale-relativistic log-Lorentz factor. Indeed, the predicted energy becomes 8.0×10^{19} for 0 Higgs doublet (excluded) and 1.7×10^{21} for 2 Higgs doublets (which will be excluded if no showers at energies larger than 3.2×10^{20} eV are observed). This is a confirmation of our result concerning the fine structure constant (see hereabove and [9]):

$$\alpha^{-1} = 137.04 \pm 0.03 + 2.11(N_H - 1), \tag{71}$$

that already strongly excludes $N_H \neq 1$.

This effect could also be used to discriminate between the Galilean scale-relativistic (i.e. standard) and Lorentzian scale-relativistic theory. Indeed, in the absence of a log-Lorentz factor, one obtains for one Higgs doublet $E = 2.0 \pm 0.12 \times 10^{19}$ eV, which is smaller than the Fly's Eye energy by a factor of 16 and lies below the GZK cutoff.

With additional Higgs doublets, one obtains $1.9 \pm 0.1 \times 10^{20}$ eV for 3 doublets and 7.2×10^{20} eV for 4 doublets. The first value is already too low and the second will be excluded if the present limiting energy is not exceeded by future detections. This is therefore a new test of the log-Lorentz factor, that is added to other previous tests such as the identification of the GUT scale with the Planck scale [5, 6].

Let us conclude this section by noting that, in the hypothesis that the existence of a new particle of mass 3.2×10^{20} eV/c^2 (the "Auger" particle) be confirmed, the distance limits set on the source of the Fly's Eye event no longer apply. This reopens the possibility of a galactic source (as supported by the arrival direction that lies close to the galactic plane, at b=9.6 deg), or of a very distant extragalactic source (3C147, a QSO of redshift 0.545, lies within the 1σ error box [47]).

3.6 Implications of scale relativity for the present Universe

The theory of scale relativity has not only consequences in the microphysical domain (Δx and $\Delta t \to 0$) and therefore for the early Universe, but also for the geometric description of the present Universe (see Refs. [6] Chap.7.1, and [9, 39]). We shall, in what follows, briefly examine the possible implications of special scale-relativistic dilation laws at cosmological scales.

We assume the validity of standard (i.e. Galilean) laws of dilation $\ln \varrho" = \ln \varrho + \ln \varrho'$ up to some transition scale λ. Beyond this scale, the law of composition of dilations takes the more general log-Lorentzian form:

$$\ln \varrho" = \frac{\ln \varrho + \ln \varrho'}{1 + \ln \varrho \ln \varrho' / \mathbb{C}^2}. \tag{72}$$

Introducing the fractal - nonfractal transition scale λ, the effect of a dilation by a factor ρ of a scale ε now yields a scale ε' given by:

$$\ln \frac{\varepsilon'}{\lambda} = \frac{\ln \rho + \ln(\varepsilon/\lambda)}{1 + \ln \rho \ln(\varepsilon/\lambda)/\ln^2(\mathbb{L}/\lambda)}. \tag{73}$$

It is easy to verify in this equation that, whatever the transition scale λ, the scale Lu remains invariant under any dilation ρ. If one starts from any scale $\varepsilon > \lambda$ and apply to it any dilation ρ, the resulting scale ε' remains smaller than $\mathbb{L}$. An infinite dilation would be needed to reach Lu. In other words, $\mathbb{L}$ is maximal scale, unreachable, invariant under dilations, that replaces the infinite of the standard theory (and now plays its physical role).

3.6.1 Cosmological constant and invariant cosmic scale

In the same way as the minimal invariant scale-relativistic scale is naturally identified with the Planck length-scale [5, 6], the maximal invariant scale can be identified with the scale of the cosmological constant, $\mathbb{L} = \Lambda^{-1/2}$.

For this purpose, one reconsiders Hawking's quantum cosmology calculation [48] in the special scale-relativity framework. In Ref. [49], Weinberg considered five different approaches to the solution of the cosmological constant problem, and concluded that Hawking's result solved the problem. Indeed, it yields a probability density for the various values of the cosmological constant proportional to:

$$P = e^{3\pi/G\Lambda}, \tag{74}$$

where Λ is the true effective cosmological constant that is measured in gravitational phenomena at long ranges. This probability density has an infinite peak for $\Lambda \to 0+$, so that this calculation solves the problem in terms of a zero effective cosmological constant in the standard framework. Let us however reconsider this conclusion in the new framework.

The cosmological constant is fundamentally a curvature term, as can be seen in Einstein's field equations:

$$R_{\mu\nu} + (\Lambda - \frac{1}{2}R)\, g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}. \tag{75}$$

Therefore it is the square of the inverse of a fundamental length, $\Lambda = 1/L^2$. The infinite peak of Hawking's probability density for $\Lambda = 0$ means an infinite peak for a running value of $L \to \infty$.

Now all this calculation implicitly pre-assumes underlying Galilean dilation laws, in which an infinite length-scale is assumed to have physical meaning. On the contrary, in scale relativity, the infinite length-scale is replaced by a maximal scale $\mathbb{L}$, which is unreachable and invariant under dilations. The operation by which one consider an ever increasing running L is precisely such a dilation under which $\mathbb{L}$ is invariant. Therefore the infinite scale in Hawking's calculation is replaced by the length-scale $\mathbb{L}$.

More specifically, Hawking's probability density writes in the scale-Galilean case:

$$P = \exp\left[\exp\left(2\,\ln\frac{L}{\lambda}\right)\right], \tag{76}$$

where λ is a static reference length-scale ($\lambda << \mathbb{L}$). It becomes in the scale-Lorentz case:

$$P = \exp\left[\exp\left(\frac{2\,\ln(L/\lambda)}{\sqrt{1 - \frac{\ln^2(L/\lambda)}{\ln^2(\mathbb{L}/\lambda)}}}\right)\right]. \tag{77}$$

Therefore, as can be easily checked in this expression, the probability density has an infinite peak for $L \to \mathbb{L}$, i.e., for a cosmological constant $\Lambda = 1/\mathbb{L}^2$.

3.6.2 Cosmological constant and Mach principle

Let us show that the very existence of the cosmological constant actually solves in a general way the problem of whether Einstein's theory of general relativity is Machian or not.

We have recalled in Sec. 3.4 that the "Mach II" principle should be implemented in cosmology provided there exists a fundamental Schwarzschild-like relation $GM/c^2R = 1$ for the Universe in its whole. Einstein [37] has introduced the cosmological constant in the gravitational field equation precisely in order to make the final theory agree with the first principles from which it was constructed, that included Mach's principle. Now all cosmological solutions of Einsteins equations based on the cosmological principle of homogeneity and isotropy at large scale are characterized by a prime integral that reads $\rho R^3 =$ constant in dust models (which are valid for the present epoch). This means that all models are characterized by an invariant energy Mc^2, which can be neither created nor destroyed. Conversely, all cosmological models which are solutions of the 1915 version of Einstein equations are non-static, i.e., are characterized by a variable scale $R(t)$, so that M/R is variable. This was the problem Einstein wanted to solve in his 1917 paper by introducing a cosmological constant, and Eddington and Dirac by compensating the variation of R by a similar variation of G.

However, the solution proposed by Einstein was to construct a particular cosmological model for which $R =$ constant. Such a model is indeed allowed when a cosmological constant is present: however, it is unique, metastable and we know since 1929 that it contradicts the observation of the expansion of the Universe (remind that when Einstein made this proposal in 1917, the universe was still identified with the "gas" of stars, which is globally static, since this is actually our Milky Way galaxy). Later, Einstein suggested that the solution was given by closed models, which are characterized by a maximal value of the scale factor. Anyway, none of these solutions to the Mach problem are satisfactory, since the the initial question was a matter of first principle concerning the whole theory.

It is actually easy to show that general relativity in its whole is Machian, in agreement with Barbour and Bertotti's view [50, 51]. Indeed, the cosmological constant is a curvature, i.e. the inverse of the square of a cosmic length. Therefore, the introduction of a fundamentally invariant cosmological constant amounts to the introduction of an invariant length $\mathbb{L}$. This length is present in Einstein's equation whatever the model, even beyond the validity of the cosmological principle. As a consequence, any cosmological model being characterized by an invariant mass M_U, a universal Machian relation can be written whatever the model:

$$\frac{2G}{c^2}\,\frac{M_U}{\mathbb{L}} = \text{constant}. \tag{78}$$

Moreover, in the special scale relativity framework, in which the length-scale $\mathbb{L}$ is identified with the maximal possible resolution scale, and replaces the infinite scale, $\mathbb{L}$ is a Universal horizon which exists independently of any model.

The theoretical prediction of a non-infinite value of $\mathbb{L} = 2.86$ Gpc (see [6] Chap. 7.1, [52, 9] and the next section), i.e. $\Lambda = 1.36 \times 10^{-56}$ cm^{-2}, and its recent observational determination $\Lambda = 3H_0^2\Omega_\Lambda/c^2 = (1.2 \pm 0.6) \times 10^{-56}$ cm^{-2} (for $\Omega_\Lambda = 0.7 \pm 0.2$ and $H_0 = 70 \pm 7$ km/s.Mpc) support such a view.

However, the problem of the consistency of the Mach principle with the strong equivalence principle remains formally set. This question will be addressed in a forthcoming work.

3.6.3 Cosmological constant and vacuum energy density

One of the most difficult open questions in present standard cosmology is the problem of the vacuum energy density and of its manifestation as an effective cosmological constant [49, 53]. This question dates back to Lemaitre's understanding that the cosmological constant term in Einstein's field equations is equivalent to the effects of a Lorentz-invariant vacuum [54]. Indeed, such a vacuum is characterized by a negative pressure which is exactly the opposite of its energy density, $p_v = -\rho_v c^2$. When they are inserted in the energy-momentum tensor, this pressure and this density combine themselves in terms of an effective positive cosmological constant $\Lambda_{\rm eff}$, such that $\rho_v = \Lambda_{\rm eff} c^2/8\pi G$. Its effects are therefore equivalent to those of a repulsive force.

This lead Lemaitre [54], then Zeldovich [55] and others to suggest that one could calculate the value of the cosmological constant from the quantum vacuum energy density. This hope was deceived, since the vacuum energy in standard quantum field theory is the sum of the zero point energies of all normal modes of the various fields, and it is therefore infinite. If one imposes an arbitrary cut-off $\nu_{\rm max}$ to the possible wave numbers, one finds the vacuum energy to vary as $\nu_{\rm max}^4$ [49]. By chosing, once again in an arbitrary way, $h\nu_{\rm max} = E_P$, the Planck energy, one finds a calculated vacuum energy density $\approx 10^{120}$ times larger than the observed cosmological cosntant.

In Refs. [52, 9], a solution to this problem has been proposed, that lead to a theoretical prediction of the value of the cosmological constant $\Lambda = 1.36 \times 10^{-56} cm^{-2}$. This value, which corresponds to a scaled cosmological constant $\Omega_\Lambda = \Lambda c^2/3H_0^2 = 0.79 \pm 0.15$ for $H_0 = 70 \pm 7$ km/s.Mpc, is now supported by recent observational determinations that have yielded $\Omega_\Lambda = 0.7 \pm 0.2$ [57, 58, 59, 60].

The scale relativity approach actually generalizes Zeldovich's [55] one; it allows one to suggest a solution to the cosmological problem; moreover, this solution gives physical meaning in an exact way to the Eddington-Dirac large number coincidence (see also Sidharth [56] about this point), i.e, what we have called hereabove the Mach III principle.

The first step toward this solution consists of considering the vacuum as fractal, i.e., as explicitly scale-dependent. As a consequence, the Planck value of the vacuum energy density (that gives rise to the 10^{120} discrepancy with observational limits) is relevant only at the Planck scale, and becomes irrelevant at the cosmological scale. We expect the vacuum energy density ρ to be solution of a (renormalisation group-like) scale differential equation:

$$\frac{d\rho}{d \ln r} = \Gamma(\rho) = a + b\rho + O(\rho^2), \tag{79}$$

where ρ has been normalized to its Planck value, so that it is always < 1, allowing us to perform a Taylor expansion of $\Gamma(\rho)$. This equation is solved as:

$$\rho = \rho_c \, [1 + \left(\frac{r_0}{r}\right)^{-b}], \tag{80}$$

where $\rho_c = -a/b$ can be identified with the cosmological energy density. We recover a combination of a power law behavior at small scales and of scale-independence at large scale (see Sect. 2.1.2), with a fractal/non-fractal transition about some scale r_0 that comes out as an integration constant.

Such a result can also be directly obtained from general relativity, by writing that the final effective cosmological constant is the sum of a geometric, invariant term from the left-hand side of Einstein equations and of the vacuum energy density contribution in the energy-momentum tensor, which is explicitly scale dependent.

Now, various contributions to the vacuum energy density can be considered. The largest comes from the vacuum energy itself. We have seen that it depends on scale as r^{-4} (as in the Casimir effect).

However, when considering the various field contributions to the vacuum density, we may always chose $< E >= 0$, i.e., the energy density of the vacuum can always be renormalized. But consider now the gravitational self-energy of vacuum fluctuations. It writes:

$$E_g = \frac{G}{c^4} \frac{< E^2 >}{r}. \tag{81}$$

The Heisenberg relation prevent from making $< E^2 >= 0$, so that this gravitational self-energy cannot vanish. This relation writes $< E^2 >^{1/2} = \hbar c/r$, so that we obtain the asymptotic high energy behavior:

$$\rho_g = \rho_P \left(\frac{\lambda_P}{r} \right)^6, \tag{82}$$

where λ_P is the Planck length and ρ_P is the Planck energy density. From this equation we can make the identification $b = -6$. Therefore we obtain the complete behavior of the vacuum energy density:

$$\rho = \rho_c \, [1 + \left(\frac{r_0}{r} \right)^6]. \tag{83}$$

For $r << r_0$ this energy density is dominated by the scale-dependent term, that decreases very rapidly with increasing scale, then becomes negligible beyond r_0. For $r >> r_0$, it becomes dominated by the invariant term ρ_c, which is identified with the geometric cosmological constant term. But now, the value of the geometric cosmological constant is given by the value reached by the quantum energy density at the transition scale r_0, so that the problem of determining the cosmological constant now amounts to determining this length-scale.

Note that all the above calculation has been performed in the standard quantum mechanical framework, i.e., (from the scale-relativity viewpoint), in the framework of Galilean scale-relativity. It is easy to generalize Eq. 83 in the case of log-Lorentzian dilation laws. One obtains:

$$\rho = \rho_c \left[1 + \left(\frac{r_0}{r} \right)^{6/\sqrt{1 - \ln^2(r/r_0)/\ln^2(l_P/r_0)}} \right]. \tag{84}$$

Anyway both formulae become equivalent at the scale r_0, which is the relevant scale as concerns the determination of the cosmological constant.

At this stage, we are already able to demonstrate the second Eddington-Dirac's large number relation (that relates cosmological scales to elementary particle scales), and to write it in an exact way and in terms of invariant quantities (i.e., we do not need varying constants to implement it in this form).

Indeed, introducing the cosmic length-scale $\mathbb{L} = \Lambda^{-1/2}$, we get the relation:

$$\mathbb{K} = \frac{\mathbb{L}}{l_P} = \left(\frac{r_0}{l_P}\right)^3 = \left(\frac{m_P}{m_0}\right)^3, \tag{85}$$

where r_0 is the Compton length associated with the mass scale m_0. Then the power 3 in the Eddington-Dirac relation is understood as coming from the power 6 of the gravitational self-energy of vacuum fluctuations and of the power 2 that connects the invariant impassable scale $\mathbb{L}$ to the cosmological constant, following the relation $\Lambda = \mathbb{L}^{-2}$.

Now a complete solution to the problem can be reached only provided the transition scale r_0 be identified. We have suggested [9] that this scale be nothing but a QCD scale, i.e., that the final value of the cosmological constant is fixed at the quark-hadron transition during the Big-Bang. Indeed, before the epoch of this quark-hadron transition, the expansion of the Universe applies to free quarks interdistances. When the distance between quarks reaches the size of hadrons, the quarks become confined, the nucleons are formed, and quarks are no longer subjected to the expansion. The radius of hadrons defines a static scale while the expansion of the Universe continues, but now between hadrons.

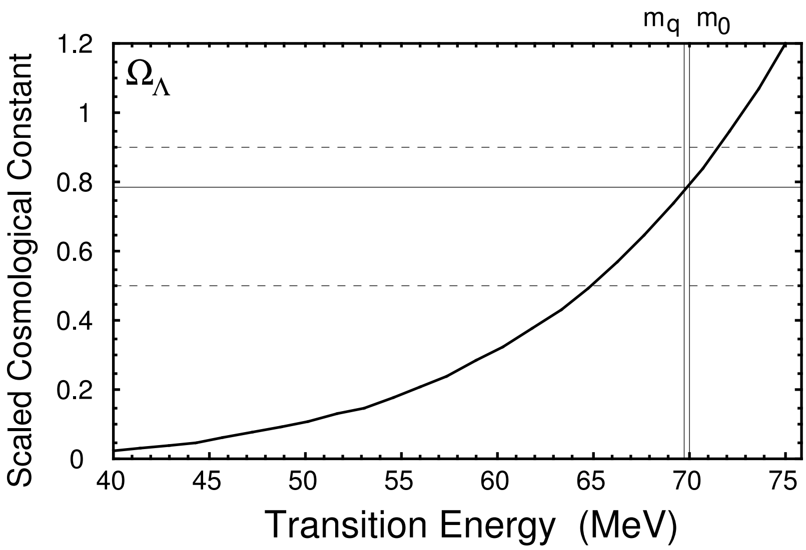

Figure 3: Variation of the value of the scaled cosmological constant $\Omega_\Lambda = \Lambda c^2/3H_0^2$ in function of the transition energy scale E_0 (see text). The dashed horizontal lines give the present observational range ($\Omega_\Lambda = 0.7 \pm 0.2$). The values of the effective mass of quarks in pions and of the energy that correspond to the classical radius of the electron are shown as vertical lines.

The end of this confinement process occurs when the lightest hadrons (the pions) are formed, since it corresponds to the largest possible interquark distance. The effective mass of quarks in the pion, $m_\pi/2 = 69.78$ MeV, and its associated Compton length therefore define a static / expansion transition scale in the scale space.

This value is conforted by a calculation of the confinement QCD scale for 6 quark flavours and our predicted value of $\alpha_3(m_Z) = 0.1155 \pm 0.0005$. We find $\lambda_{QCD} = 71 \pm 2$

MeV from the extrapolation of the running α_3 behavior beyond the Z scale (see Fig. 1). Note that several other fundamental scales of physics fall very close to this energy, such as the classical radius of the electron r_e, that yields the e^+e annihilation cross section at the energy of the electron mass and corresponds to an energy $E_e = 70.02$ MeV, or the diameter of nucleons, that corresponds to an energy 2×64 MeV. One may wonder whether this coincidence of the classical electron radius and the QCD scale could be the manifestation of a strong-electroweak partial unification already at those "low" energy scales.

Therefore we have suggested to identify the transition scale r_0 with the 70 MeV scale. Remark in this context that there is another fundamental justification for relating the quark confinement scale and the cosmic scale $\mathbb{L}$ by a Dirac-Eddington large-number relation: it is the fact that, in the special scale-relativity framework, $\mathbb{L}$ is a confinement scale for the whole universe, while r_0 is a confinement scale for the subset of the Universe which is contained inside hadrons. We obtain from Eq. 85 [6, 9]:

$$\mathbb{K} = 5.302 \times 10^{60}, \tag{86}$$

allowing us to predict a cosmological constant :

$$\Lambda = 1.362 \times 10^{-56} \text{cm}^{-2} \tag{87}$$

i.e. a scaled cosmological constant

$$\Omega_\Lambda = 0.389\, h^{-2}, \tag{88}$$

where $h = H_0/100$ km/s.Mpc. (Note the small correction to the previous publications that gave 0.36 h^{-2}).

Now the Hubble constant has been recently determined with an improved precision to be $H_0 = 71 \pm 7$ km/s.Mpc. Therefore our theoretical prediction yields a scaled cosmological constant $\Omega_\Lambda = 0.77 \pm 0.15$. Such a value is now supported by experimental determinations. Indeeed, recent measurements using the Hubble diagram of SNe I [59, 57, 58], the angular power spectrum of the cosmic microwave radiation [60] and gravitational lensing [61], have yielded a self-consistent value 0.7 ± 0.2. Note that this result was anticipated twenty years ago by an analysis of the Hubble diagram of infrared ellipticals, that had yielded $\Omega_\Lambda = 0.9 \pm 0.6$ [62].

3.6.4 Neutrino mass

One of the new ways offered by the scale-relativity approach for identifying fundamental mass-scales is the interpretation of transitions between a fractal, nonderivable, scale-dependent regime and a classical, scale-independent, derivable regime as being Compton-lengths (in rest frame). Such a transition between the $1/r^6$ gravitational self-energy density of quantum fluctuations and the constant energy density linked to the geometric cosmological constant has been identified in the previous section with the effective quark mass in pions (i.e., the minimal possible effective quark mass).

Now one can consider again the energy density contribution, that varies as $1/r^4$ (see Fig. 4). It will therefore cross the cosmological constant contribution at a scale that is the "middle" of the Universe in scale space, given by:

$$\frac{\lambda_s}{l_P} = \left(\frac{\mathbb{L}}{l_P}\right)^{1/2}. \tag{89}$$

This Compton length takes the value $\lambda_s = 37.22\ \mu$m from our determination of the cosmic scale $\mathbb{L} = 5.302 \times 10^{60}\, l_P$. From the measurement of the cosmological constant, $\Omega_\Lambda = 0.7 \pm 0.2$, one obtains $\lambda_s = 37 \pm 5\mu$m. It corresponds to a mass:

$$m_s = \frac{m_P}{\sqrt{5.302} \times 10^{30}} = 5.3 \times 10^{-3}\text{eV}. \tag{90}$$

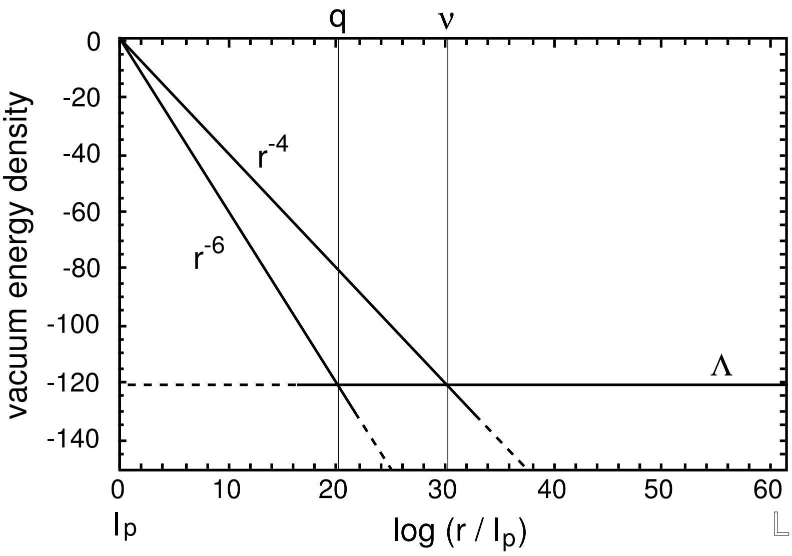

Figure 4: Variation of the vacuum energy density [in $(l_P/r)^4$] and of the gravitational self-energy density of quantum vacuum fluctuations [in $(l_P/r)^6$] in the framework of Galilean scale relativity. We have suggested that the r^{-6} gravitational self-energy contribution crosses the geometric cosmological constant contribution at a scale of 70 MeV (quark confinement transition). This yields a cosmological constant $\Lambda = 1/\mathbb{L}^2$, where $\mathbb{L} = 5.3 \times 10^{60}\ l_P$. Therefore the vacuum energy contribution crosses Λ at a scale of 5.3×10^{-3} eV, which is the typical neutrino mass needed to explain solar neutrino oscillations.

Is such a small mass scale relevant to some physical or astrophysical process ? There is now convincing evidence that neutrinos have nonzero masses (see e.g. [42]). Indeed, there are three reported indications that neutrinos oscillate and therefore have mass (atmospheric neutrinos, solar neutrinos and neutrinos from stopped π^+'s in the LSND experiment). For example, the results of the three experiments can be accomodated in a four neutrino scheme including a sterile neutrino, i.e., that does not participate in the normal weak interaction [42].

In particular, the solar neutrino observations are understood in terms of a squared mass difference $m_\nu^2 - m_{\nu_e}^2 \approx 10^{-5}$ eV^2. Provided $m_{\nu_e} \approx 0$ [42], this yields a mass $m_\nu \approx 3 \times 10^{-3}$ eV for the other neutrino state. The above mass scale at 5.3×10^{-3} is therefore compatible with such a neutrino mass. It also agrees with the estimate derived by Sidharth from another argument ($\approx 10^{-8} m_e$) [63].

4 Conclusions and prospects

The present contribution was devoted to the study of the cosmological consequences of the scale-relativistic generalization of dilations laws. Strictly these consequences should be studied in the framework of a complete theory which would be both motion-general-relativistic and scale-general-relativistic. Such a general theory is still in construction, so that only hints about its expected conclusions have been given here. In particular, a fundamental study is still needed in order to obtain a proper scale-relativistic generalization of Einstein's equations. Such a generalization, which we shall attempt in a future work, will allow one to obtain a genuine special-scale-relativistic generalization of what is already a fundamental scale behaviour of cosmology, namely, the expansion of the universe as described by Hubble's law.

However, even though the theory has not yet reached such a stage of development, it already allows a renewed understanding of the nature of the cosmological constant and it connects it to several observable and measurable features. Reversing the argument, these relations become new consistent ways to measure the cosmological constant: namely, one can use the flatness condition, the gravitational coupling constant / length-scale condition, the quark Compton-length / cosmic length relation, the sterile neutrino Compton-length / cosmic length relation and the mass of the Universe / cosmic length relation. In previous works [6, 52, 9], we have shown that a value of the cosmological constant consistent with the other estimates may be also determined from the variation with scale of the slope of the galaxy-galaxy correlation function.

Let us conclude by remarking that not all structures in the (length-scale versus couplings) diagram leading to new predictions of fundamental mass scales relevant to astrophysical processes have been considered in the present contribution. Indeed, we can also consider the solution of the equation $\alpha_2^{-1}(E) = \mathbb{C}(E)$ (see Fig. 1), where $\alpha_2(E)$ is the running SU(2) coupling, which is $E \approx 75$ TeV. Such an energy is of the same order as the highest energy cut-off $E_{cut} = 28$ TeV in the gamma ray spectrum of Mkn 501 [64], which has up to now received no standard explanation. Another relevant mass-charge relation concerns the strong coupling: $\alpha_3^{-1}(E) = \mathbb{C}(E)$. The solution of this equation is $E = 3 \times 10^{17}$ eV, and it could help explaining the "knee" in the (flux versus energy) diagram of high-energy cosmic rays.

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References

[1] Galilei Galileo. Dialogue concerning the two chief world systems. Berkeley: University of California Press; 1967.

[2] Poincaré H. C.R. Acad. Sci. Paris 1905; t. CXL: 1504.

[3] Einstein A. Annalen der Physik 1905; 17: 891.

[4] Einstein A. Annalen der Physik 1916; 49: 769.

[5] Nottale L. Int. J. Mod. Phys. 1992;A7:4899.

[6] Nottale L. Fractal Space-Time and Microphysics: Toward a Theory of Scale Relativity. London: World Scientific; 1993.

[7] Nottale, L. Chaos, Solitons and Fractals 1999; 10: 459.

[8] Nottale L. in 'Relativity in General", E.R.E. 93 (Spanish Relativity Meeting) Salas, Ed. J. Diaz Alonso and M. Lorente Paramo (Frontières), p.121.

[9] Nottale L. Chaos, Solitons & Fractals 1996;7:877.

[10] Nottale L. Int. J. Mod. Phys. 1989;A4:5047.

[11] Nottale L. Astron. Astrophys. 1997;327:867.

[12] Ord G. N. J. Phys. A: Math. Gen. 1983;16:1869.

[13] Nottale L. and Schneider J. Math. Phys. 1984;25:1296.

[14] El Naschie M.S. Chaos, Solitons & Fractals 1992;2:211.

[15] Quantum mechanics, Diffusion and Chaotic Fractals, eds. M.S. El Naschie, O.E. Rossler and I. Prigogine, (Pergamon).

[16] Da Rocha D., Nottale L. Chaos, Solitons and Fractals 2002; this volume.

[17] Dubrulle B. Eur. Phys. J. B 2000; 14: 757.

[18] Nottale L. in Traité IC2. Fractals et Lois d'échelles. eds. P. Abry, P. Goncalvès et J. Levy Vehel, Hermès 2001, vol.2, Chap. 7, p. 235.

[19] Célérier M.N. & Nottale L. Phys. Rev A 2002, submitted, hep-th/0112213.

[20] Randall L. and Sundrum R. Phys. Rev. Lett. 1999; 83: 4690.

[21] Weyl H., Sitz. Preus. Akad. d. Wiss. (1918). English translation in The Principle of Relativity, (Dover publications), p. 201-216.

[22] Nottale L. in Science of the Interface, Proceedings of International Symposium in honor of Otto Rössler, ZKM Karlsruhe, 18-21 May 2000, Eds. H. Diebner, T. Druckney and P. Weibel, p. 38. Genista Verlag, Tübingen, 2001.

[23] Nottale L. in Frontiers of Fundamental Physics, Proceedings of Birla Science Center Fourth International Symposium, edited by B. G. Sidharth and M. V. Altaisky, (Kluwer Academic, 2001) p. 65.

[24] Nottale, L. Chaos, Solitons and Fractals 2001;12: 1577.

[25] Einstein A. Ann. Phys. 1905; 27: 132.

[26] Finkel R.W. Phys. Rev. A 1987; 35: 1486.

[27] Nottale L. in Particle Astrophysics : the early universe and cosmic structures, J.M. Alimi et al. eds, Proceedings of the XXVth Rencontres de Moriond, Frontières 1990, p. 13.

[28] Nottale L., Schumacher G., Lefèvre E.T. Astron. Astrophys. 2000; 361: 379.

[29] Agop M., Ioannou P.D., Buzea C. Chaos, Solitons and Fractals 2002; 13: 1137.

[30] Markowich P.A., Rein G., Wolansky G. 2001, arXiv:math-ph/0101020.

[31] Nottale L., Tran Minh N., Astron. Astrophys. 2002, submitted.

[32] Nottale, L., in “Chaos and Diffusion in Hamiltonian Systems”, ed. D. Benest and C. Froeschlé 1994; Frontières: p. 173.

[33] Nottale L. Astron. Astrophys. Lett 1996;315:L9.

[34] Nottale L., Schumacher G., Gay J. Astron. Astrophys. 1997;322:1018.

[35] Hermann R., Schumacher G., Guyard R. Astron. Astrophys. 1998;335:281.

[36] Nottale L., Schumacher G. in Fractals and Beyond: complexities in the sciences, ed. M.M. Novak. 1998 World Scientific; p. 149.

[37] Einstein A. Sitzungsberichte der Preussichen Akad. d. Wissenschaften 1917, translated in The principle of relativity. Dover, p. 177.

[38] Sciama D. W. MNRAS 1953; 113: 34.

[39] Nottale L. Ciel et Terre. 1998; 114: 63.

[40] Weinberg S. Gravitation and Cosmology. New-York: John Wiley and Sons; 1972.

[41] Barbour J. Nature 1974; 249: 328.

[42] Particle Data Group, 2000, The European Physical Journal, C15, 1 (http://www-pdg.lbl.gov/).

[43] Erler J. Phys. Rev. D 1999; 59: 054008.

[44] Bird D.J. et al. Astrophys. J. 1994; 424: 491.

[45] Bird D.J. et al. Astrophys. J. 1995; 441: 144.

[46] Bonazzola S. and Peter P. Astropart. Phys. 1997; 7: 161.

[47] Elbert J.W. and Sommers P. Astrophys. J. 1995; 441: 151.

[48] Hawking S. W. Phys. Lett. B 1984; 134: 403.

[49] Weinberg S. Rev. Mod. Phys. 1989; 61: 1.

[50] Barbour J. and Bertotti B. Proc. R. Soc. Lond. A 1982; 382: 295.

[51] Barbour J. and Pfister H., editors. Mach principle: From Newton's bucket to quantum gravity. Birkhäuser. Boston 1995.

[52] Nottale L. in Clustering of the Universe, S. Maurogordato et al. eds. Proceedings of the XXXth Rencontres de Moriond, Frontières 1995, p. 523.

[53] Carroll S.M. and Press W.H. Ann. Rev. Astron. Astrophys. 1992; 30: 499.

[54] Lemaitre G. Proc. Nat. Acad. Sci. 1934; 20: 12.

[55] Zeldovich Ya. B. JETP Lett 1967; 6: 316.

[56] Sidharth B.G. The Chaotic Universe. New York: Nova Science Publishers; 2001.

[57] Perlmutter S., Aldering G., Della Valle M. et al. Nature 1998; 391: 51.

[58] Riess A.G., Filippenko A.V., Challis P. et al. AJ 1998; 116: 1009.

[59] Garnavich P.M., Jha S., Challis P. et al. ApJ 1998; 509: 74.

[60] de Bernardis P. et al. Nature 2000; 404: 955.

[61] Van Waerbeke et al. Astron. Astrophys. 2001; 374, 757.

[62] Nottale, L., 1982, in "L'Univers", Ecole de Goutelas, E. Schatzman ed., "Clusters of galaxies and cosmological tests"

[63] Sidharth B.G. Chaos, Solitons & Fractals 2001; 12: 1101.

[64] Aharonian F.A. et al. Astron. Astrophys.1999; 349: 11.