# Scale-relativity and quantization of the universe

## I. Theoretical framework

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**Abstract.** The theory of scale relativity extends Einstein's principle of relativity to scale transformations of resolutions. It is based on the giving up of the axiom of differentiability of the space-time continuum. The new framework generalizes the standard theory and includes it as a special case. Three consequences arise from this withdrawal: (i) The geometry of space-time must be fractal, i.e., explicitly resolution-dependent. This allows us to include resolutions in the definition of the state of the reference system, and to require scale-covariance of the equations of physics under scale transformations. (ii) The geodesics of the non-differentiable space-time are themselves fractal and in infinite number. This divergence strongly suggests we undertake a statistical, non-deterministic description. (iii) Time reversibility is broken at the infinitesimal level. This can be described in terms of a two-valuedness of the time derivative, which we account by using complex numbers. We finally combine these three effects by constructing a new tool, the scale-covariant derivative, which transforms classical mechanics into a generalized, quantum-like mechanics.

Scale relativity was initially developed in order to re-found quantum mechanics on first principles (while its present foundation is axiomatic). However, the scale-relativistic approach is expected to apply not only at small scales, but also at very large space- and time-scales, although with a different interpretation. Indeed, we find that the scale symmetry must be broken at two (relative) scales, so that the scale axis is divided in three domains: (i) the quantum, scale-dependent microphysical domain, (ii) the classical, intermediate, scale-independent domain, (iii) but also the macroscopic, cosmological domain which becomes scale-dependent again and may then be described on very large time-scales (beyond a predictability horizon) in terms of a non-deterministic, statistical, quantum-like theory. In the new framework, we definitively give up the hope to predict individual trajectories on very large time scales. This leads us to describing their virtual families in terms of complex probability amplitudes, which are solutions of generalized Schrödinger equations. The squared modulus of these probability amplitudes yields probability densities, whose peaks are interpreted as a

tendency for the system to make structures. Since the quantizations in quantum mechanics appear as a direct consequence of the limiting conditions and of the shape of the input field, the theory thus naturally provides self-organization of the system it describes, in connection with its environment.

In the present first paper of this series, we first recall the structure of the scale-relativity theory, then we apply our scale-covariant procedure to various equations of classical physics that are relevant to astrophysical processes, including the equation of motion of a particle in a gravitational field (Newtonian and Einsteinian), in an electromagnetic field, the Euler and Navier-Stokes equations, the rotational motion of solids, dissipative systems, and first hints on field equations themselves. In all these cases, we obtain new generalized Schrödinger equations which allow quantized solutions. In scale-relativity therefore, the underlying fractal geometry of space-time plays the role of a universal structuring "field". In subsequent papers of this series, we shall derive the solutions of our equations, then show that several new theoretical predictions can be made, and that they can be successfully checked by an analysis of the observational data.

## 1. Introduction

The theory of scale relativity (Nottale 1993a) is founded on the realization that the whole of present physics relies on the implicit assumption of differentiability of the space-time continuum. Giving up the a priori hypothesis of the differentiability of coordinates has important physical consequences: one can indeed demonstrate (Nottale 1993a, 1994a, 1995a) that a continuous but nondifferentiable space-time is necessarily *fractal*. Here the word fractal (Mandelbrot 1975, 1983) is taken in a general meaning, as defining a set, object or space that shows structures

at all scales. More precisely, one can demonstrate that a continuous but nondifferentiable function is explicitly resolution-dependent, and that its length $\mathscr{L}$ tends to infinity when the resolution interval tends to zero, i.e. $\mathscr{L} = \mathscr{L}(\epsilon)_{\epsilon \to 0} \to \infty$ (see Appendix A). This theorem naturally leads to the proposal that the concept of *fractal space-time* (Nottale 1981; Nottale and Schneider 1984; Ord 1983; Nottale 1989, 1993a; El Naschie 1992) is the geometric tool adapted to the research of such a new description based on non-differentiability.

It is important to be by now more specific about the precise meaning of the withdrawal of the axiom of differentiability. That does not mean that we a priori assume that the coordinates are not differentiable with certainty, but instead that we consider a generalized framework including all continuous functions, those which are differentiable and those which are not. Thus this framework includes the usual differentiable functions, but as very particular and rare cases. It is an extension of the usual framework, so that the new theory is expected, not to contradict, but instead to generalize the standard theory, since standard differentiable physics will be automatically included in it as a special case. An historical example of such an extension is the passage to curved spacetimes in general relativity, which amounts to giving up the previous implicit assumption of flatness of Euclidean geometry, and which anyway includes flat spacetimes in its description.

Since a nondifferentiable, fractal space-time is explicitly resolution-dependent, the same is a priori true of all physical quantities that one can define in its framework. (Once again, this means that we shall formally introduce such a scale-dependence as a generalization, but that the new description will also include the usual scale-independence as a special case, in a way similar to the relations between statics and kinematics: statics is a special, degenerate case of the laws of motion). We thus need to complete the standard laws of physics (which are essentially laws of motion and displacement in classical physics) by laws of scale, intended to describe the new resolution dependence. We have suggested (Nottale 1989, 1992, 1993a) that the principle of relativity of Galileo and Einstein, that is known since Descartes and Huygens to be a constructive principle for motion laws, can be extended to constrain also these new scale laws.

Namely, we generalize Einstein's (1916) formulation of the principle of relativity, by requiring *that the laws of nature be valid in any reference system, whatever its state*. Up to now, this principle has been applied to changes of state of the coordinate system that concerned the origin, the axes orientation, and the motion (measured in terms of velocity, acceleration, ...) .

In scale relativity, the space-time resolutions are not only a characteristic of the measurement apparatus, but acquire a universal status. They are considered as essential variables, inherent to the physical description. We define them as characterizing the "state of scale" of the reference system, in the same way as the velocity characterizes its state of motion. The principle of scale relativity consists of applying the principle of relativity to such a scale-state. Then we set a principle of *scale-covariance*, requiring that the equations of physics keep their simplest form under resolution transformations (dilations and contractions).

The domains of application of this theory are typically the asymptotic domains of physics, small length-scales and small time-scales $\Delta x \to 0$, $\Delta t \to 0$ (microphysics), large length-scales $\Delta x \to \infty$ (cos mology), but also large time-scales $\Delta t \to \infty$. The present series of papers particularly addresses this last domain.

Initially, the theory of scale relativity was mainly an attempt at refounding quantum mechanics on first principles (Nottale 1993a). We have demonstrated that the main axioms of quantum mechanics can be recovered as consequences of the principle of scale relativity, and that the behavior of the quantum world can be understood as the various manifestations of the non-differentiability and fractality of space-time at small scales (Nottale 1993a, 1994a,b, 1995c, 1996a). Moreover, the theory allows one to generalize standard quantum mechanics. Indeed, we have shown that the usual laws of scale (power law, self-similar, constant fractal dimension) have the status of "Galilean" scale laws, while a full implementation of the principle of scale relativity suggests that they could be medium scale approximations of more general laws which take a Lorentzian form (Nottale 1992, 1993a). In such a "special scale relativity" theory, the Planck length- and time-scale becomes a minimal, impassable scale, invariant under dilations and contractions, which replaces the zero point (since owning all its physical properties) and plays for scales the same role as played by the velocity of light for motion (see Appendix B). In this new framework, several still unsolved problems of fundamental physics find simple and natural solutions: new light is brought on the nature of the Grand Unification scale, on the origin of the electron scale and of the electroweak scale, on the scale-hierarchy problem, and on the values of coupling constants (Nottale 1993a, 1994a, 1996a); moreover, our scale-relativistic interpretation of gauge invariance allowed us to give new insights on the nature of the electric charge, then to predict new mass-charge relations for elementary particles (Nottale 1994a,b, 1996a).

The same approach has been applied to the cosmological domain, leading to similar conclusions. Namely, new special scale-relativistic dilation laws can be constructed, in terms of which there exists a maximal length-scale of resolution, impassable and invariant under dilations. Such a scale can be identified with the scale $\mathbb{L}$ of the cosmological constant ($\Lambda = \mathbb{L}^{-2}$). It would own all the physical properties of the infinite. Its existence also solves several fundamental problems in cosmology, including the problem of the vacuum energy density, the value of the cosmological constant, the value of the index of the galaxy-galaxy correlation function and the transition scale to uniformity (Nottale 1993a, 1995d, 1996a).

In the present series of papers, we shall not consider the consequences of this new interpretation of the cosmological constant, which mainly apply to the domain of very large scales ($z \gtrsim 1$). This case has already been briefly considered in (Nottale 1993a Chap. 7, 1995d) and will be the subject of a particularly devoted, more detailed work (Nottale 1997). We shall instead specialize our study here to the typical scales where structures are observed, which remain small compared with the size of the Universe, but yet are of cosmological interest. For

such scales, Galilean scale laws (i.e., standard self-similar laws with constant fractal dimension) remain a good approximation of the more general scale-relativistic laws.

We are mainly concerned here with the physical description of systems when they are considered on very large time-scales. As we shall see, this question is directly related to the problem of chaos. It is indeed now widely known that most classical equations describing the evolution of natural systems, when integrated on sufficiently large times, have solutions that show a chaotic behavior. The consequence of strong chaos is that, at time-scales very large compared with the "chaos time", $\Delta t \gg \tau$ (the inverse Lyapunov exponent), i.e., beyond the horizon of predictability, there is a complete loss of information about individual trajectories. Basing ourselves on the existence of such predictability horizons, we have suggested (Nottale 1993a, Chap. 7) that the universal emergence of chaos in natural systems was the signature of new physics on very large time scales, and that chaotic systems could be described beyond the horizon by a new, quantum-like, *non-deterministic* theory, since the classical equations become unusable for $\Delta t \gg \tau$.

But we shall suggest in the present paper an even more profound connection between chaos and scale relativity. After all, chaos has been discovered (Poincaré 1892) as a general, empirical property of the solutions of most classical equations of physics and chemistry, when applied to natural systems in all their complexity. But this does not mean that we really understand its origin. On the contrary, it is rather paradoxical that deterministic equations, built in the framework of a causal way of thinking (one gives oneself initial conditions in position and velocity, then the evolution of the system is predicted in a totally deterministic way), finally lead to a complete loss of predictability of individual trajectories. Our first proposal (Nottale 1993a,b) has therefore been (following the above reasoning) to jump to a non-deterministic description, that would act as a large-time scale *approximation*. We suggest in the present paper to reverse the argument, and to take into account the particular chaos that takes its origin in the underlying, non-differentiable and fractal character of space-time, since, as we shall see, it is expected to become manifest not only at small scales but also at very large length-scales and *time-scales*. The advantages of this viewpoint reversal are important:

(i) The breaking of the reflection invariance($dt \rightarrow -dt$), which is one of the principal new effect of nondifferentiability (see Sect. 3.1.3), find its complete justification only when acting at the space-time level, not only that of fractal trajectories in a smooth space-time.

(ii) The chaotic behavior of classical equations could now be understood (or at least related to a first principle approach): these equations would actually be incomplete versions of more general equations, that would be classical and deterministic at small scales, but would become quantum-like and non-deterministic at large-time scales (see Sect. 3).

(iii) The additional conclusion in the scale-relativistic framework is that the new structuring behavior may be universal. We shall see from a comparison with observational data (Paper III) that the observed structures show indeed universal proper-

ties, since we find that identical structuring laws are observed at scales which range from the Solar System scale to the cosmological scales, and since these laws are written in terms of a unique new fundamental constant (Nottale 1996b,c; Nottale, Schumacher and Gay 1997).

In the present paper, we shall first be more specific about our motivations for constructing such a new theory (Sect. 2), then we shall describe our general method (Sect. 3), and apply it to the fundamental equations used in several domains of fundamental physics having astrophysical implications (Sect. 4). Additional informations, in particular about the general framework of which the present developments are a subset, are given in Appendices A and B. Paper II will be specially devoted to the application of our theory to gravitational structures, and Paper III to a first comparison of our theoretical predictions with observational data.

## 2. Motivation

### 2.1. Chaotic systems beyond their horizon of predictability

Consider a strongly chaotic system, i.e., the gap between any couple of trajectories diverges exponentially with time. Let us place ourselves in the reference frame of one trajectory, that we describe as uniform motion on the $z$ axis:

$$x = 0, \ y = 0, \ z = at. \tag{1}$$

The second trajectory is then described by the equations:

$$x = \delta x_0(1 + e^{t/\tau}), \ y = \delta y_0(1 + e^{t/\tau}), \ z = at + \delta z_0(1 + e^{t/\tau}), \tag{2}$$

where we have assumed a single Lyapunov exponent $1/\tau$ for simplicity of the argument. Let us eliminate the time between these equations. We obtain:

$$y = \frac{\delta y_0}{\delta x_0}x, \ z = \frac{\delta z_0}{\delta x_0}x + a\tau \ln(\frac{x}{\delta x_0} - 1). \tag{3}$$

As schematized in Fig. 1, this means that the relative motion of one trajectory with respect to another one, when looked at with a *very long time resolution* (i.e., $\Delta t \gg \tau$, right diagram in Fig. 1), becomes non-differentiable at the origin, with different backward and forward slopes. Moreover, *the final direction of the trajectory in space is given by the initial "uncertainty vector"* $\epsilon = (\delta x_0, \delta y_0, \delta z_0)$. Then chaos achieves a kind of amplification of the initial uncertainty. But the orientation of the uncertainty vector $\epsilon$ being completely uncontrolable (it can take its origin at the quantum scale itself), the second trajectory can emerge with any orientation with respect to the first. If we now start from a continuum of different values $\delta x_0$, the breaking point in the slope (Fig. 1) occurs anywhere, and the various trajectories become describable by non-differentiable, fractal paths.

In the end, beyond the horizon of predictability, the information about the behavior of the trajectory at $t < 0$ is completely lost; this strongly suggests that we switch to a statistical description. Indeed, assume that we are looking at the evolution of the system during a very large time scale (of the order of the age
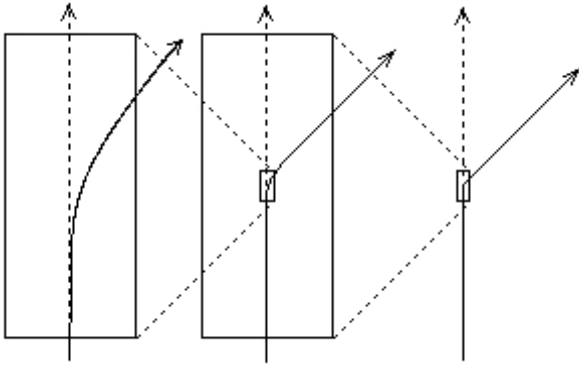
**Fig. 1.** Schematic representation of the relative evolution in space of two initially nearby chaotic trajectories seen at three different time scales, $\tau$, $10\,\tau$ and $100\,\tau$ (from Nottale, 1993a).

of the Universe) with a time-resolution $\approx 10\tau$. Each successive event can be considered as totally independent of its preceding one, because of this information loss. Such an independence of the various events leads to describing the trajectories in terms of a Markov process. In other words, even if the basic equations remain deterministic, it is not the case of their solutions. We can then wonder whether the classical equations remain adapted to the physical description on very large time scales, and we are led to suggesting the alternative starting point of inherently statistical systems. Moreover, we shall see that a description in terms of classical probabilities seems to be incomplete, and that, at time resolutions larger than the horizon, we need a quantum-like description in terms of *probability amplitudes*.

One must keep in mind that such a large time-scale description would be no longer valid at small time-scales, since when going back to $\Delta t \ll \tau$ (left diagram in Fig. 1), differentiability is recovered. This is in accordance with the scale-relativistic framework, in which physics, including its fundamental equations and their interpretation, is now explicitly scale-dependent. In particular, the physical laws can be subjected to a kind of phase transition around some symmetry breaking scales, as we shall see in what follows.

## 2.2. Giving up differentiability of space-time

There is a fundamental reason for jumping to a non-deterministic, scale-relativistic physical description at small *and large* scales. Since more than three centuries, physics relies on the assumption that space-time coordinates are a priori differentiable. However, it was demonstrated by Feynman (see Feynman & Hibbs 1965) that the typical paths of quantum mechanical particles are continuous but non-differentiable. Now, one of the most powerful avenue for reaching a genuine understanding of the laws of nature has been to construct them, not from setting additional hypotheses, but on the contrary by attempting to give up some of them, i.e., by going to increased generality. From that point of view, of which Einstein was a firm supporter, the laws and structures of nature are simply the most general laws and structures that are physically possible. It culminated in the

principle of "general" relativity and in Einstein's explanation of the nature of gravitation as the various manifestations of the Riemannian geometry of space-time (i.e., of the giving up of flatness).

However, in the light of the above remark, Einstein's principle of relativity is not yet fully general, since it applies to coordinate transformations that are continuous and at least two times differentiable. The aim of the theory of scale-relativity is to look for the laws and structures that would be the manifestations of still more general transformations, namely, continuous ones (that can be differentiable or not). In such a construction the standard theory will automatically be recovered as a special case, since differentiable spaces are a particular subset of the set of all continuous spaces.

In that quest, the first step consists of realizing that a continuous but non-differentiable space-time is necessarily fractal, i.e., explicitly resolution-dependent (see Appendix A). This leads us to introduce new intrinsic scale variables in the very definition of physical quantities (among which the coordinates themselves), but also to construct the differential equations (in the "scale space") that would describe this new dependence. In other words, the search for the laws of a non-differentiable physics can be brought back to the search of a completion of the laws of motion by new laws of scale and laws of motion/scale coupling.

The remaining of the present section (completed by Appendix B) is aimed at giving to the reader a hint of the general structure of the scale-relativity theory. We shall see that, since the new scale equations are themselves constrained by the principle of relativity, the new concepts fit well established structures. Namely, the so-called symplectic structure of most physical theories (including thermodynamics, see Peterson 1979), i.e., the Poisson bracket / Euler-Lagrange / Hamilton formulation, can be also used to construct scale laws. Under such a viewpoint, scale invariance is recovered as corresponding to the "free" case (the equivalent of what inertia is for motion laws).

## 2.3. Scale invariance and Galilean scale relativity

Scaling laws have already been discovered and studied at length in several domains of science. A power-law scale dependence is frequently encountered in a lot of natural systems, it is described geometrically in terms of fractals (Mandelbrot 1975, 1983), and algebrically in terms of the renormalization group (Wilson 1975, 1979). As we shall see now, such simple scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws.

In most present use and applications of fractals, the fractal dimension $D$ is defined from the variation with resolution of the main fractal variable (e.g., the length $\mathscr{L}$ of a fractal curve which plays here the role of a fractal curvilinear coordinate, the area of a fractal surface, etc...). Namely, if $D_T$ is the topological dimension ($D_T = 1$ for a curve, 2 for a surface, etc...), the scale dimension $\delta = D - D_T$ is defined, following Mandelbrot, as:

$$\delta = \frac{d\ln\mathscr{L}}{d\ln(\lambda/\epsilon)}. \tag{4}$$

When $\delta$ is constant, we obtain a power-law resolution dependence $\mathscr{L} = \mathscr{L}_0(\lambda/\epsilon)^\delta$. The Galilean structure of the group of scale transformation that corresponds to this law can be verified in a straightforward manner from the fact that it transforms in a scale transformation $\epsilon \rightarrow \epsilon'$ as

$$\ln \frac{\mathscr{L}(\epsilon')}{\mathscr{L}_0} = \ln \frac{\mathscr{L}(\epsilon)}{\mathscr{L}_0} + \delta(\epsilon) \ln \frac{\epsilon}{\epsilon'}, \tag{5}$$

$$\delta(\epsilon') = \delta(\epsilon). \tag{6}$$

This transformation has exactly the structure of the Galileo group, as confirmed by the law of composition of dilations $\epsilon \rightarrow \epsilon' \rightarrow \epsilon''$, which writes

$$\ln \rho'' = \ln \rho + \ln \rho', \tag{7}$$

with $\rho = \epsilon'/\epsilon$, $\rho' = \epsilon''/\epsilon'$ and $\rho'' = \epsilon''/\epsilon$.

### 2.4. Lagrangian approach to scale laws

We are then naturally led, in the scale-relativistic approach, to reverse the definition and the meaning of variables. The scale dimension $\delta$ becomes, in general, an essential, fundamental *variable*, that remains now constant only in very particular situations (namely, in the case of scale invariance, that corresponds to "scale-freedom"). It plays for scale laws the same role as played by time in motion laws. The resolution can now be defined as a derived quantity in terms of the fractal coordinate and of the scale dimension:

$$\bar{V} = \ln(\lambda/\epsilon) = \frac{d \ln \mathscr{L}}{d\delta}. \tag{8}$$

Our identification of standard fractal behavior as Galilean scale laws can now be fully justified. We assume that, as in the case of motion laws, scale laws can be constructed from a Lagrangian approach. A scale Lagrange function $\bar{L}(\ln \mathscr{L}, \bar{V}, \delta)$ is introduced, from which a scale-action is constructed:

$$\bar{S} = \int_{\delta_1}^{\delta_2} \bar{L}(\ln \mathscr{L}, \bar{V}, \delta) d\delta. \tag{9}$$

The action principle, applied on this action, yields a scale-Euler-Lagrange equation that writes:

$$\frac{d}{d\delta} \frac{\partial \bar{L}}{\partial \bar{V}} = \frac{\partial \bar{L}}{\partial \ln \mathscr{L}}. \tag{10}$$

The simplest possible form for the Lagrange function is the equivalent for scales of what inertia is for motion, i.e., $\bar{L} \propto \bar{V}^2$ and $\partial \bar{L}/\partial \ln \mathscr{L} = 0$ (no scale "force", see Appendix B). Note that this form of the Lagrange function becomes fully justified, as in the case of motion laws, once one jumps to special scale-relativity (Nottale 1992) and then goes back to the Galilean limit (see Appendix B). The Lagrange equation writes in this case:

$$\frac{d\bar{V}}{d\delta} = 0 \Rightarrow \bar{V} = cst. \tag{11}$$

The constancy of $\bar{V} = \ln(\lambda/\epsilon)$ means here that it is independent of the scale-time $\delta$. Then Eq. (8) can be integrated in terms of the usual power law behavior, $\mathscr{L} = \mathscr{L}_0(\lambda/\epsilon)^\delta$. This reversed viewpoint has several advantages which allow a full implementation of the principle of scale relativity:

(i) The scale dimension takes its actual status of "scale-time", and the logarithm of resolution $\bar{V}$ its status of "scale-velocity", $\bar{V} = d \ln \mathscr{L}/d\delta$. This is in accordance with its scale-relativistic definition, in which it characterizes the state of scale of the reference system, in the same way as the velocity $v = dx/dt$ characterizes its state of motion.

(ii) This leaves open the possibility of generalizing our formalism to the case of four independent space-time resolutions, $\bar{V}^\mu = \ln(\lambda^\mu/\epsilon^\mu) = d \ln \mathscr{L}^\mu/d\delta$.

(iii) Scale laws more general than the simplest self-similar ones can be derived from more general scale-Lagrangians (Appendix B).

It is essentially Galilean scale relativity that we shall consider in the present series of papers. Before developing it further, we recall, however, that the Galilean law is only the simplest case of scale laws that satisfy the principle of scale relativity. We shall, in Appendix B, give some hints about its possible generalizations, since they determine the general framework of which Galilean scale relativity is only a subset.

### 2.5. Scale-symmetry breaking

An important point concerning the scale symmetry, which is highly relevant to the present study is that, as is well-known from the observed scale-independence of physics at our own scales, and as we shall demonstrate in more detail in Sect. 3, the scale dependence is a spontaneously broken symmetry (Nottale 1989, 1992, 1993a). Let us recall the simple theoretical argument that leads to this result and to its related consequence that space-time is expected to become fractal at small but also at large space-time scales.

In the general framework of a continuous space-time (not necessarily differentiable), we expect a general curvilinear coordinate to be explicitly resolution-dependent (Appendix A), i.e. $\mathscr{L} = \mathscr{L}(\epsilon)$. We assume that this new scale dependence is itself solution of a differential equation *in the scale space*. The simplest scale differential equation one can write is a first order equation where the scale variation of $\mathscr{L}$ depends on $\mathscr{L}$ only, $d\mathscr{L}/d \ln \epsilon = \beta(\mathscr{L})$. The function $\beta(\mathscr{L})$ is a priori unknown but, always taking the simplest case, we may consider a perturbative approach and take its Taylor expansion. We obtain the equation:

$$\frac{d\mathscr{L}}{d \ln \epsilon} = a + b\mathscr{L} + c\mathscr{L}^2 + \dots \tag{12}$$

Disregarding for the moment the quadratic term, this equation is solved in terms of a standard power law of power $\delta = -b$, broken at some scale $\lambda$, as illustrated in Fig. 2 ($\lambda$ appears as a constant of integration):

$$\mathscr{L} = \mathscr{L}_0 \left[1 + (\frac{\lambda}{\epsilon})^\delta\right]. \tag{13}$$

Depending on the sign of $\delta$, this solution represents either a small-scale fractal behavior (in which the scale variable is a resolution), broken at larger scales, or a large-scale fractal behavior (in which the scale variable $\epsilon$ would now represent a changing window for a fixed resolution $\lambda$), broken at smaller scales.

The symmetry between the microscopic and the macroscopic domains can be even more directly seen from the properties of Eq. (12). Let us indeed transform the two variables $\mathscr{L}$ and $\epsilon$ by *inversion*, i.e. $\mathscr{L} \to \mathscr{L}' = 1/\mathscr{L}$ and $\epsilon \to \epsilon' = 1/\epsilon$, we find that Eq. (12) becomes:

$$\frac{d\mathscr{L}'}{d \ln \epsilon'} = c + b\mathscr{L}' + a\mathscr{L}'^2 + \ldots \tag{14}$$

This is exactly the same equation up to the exchange of the constants $a$ and $c$. In other words, Eq. (12) is covariant (i.e. form invariant) under the inversion transformation, which transforms the small scales into the large ones and reciprocally, but also the upper symmetry breaking scale into a lower one. Hence the inversion symmetry, which is clearly not achieved in nature at the level of the observed structures, may nevertheless be an exact symmetry at the level of the fundamental laws. This is confirmed by directly looking at the solutions of Eq. (12) keeping now the quadratic term, since they may include two transitions separating the scale space into 3 domains.

The symmetry breaking is also an experimental fact. The scale symmetry is indeed broken at small scales by the mass of elementary particles, i.e., by the emergence of their de Broglie length:

$$\lambda_{dB} = \hbar/mv, \tag{15}$$

and at large length-scales by the emergence of static structures (galaxies, groups, cluster cores) of typical sizes

$$\lambda_g \approx \frac{1}{3} \frac{Gm}{< v^2 >}, \tag{16}$$

beyond which the general scale dependence shows itself in particular by the expansion of the Universe but also by the fractal-like observed distribution of structures in the Universe. The effect of these two symmetry breakings is to separate the scale space into three domains (see Figs. 2 and 5), a microphysical quantum domain (scale-dependent), a classical domain (scale-independent), and a "cosmological" domain (scale-dependent again). Remark that the existence of the classical, scale-independent domain does not disprove the universality of the principle of scale relativity, since this intermediate domain actually plays for scale laws the same role as *statics* plays for motion laws: namely, it corresponds to a degeneration of the scale laws. It is easy to include the symmetry breaking in our description, by accounting for the fact that the origin of a fractal coordinate is arbitrary. As we shall see in more detail in what follows, the spontaneous symmetry breaking is the result of translation invariance, i.e., of the coexistence of scale laws and of motion/displacement laws (see Sect. 3.1.1 and Fig. 3).
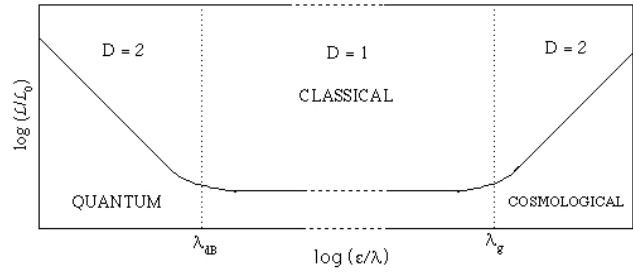


**Fig. 2.** Typical behavior of the solutions to the simplest linear scale differential equation (see Sect. 3.1.1). One obtains an asymptotic fractal (power-law resolution-dependent) behavior at either large or small scales, and a transition to scale-independence toward the classical domain (intermediate scales). The transitions are given by the Compton-de Broglie scale in the microscopic case and by the typical static radius of objects (galaxy radii, cluster cores) in the macroscopic case. Note that the microscopic and macroscopic plots actually correspond to two different kinds of experiments: in the microscopic case, the "window" $\lambda$ is kept constant while the "resolution" $\epsilon$ is changed, leading to an increase toward small scales, $\mathscr{L} = \mathscr{L}_0(\lambda/\epsilon)^\delta$; in the macroscopic case, the fractal behavior shows itself by increasing the window $\epsilon$ for a fixed resolution $\lambda$, this leading to an increase toward large scales, $\mathscr{L} = \mathscr{L}_0(\epsilon/\lambda)^\delta$.

Simply replacing the fractal coordinate $\mathscr{L}$ by $\mathscr{L} - \mathscr{L}_0$ in the pure scale-invariant law $\mathscr{L} = \mathscr{L}_0(\lambda/\epsilon)^\delta$, we recover the broken law

$$\mathscr{L} = \mathscr{L}_0 \left[1 + \left(\frac{\lambda}{\epsilon}\right)^\delta\right], \tag{17}$$

which becomes scale-independent for $\epsilon \gg \lambda$ when $\delta > 0$, and for $\epsilon \ll \lambda$ when $\delta > 0$. We then expect the three domains and the two transitions of Figs. 2 and 5 (see Sect. 3 for more detail). Note that in these figures, the scale dimension is an *effective* scale dimension $\delta_{\text{eff}}$ which includes the transition in its definition, i.e. $(\lambda/\epsilon)^{\delta_{\text{eff}}} = 1 + (\lambda/\epsilon)^\delta$.

## 3. Theoretical framework

### 3.1. Description of a non-differentiable and fractal space-time

Giving up differentiability of the space-time coordinates has three main consequences: (i) the explicit scale-dependence of physical quantities on space-time resolutions, that implies the construction of new fundamental laws of scale; (ii) the multiplication to infinity of the number of geodesics, that suggests jumping to a statistical and probabilistic description; (iii) the breaking of the time symmetry ($dt \leftrightarrow -dt$) at the level of the space-time geometry, that implies a "two-valuedness" of velocities which we represent in terms of a complex and non-classical new physics. The aim of the present section is to explain in more detail how the giving up of differentiability leads us to introduce such new structures.

### 3.1.1. New scale laws

Strictly, the nondifferentiability of the coordinates means that the velocity $V = dX/dt$ is no longer defined. However, as recalled in the introduction, the combination of continuity and nondifferentiability implies an explicit scale-dependence of the various physical quantities (Nottale 1993a, 1994a). Therefore the basis of our method consists in replacing the classical velocity by a function that depends explicitly on resolution, $V = V(\epsilon)$. Only $V(0)$ is now undefined, while $V(\epsilon)$ is now defined for any non-zero $\epsilon$. Consider indeed the usual expression for the velocity:

$$\frac{dX}{dt} = \lim_{dt \to 0} \frac{X(t + dt) - X(t)}{dt}. \tag{18}$$

In the nondifferentiable case, the limit is undefined. This means that, when $dt$ tends to zero, either the ratio $dX/dt$ tends to infinity, or it fluctuates without reaching any limit. The solution proposed in scale relativity to this problem is very simple. We replace the differential $dt$ by a scale variable $\delta t$, and we consider now $\delta X/\delta t$ as an explicit function of this variable:

$$V(t, \delta t) = \frac{X(t + \delta t) - X(t)}{\delta t} \bmod \mathscr{R}. \tag{19}$$

Here $V(t, \delta t)$ is a "fractal function" (see Nottale, 1993a, chap. 3.8). It is defined modulo some equivalence relation $\mathscr{R}$ which expresses that the variable $\delta t$ has the physical meaning of a resolution: $f(t, \delta t) = g(t, \delta t) \bmod \mathscr{R} \Leftrightarrow \forall \delta t, \forall t, |f(t, \delta t) - g(t, \delta t)| \leq \epsilon(\delta t)$, where $\epsilon(\delta t)$ is the resolution in $f$ and $g$ which corresponds to the resolution $\delta t$ in $t$ (e.g., in the case of a constant fractal dimension $D$, $\epsilon \approx \delta t^{1/D}$. This means that we no longer work with the limit $\delta t \to 0$, which is anyway devoid of observable physical meaning (since an infinite energy-momentum would be needed to reach it, according to quantum mechanics), and that we replace this limit by a description of the various structures which appear during the zoom process toward the smaller scales. Our tool can be thought of as the theoretical equivalent of what are wavelets in fractal and multifractal data analysis (see e.g. Arneodo et al. 1988, Argoul et al. 1989, Farge et al. 1996).

The advantage of our method is now that, for a given value of the resolution $\delta t$, differentiability in $t$ is recovered. The non-differentiability of a fractal function $f(t, \delta t)$ means that $\partial f(t, 0)/\partial t$ does not exist. But $\partial f(t, \delta t)/\partial t$ exists for any given value of the resolution $\delta t$, which allows us to recover a differential calculus even when dealing with non-differentiability. However, one should be cautious about the fact that the physical description and the mathematical description are no longer always coincident. Indeed, once $\delta t$ given, one can write a *mathematical* differential equation involving terms like $\partial f(t, \delta t)/\partial t$. In such an equation, one can make $\partial t \to 0$ and then use the standard mathematical methods to solve for it and determine $f(t, \delta t)$. But it must be understood that this is a purely mathematical intermediate description with no physical counterpart, since for the real system under consideration, the very consideration of an interval $dt < \delta t$ changes the function $f$ (such a

behavior is experimentally well-known in quantum systems). As a consequence of this analysis, there is a particular subspace of description where the physics and the mathematics coincide, namely, when making the particular choice $dt = \delta t$. We shall work in what follows with such an identification of the time differential and of the new time resolution variable.

A consequence of the new description is that the current equations of physics are now incomplete, since they do not describe the variation of the various physical quantities in scale transformations $\delta t \to \delta t'$. The scale-dependence of the velocity suggests that we complete the standard equations of physics by new differential equations of scale.

In order to work out such a completion, let us apply to the velocity and to the differential element (now interpreted as a resolution) the reasoning already touched upon in Sect. 2. The simplest possible equation that one can write for the variation of the velocity $V(t, dt)$ in terms of the new scale variable $dt$ is:

$$\frac{\partial V}{\partial \ln dt} = \beta(V), \tag{20}$$

i.e., a first order, renormalization-group-like differential equation, written in terms of the dilatation operator

$$\tilde{D} = \partial/\partial(\ln dt), \tag{21}$$

in which the infinitesimal scale-dependence of $V$ is determined by the "field" $V$ itself. The $\beta$-function here is a priori unknown, but we can use the fact that $V < 1$ (in motion-relativistic units) to expand it in terms of a Taylor expansion. We obtain:

$$\frac{\partial V}{\partial \ln dt} = a + bV + O(V^2), \tag{22}$$

where $a$ and $b$ are "constants" (independent of $dt$ but possibly dependent on space-time coordinates). Setting $b = (1/D) - 1$, we obtain the solution of this equation under the form:

$$V = v + \mathscr{W} = v \left[ 1 + \zeta \left( \frac{\tau}{dt} \right)^{1 - 1/D} \right], \tag{23}$$

where $v$ is a mean velocity and $\mathscr{W}$ a fractal fluctuation that is explicitly scale-dependent, and where $\tau$ and $\zeta$ are chosen such that $< \zeta >= 0$ and $< \zeta^2 >= 1$.

We recognize here the combination of a typical fractal behavior with fractal dimension $D$, and of a breaking of the scale symmetry at scale $\tau$, that plays the role of an upper fractal / nonfractal scale transition (since $V \approx v$ when $dt \gg \tau$ and $V \approx \mathscr{W}$ when $dt \ll \tau$). As announced in Sect. 2, the symmetry breaking is not added artificially here to the scale laws, but is obtained as a natural consequence of the scale-relativistic approach, in terms of solutions to Eq. (22). It is now clear from Eq. (23) (see also Fig. 3) that the symmetry breaking comes from a confrontation of the motion behavior (as described by the $v$ component of $V$) with the scale behavior (as described by the $\mathscr{W}$ component of $V$). Their relative sizes determine the scale of the transition (Fig. 3).

Concerning the value of the fractal dimension, recall that $D = 2$ plays the role of a critical dimension in the whole theory,
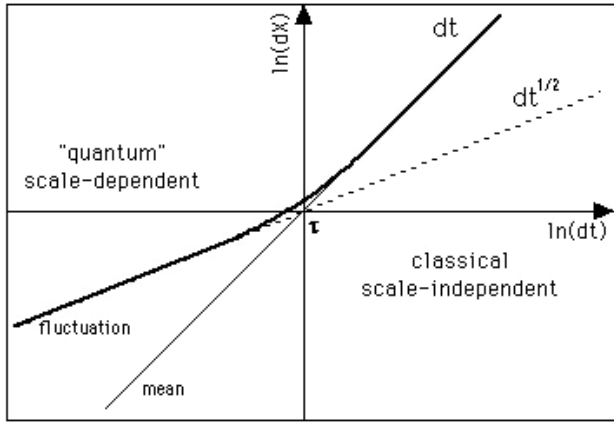
**Fig. 3.** The quantum-microscopic to classical transition is understood in the scale-relativistic approach as a spontaneous symmetry breaking: the "classical" term $dx = v\,dt$ becomes dominant beyond some upper scale while the "fractal" term $d\xi \propto dt^{1/2}$ (here of critical fractal dimension 2) is dominant toward the small scale in absolute value, though it vanishes in the mean. Note that in the cosmological-macroscopic case which is the subject of the present paper, there is an additional transition to classical laws toward the small scales, while the upper classical domain is sent to infinity (since $\tau = \tau_0 / < v >^2$, with $< v >= 0$).

(see Nottale 1993a and refs. therein, Nottale 1995a). In this case we find in the asymptotic scaling domain that $\mathcal{W} \propto (dt/\tau)^{-1/2}$, in agreement with Feynman and Hibbs (1965).

Let us finally write the expression for the elementary displacement derived from the above value of the velocity. We shall now consider the two inverse cases identified in Sect. 2.5, i.e. not only the case where the scaling domain is at small scales (standard quantum mechanics) but also the case where it lies at large scales, which is the relevant situation in the present paper. In both cases, the elementary displacement $dX$ in the scaling domain can be written under the sum of two terms,

$$dX = dx + d\xi, \tag{24}$$

with

$$dx = v\,dt, \tag{25}$$

$$d\xi = \zeta\,\tau_0^{1-(1/D)} dt^{1/D}, \tag{26}$$

where $\tau_0$ is a constant. The comparison with Eq. (23) allows to show that the transition scale is therefore $\tau = \tau_0/v^2$ (Nottale 1994a) when $D = 2$. In the scaling regime ($dt < \tau$) both terms are relevant, since $d\xi$ vanishes in the mean, i.e. $< d\xi >= 0$, but $< d\xi^2 >^{1/2} \gg dx$ (left of Fig. 3): we shall see in what follows that the fluctuation, in spite of its vanishing in the mean, plays nevertheless an essential role in the laws of average motion. When applied to atomic and elementary particle physics (microscopic case), we find that the fluctuation becomes dominated at larger scales ($dt > \tau$) by the classical term $dX = dx$, and the system becomes classical beyond the de Broglie scale $\tau$ (since $\tau_0$ is the Compton scale in this case). When applied to the
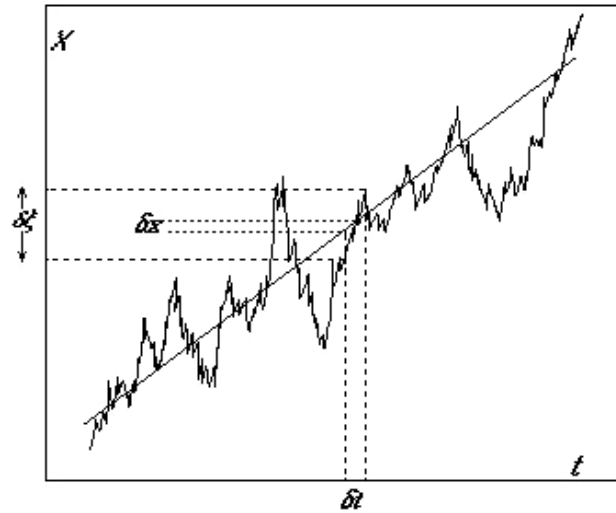


**Fig. 4.** Relation between differential elements on a fractal function. While the average, "classical" variation $\delta x =< \delta X >$ is of the same order as the abscissa differential $\delta t$, the fluctuation is far larger and depends on the fractal dimension $D$ as: $d\xi \propto \delta t^{1/D}$.

macroscopic case, the situation is different, since: (i) there is a new transition to classical behavior below some smaller scale $\tau'$, in accordance with the solutions of Eq. (12); (ii) the upper transition scale $\tau$ is expected to be pushed to infinity, since the theory will be preferentiably applied to bound systems such that the classical average velocity $< v >= 0$ (hydrogen atom-like systems), while it will a priori be irrelevant for free systems.

Equations (24-25-26) can be used to recover a fundamental, well-known formula relating the space-resolution and the time-resolution in the asymptotic domain $\delta t \ll \tau$ on a fractal curve (see Fig. 4)

$$(\delta\xi/\lambda)^D = (\delta t/\tau), \tag{27}$$

in which the length scale $\lambda$ and the time-scale $\tau$ are naturally introduced for dimensional reasons.

In the present series of paper, only the above simplest scale-laws with fractal dimension $D = 2$ will be developed. However, as recalled in Sect. 2 and Appendix B, these laws can be identified with "Galilean" approximations of more general scale-relativistic laws in which the fractal dimension becomes itself variable with scale (Nottale 1992, 1993a, 1995d). Such special scale-relativistic laws are expected to apply toward the very small and very large scales (see Appendix B).

### 3.1.2. Infinity of geodesics

The above description applies to an individual fractal trajectory. However, we are not interested here in the description of fractal trajectories in a space that would remain Euclidean or Riemannian, but in the description of a *fractal space* and of its geodesics. The trajectories are then fractal as a consequence of the fractality of space itself. This problem is analogous to the

jump from flat to curved space-time in Einstein's general relativity. One can work in a curvilinear coordinate system in flat space-time, and this introduces a GR-like metric element, but this apparent new structure is trivial and can be cancelled by coming back to a Cartesian coordinate system; on the contrary the curvature of space-time itself implies structures (described e.g. by the curvature invariants) that are new and irreducible to the flat case, since no coordinate system can be found where they would be cancelled (except locally). The same is true when jumping, as we attempt here, from a differentiable (Riemannian) manifold to a nondifferentiable (non Riemannian) manifold. We expect the appearance of new structures that would be also new and irreducible to the old theory. Two of these new geometric properties will be now described (but it is clear that this is only a minimal description, and that several other features will have to be introduced for a general description of nondifferentiable spacetimes).

One of the geometric consequences that is specific of the nondifferentiability and of the subsequent fractal character of space itself (not only of the trajectories), is that there will be an infinity of fractal geodesics that relate any couple of points in a fractal space (Nottale 1989, 1993a). The above description of an individual fractal trajectory is thus insufficient to account for the properties of motion in a fractal space. This is an important point, since, as recalled in the introduction, our aim here is to recover a physical description of motion and scale laws, even in the microscopic case, by using only the geometric concepts and methods of general relativity (once generalized, using new tools, to the nondifferentiable case). These basic concepts are the geometry of space-time and its geodesics, so that we have suggested (Nottale 1989) that the description of a quantum mechanical particle (including its property of wave-corpuscle duality) could be reduced to the various geometric properties of the ensemble of fractal geodesics of the fractal space-time that correspond to a given state of this "particle" (defined here as a geometric property of a subset of all geodesics). In such an interpretation, we do not have to endow the "particle" with internal properties such as mass, spin or charge, since the "particle" is identified with the geodesics themselves (not with a point mass which would follow them), and since these "internal" properties can be defined as geometric properties of the fractal geodesics themselves. As a consequence, any measurement is interpreted as a sorting out of the geodesics, namely, after a measurement, only the subset of geodesics which share the geometrical property corresponding to the measurement result is remaining (for example, if the "particle" has been observed at a given position with a given resolution, this means that the geodesics which pass through this domain have been selected).

This new interpretation of what are "particles" ensures the validity of the Born axiom and of the Von Neumann axiom (reduction of wave function) of quantum mechanics. This is confirmed by recent numerical simulations by Hermann (1997), that have indeed shown that one can obtain solutions to the Schrödinger equation without using it, directly from the elementary process introduced in scale relativity. Moreover, a many-particle simulation of quantum mechanics has been performed

by Ord (1996a,b) in the fractal space-time framework. He finds, in agreement with our own results, that the Schrödinger equation may describe ensembles of classical particles moving on fractal random walk trajectories, so that it has a straighforward microscopic model which is not, however, appropriate for standard quantum mechanics.

This point is also a key to understanding the differences between the microscopic and macroscopic descriptions, which implies a fundamental difference of interpretation of the final quantum-like equations and of their solutions. The two main differences are the following:

(i) In microphysics, we identify the particle to the geodesics themselves, while in macrophysics there is a macroscopic object that follows the geodesic. Elementary particles thus become a purely geometric and extended concept. This allows to recover quantum mechanical properties like indiscernability, identity and non locality in the microphysical domain, but not in the macrophysical one.

(ii) In microphysics we assume that non-differentiability is unbroken toward the smaller scales, i.e. that there is no underlying classical theory, or in other words that the quantum theory is complete (in the sense of no hidden parameter), so that the Bell inequalities can be violated. On the contrary, we know by construction that our quantum-like macroscopic theory is subjected to a kind of "phase transition" that transforms it to a classical theory at smaller scales. Non-differentiability is only a large scale approximation, so that our macroscopic theory is a hidden parameter theory, that is therefore not expected to violate Bell's inequalities.

The infinity of geodesics leads us to jump to a statistical description, i.e., we shall in what follows consider averages on the set of geodesics, not on an a priori defined probability density as in stochastic theories. Namely, two kinds of averaging processes are relevant in our description:

(i) Each geodesic can be smoothed out with time-resolution larger than $\tau$ (which plays the role of a fractal / nonfractal transition). At scales larger than $\tau$, the fluctuation $d\xi$ becomes far smaller than the mean $dx$, making each trajectory no longer fractal (line in Fig. 4).

(ii) One can subsequently take the average of the velocity on the infinite set of these "classical" geodesics that pass through a given point.

In what follows, the decomposition of $dX$ in terms of a mean, $< dX >= dx = v\,dt$, and a fluctuation respective to the mean, $d\xi$ (such that $< d\xi >= 0$ by definition) will be made using both averaging processes. Since all geodesics are assumed to share the same statistical fractal geometric properties, the form of Eqs. (25-27), is conserved. We stress once again the fact that the various expectations are taken in our theory on the set of geodesics, not on a previously given probability density. The probability density will be introduced as the density of the fluid of geodesics, ensuring by construction the Born interpretation of the theory.

We also recall again that, in the particular domain of application with which we are concerned in the present series of papers (macroscopic large scale systems), two particular features are

relevant: (i) a lower transition scale must be introduced, as recalled above and as predicted from Eq. (12); (ii) the average classical velocity must be zero, implying an infinite upper fractal / nonfractal transition (see the discussion in Sect. 5). Remark, however, (see Sects. 3.1.3 and 3.2) that we will be led to introduce two average velocities, a forward one $v_+$ and a backward one $v_-$, in terms of which the classical average velocity writes $V = (v_+ + v_-)/2$. Therefore its vanishing does not mean the vanishing of $v_+$ and $v_-$ individually.

### 3.1.3. Differential time symmetry breaking

The nondifferentiable nature of space-time implies an even more dramatic consequence, namely, a breaking of *local* time reflection invariance. Remark that such a discrete symmetry breaking can *not* be derived from only the fractal or nondifferentiable nature of *trajectories*, since it is a consequence of the irreducible nondifferentiable nature of *space-time* itself.

Consider indeed again the definition of the derivative of a given function with respect to time:

$$\frac{df}{dt} = \lim_{dt \to 0} \frac{f(t+dt) - f(t)}{dt} = \lim_{dt \to 0} \frac{f(t) - f(t-dt)}{dt}. \qquad (28)$$

The two definitions are equivalent in the differentiable case. One passes from one to the other by the transformation $dt \to -dt$ (local time reflection invariance), which is therefore an implicit discrete symmetry of differentiable physics. In the nondifferentiable situation considered here, both definitions fail, since the limits are no longer defined. The scale-relativistic method solves this problem in the following way.

We have attributed to the differential element $dt$ the new meaning of a variable, identified with a time-resolution, $dt = \delta t$ as recalled hereabove ("substitution principle"). The passage to the limit is now devoid of physical meaning (since quantum mechanics itself tells us that an infinite momentum and an infinite energy would be necessary to make explicit measurements at zero resolution interval). In our new framework, the physics of the problem is contained in the behavior of the function during the "zoom" operation on $\delta t$. The two functions $f'_+$ and $f'_-$ are now defined as explicit functions of $t$ and of $dt$:

$$f'_+(t, dt) = \frac{f(t+dt) - f(t)}{dt}, \qquad (29)$$

$$f'_-(t, dt) = \frac{f(t) - f(t-dt)}{dt}. \qquad (30)$$

When applied to the space variable, we get for each geodesic two velocities that are fractal functions of resolution, $V_+[x(t), t, dt]$ and $V_-[x(t), t, dt]$. In order to go back to the classical domain, we first smooth out each geodesic with balls of radius larger than $\tau$: this defines two classical velocity fields now independent of resolution, $V_+[x(t), t, dt > \tau] = V_+[x(t), t]$ and $V_-[x(t), t, dt > \tau] = V_-[x(t), t]$; then we take the average on the whole set of geodesics. We get two mean velocities $v_+[x(t), t]$ and $v_-[x(t), t]$, but after this double averaging process, there is no reason for these two velocities to be equal, contrarily to what happens in the classical, differentiable case.

In summary, while the concept of velocity was classically a one-valued concept, we must introduce, if space-time is nondifferentiable, two velocities instead of one even when going back to the classical domain. Such a two-valuedness of the velocity vector is a new, specific consequence of nondifferentiability that has no classical counterpart (in the sense of differential physics), since it finds its origin in a breaking of the discrete symmetry $(dt \to -dt)$. This symmetry was considered self-evident up to now in physics, so that it has not been analysed on the same footing as the other well-known symmetries. It is actually independent from the time reflection symmetry T, even though it is clear that the breaking of this "dt symmetry" implies a breaking of the T symmetry at this level of the description.

Now we have no way to favor $v_+$ rather than $v_-$. Both choices are equally qualified for the description of the laws of nature. The only solution to this problem is to consider both the forward $(dt > 0)$ and backward $(dt < 0)$ processes together. The number of degrees of freedom is doubled with respect to the classical, differentiable description (6 velocity components instead of 3).

A simple and natural way to account for this doubling of the needed information consists in using complex numbers and the complex product. As we shall recall hereafter, this is the origin of the complex nature of the wave function in quantum mechanics, since the wave function can be identified with the exponential of the complex action that is naturally introduced in such a theory. One can indeed demonstrate (Nottale 1997) that the choice of complex numbers to represent the two-valuedness of the velocity is not an arbitrary choice, since it achieves a covariant description of the new mechanics: namely, it ensures the Euler-Lagrange equations to keep their classical form and allows one not to introduce additional terms in the Schrödinger equation. Note also that the new complex process, *as a whole*, recovers the fundamental property of microscopic reversibility.

### 3.2. Scale-covariant derivative

Finally, we can describe (in the scaling domain) the elementary displacement $dX$ for both processes as the sum of a mean, $< dx_\pm >= v_\pm \, dt$, and a fluctuation about this mean, $d\xi_\pm$ which is then by definition of zero average, $< d\xi_\pm >= 0$, i.e.:

$$dX_+(t) = v_+ \, dt + d\xi_+(t); \; dX_-(t) = v_- \, dt + d\xi_-(t). \qquad (31)$$

Consider first the average displacements. The fundamental irreversibility of the description is now apparent in the fact that the average backward and forward velocities are in general different. So mean forward and backward derivatives, $d_+/dt$ and $d_-/dt$ are defined. Once applied to the position vector $x$, they yield the *forward and backward mean velocities*, $\frac{d_+}{dt} x(t) = v_+$ and $\frac{d_-}{dt} x(t) = v_-$.

Concerning the fluctuations, the generalization of the fractal behavior (Eq. 26) to three dimensions writes

$$< d\xi_{\pm i} \, d\xi_{\pm j} >= \pm 2 \mathscr{D} \, \delta_{ij} \, dt, \qquad (32)$$

$\mathscr{D}$ standing for a fundamental parameter that characterizes the new scale law at this simple level of description (see Sect. 4.6

for a first generalization). The $d\xi(t)$'s are of mean zero and mutually independent. If one assumes them to be also Gaussian, our process becomes a standard Wiener process. But such an assumption is not necessary in our theory, since only the property (Eq. (32)) will be used in the calculations.

Our main tool now consists of recovering local time reversibility in terms of a new *complex* process (Nottale 1993a): we combine the forward and backward derivatives in terms of a complex derivative operator

$$\frac{\acute{d}}{dt} = \frac{(d_+ + d_-) - i(d_+ - d_-)}{2\,dt},\tag{33}$$

which, when applied to the position vector, yields a complex velocity

$$\mathscr{V} = \frac{\acute{d}}{dt}x(t) = V - iU = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}.\tag{34}$$

The real part $V$ of the complex velocity $\mathscr{V}$ generalizes the classical velocity, while its imaginary part, $U$, is a new quantity arising from non-differentiability (since at the classical limit, $v_+ = v_-$, so that $U = 0$).

Equation (32) now allows us to get a general expression for the complex time derivative $\acute{d}/dt$. Consider a function $f(x(t),t)$. Contrarily to what happens in the differentiable case, its total derivative with respect to time contains finite terms up to higher order (Einstein 1905). In the special case of fractal dimension 2, only the second order intervenes. Indeed its total differential writes

$$df = \frac{\partial f}{\partial t}dt + \nabla f.dX + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}dX_i dX_j.\tag{35}$$

Classically the term $dX_i dX_j/dt$ is infinitesimal, but here its average reduces to $< d\xi_i\ d\xi_j > /dt$, so that the last term of Eq. (35) will amount to a Laplacian thanks to Eq. (32). Then

$$d_\pm f/dt = (\partial/\partial t + v_\pm.\nabla \pm \mathscr{D}\Delta)f.\tag{36}$$

By inserting these expressions in Eq. (33), we finally obtain the expression for the complex time derivative operator (Nottale 1993a):

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathscr{V}.\nabla - i\mathscr{D}\Delta.\tag{37}$$

The passage from classical (differentiable) mechanics to the new nondifferentiable mechanics can now be implemented by a unique prescription: Replace the standard time derivative $d/dt$ by the new complex operator $\acute{d}/dt$. In other words, this means that $\acute{d}/dt$ plays the role of a *scale-covariant derivative* (in analogy with Einstein's general relativity where the basic tool consists of replacing $\partial_j A^k$ by the covariant derivative $D_j A^k = \partial_j A^k + \Gamma^k_{jl} A^l$).

## 3.3. Scale-covariant mechanics

Let us now give the main steps by which one may generalize classical mechanics using this scale-covariance. We assume that any mechanical system can be characterized by a Lagrange function $\bar{\mathscr{L}}(x, \mathscr{V}, t)$, from which an action $\mathscr{S}$ is defined:

$$\mathscr{S} = \int_{t_1}^{t_2} \bar{\mathscr{L}}(x, \mathscr{V}, t)dt.\tag{38}$$

Our Lagrange function and action are a priori complex and are obtained from the classical Lagrange function $L(x, dx/dt, t)$ and classical action $S$ precisely from applying the above prescription $d/dt \rightarrow \acute{d}/dt$. The action principle (which is no longer a "least-action principle", since we are now in a complex plane, but remains a "stationary-action principle"), applied on this new action with both ends of the above integral fixed, leads to generalized Euler-Lagrange equations (Nottale 1993a)

$$\frac{\acute{d}}{dt}\frac{\partial \bar{\mathscr{L}}}{\partial \mathscr{V}_i} = \frac{\partial \bar{\mathscr{L}}}{\partial x_i},\tag{39}$$

which are exactly the equations one would have obtained from applying the scale-covariant derivative ($d/dt \rightarrow \acute{d}/dt$) to the classical Euler-Lagrange equations themselves: this result demonstrates the self-consistency of the approach and vindicates the use of complex numbers. Other fundamental results of classical mechanics are also generalized in the same way. In particular, assuming homogeneity of space in the mean leads to defining a generalized *complex* momentum and a complex energy given by

$$\mathscr{P} = \frac{\partial \bar{\mathscr{L}}}{\partial \mathscr{V}},\ \ \mathscr{E} = \mathscr{P}\mathscr{V} - \bar{\mathscr{L}}.\tag{40}$$

If one now considers the action as a functional of the upper limit of integration in Eq. (38), the variation of the action from a trajectory to another nearby trajectory, when combined with Eq. (39), yields a generalization of other well -known relations of classical mechanics:

$$\mathscr{P} = \nabla\mathscr{S},\ \ \mathscr{E} = -\partial\mathscr{S}/\partial t.\tag{41}$$

We shall now apply the scale-relativistic approach to various domains of physics which are particularly relevant to astrophysical problems.

## 4. Scale-covariant equations of physics

### 4.1. Generalized Newton-Schrödinger equation: particle in scalar field

#### 4.1.1. Lagrangian approach

Let us now specialize our study, and consider Newtonian mechanics, i.e., the general case when the structuring field is a scalar field. The Lagrange function of a closed system, $L = \frac{1}{2}mv^2 - \Phi$, is generalized as $\bar{\mathscr{L}}(x, \mathscr{V}, t) = \frac{1}{2}m\mathscr{V}^2 - \Phi$, where

$\Phi$ denotes a scalar potential. The Euler-Lagrange equations keep the form of Newton's fundamental equation of dynamics

$$m\frac{d}{dt}\mathscr{V} = -\nabla\Phi, \tag{42}$$

which is now written in terms of complex variables and complex operators. In the case when there is no external field, the scale-covariance is explicit, since Eq. (42) takes the form of the equation of inertial motion, $d\mathscr{V}/dt = 0$. The complex momentum $\mathscr{P}$ now reads:

$$\mathscr{P} = m\mathscr{V}, \tag{43}$$

so that from Eq. (41) we arrive at the conclusion that, in this case, the *complex velocity $\mathscr{V}$ is a gradient*, namely the gradient of the complex action:

$$\mathscr{V} = \nabla\mathscr{S}/m. \tag{44}$$

We may now introduce a complex wave function $\psi$ which *is nothing but another expression for the complex action $\mathscr{S}$*,

$$\psi = e^{i\mathscr{S}/2m\mathscr{D}}. \tag{45}$$

It is related to the complex velocity as follows:

$$\mathscr{V} = -2i\mathscr{D}\nabla(\ln\psi). \tag{46}$$

From this equation and Eq. (43), we obtain:

$$\mathscr{P}\psi = -2im\mathscr{D}\nabla\psi, \quad \mathscr{E}\psi = 2im\mathscr{D}\partial\psi/\partial t, \tag{47}$$

which is the *correspondence principle* of quantum mechanics for momentum and energy, but here demonstrated and written in terms of exact equations. We have now at our disposal all the mathematical tools needed to write the fundamental equation of dynamics (Eq. (42)) in terms of the new quantity $\psi$. It takes the form

$$2i\mathscr{D}m\frac{d}{dt}(\nabla\ln\psi) = \nabla\Phi. \tag{48}$$

Standard calculations with differential operators (Nottale 1993a) transform this expression to:

$$\frac{d}{dt}\mathscr{V} = -2\mathscr{D}\nabla\{i\frac{\partial}{\partial t}\ln\psi + \mathscr{D}\frac{\Delta\psi}{\psi}\} = -\nabla\Phi/m. \tag{49}$$

Integrating this equation finally yields

$$\mathscr{D}^2\Delta\psi + i\mathscr{D}\frac{\partial}{\partial t}\psi - \frac{\Phi}{2m}\psi = 0, \tag{50}$$

up to an arbitrary phase factor $\alpha(t)$ which may be set to zero by a suitable choice of the phase of $\psi$. In the very particular case when $\mathscr{D}$ is inversely proportional to mass, $\mathscr{D} = \hbar/2m$, we recover the standard form of Schrödinger's equation:

$$\frac{\hbar^2}{2m}\Delta\psi + i\hbar\frac{\partial}{\partial t}\psi = \Phi\psi, \tag{51}$$

and this theory (assuming *complete* nondifferentiability) yields quantum mechanics (Nottale 1993a).

It is remarkable that, in this approach, we have obtained the Schrödinger equation *without introducing a probability density* (since expectations are taken on the beam of virtual geodesics) and without writing any Kolmogorov nor Fokker-Planck equation. In this regard our theory differs profoundly from Nelson's (1966, 1984) stochastic mechanics, in which one works with a *real* Newton equation and with real backward and forward Fokker-Planck equations; these equations are combined to yield two real equations, which are finally identified with the real part and the imaginary part of the complex Schrödinger equation. In our theory, we use only one complex equation of dynamics from the beginning of our calculation; as a consequence, the real and imaginary parts of our Schrödinger equation is not a pasting of two real equations, but instead involve combinations of terms through the complex product, so that obtaining in this way a Schrödinger equation was not a priori evident.

The statistical meaning of the wave function (Born postulate) can now be deduced from the very construction of the theory. Even in the case of only one particle the virtual family of geodesics is infinite (this remains true even in the zero particle case, i.e. for the vacuum field). The particle is one random geodesic of the family, and its probability to be found at a given position must be proportional to the density of the fluid of geodesics. This density can now be easily calculated from our variables, since the imaginary part of Eq. (50) writes:

$$\partial(\psi\psi^\dagger)/\partial t + \text{div}(\psi\psi^\dagger V) = 0, \tag{52}$$

where $V$ is the real part of the complex velocity, and has already been identified with the classical velocity (at the classical limit). This equation is recognized as the equation of continuity, implying that $\rho = \psi\psi^\dagger$ represents the fluid density which is proportional to the density of probability, and then ensuring the validity of Born's postulate. The remarkable new feature here that allows us to obtain such a result is that the equation of continuity is not written as an additional a priori equation, but is now a part of our generalized equation of dynamics.

### 4.1.2. Fractal potential and Energy equation

Let us reexpress the effect of the fractal fluctuation in terms of an effective "force". We shall separate the two effects of nondifferentiability, namely, *doubling of time derivative* expressed in terms of complex numbers, and *fractalization*, expressed by the occurence of nonclassical second order terms in the total time derivative, then treat them in a different way.

We are led in the following calculation by the well-known way allowing to recover a Newtonian, force-like interpretation of the equation of geodesics in Einstein's general relativity theory. Start with the covariant form of the geodesics equations, $D^2x^\mu/ds^2 = 0$, develop the covariant derivative and obtain $d^2x^\mu/ds^2 + \Gamma^\mu_{\nu\rho}(dx^\nu/ds)(dx^\rho/ds) = 0$, which generalizes Newton's equation, $m\ d^2x^i/dt^2 = F^i$ in terms of a "force" $-m\Gamma^\mu_{\nu\rho}(dx^\nu/ds)(dx^\rho/ds)$.

Once complex numbers are introduced ($V \rightarrow \mathscr{V}$), we write the time derivative as a partially covariant derivative:

$$\frac{\tilde{d}}{dt} = \frac{\partial}{\partial t} + \mathscr{V}.\nabla. \tag{53}$$

The equation of a free particle still takes the form of Newton's equation of dynamics, but including now a right-hand member:

$$\frac{\tilde{d}}{dt}\mathscr{V} = i\mathscr{D}\Delta\mathscr{V}. \tag{54}$$

This right-hand member can be identified with a complex "fractal force" divided by $m$, so that:

$$\mathscr{F} = im\mathscr{D}\Delta\mathscr{V}. \tag{55}$$

In our scale-relativistic, fractal-space-time approach, this "force" is assumed to come from the very structure of space-time. When applied to the microphysical domain, we can require it to be universal, independent of the mass of the particle. Then $2m\mathscr{D}$ must be a universal constant:

$$2m\mathscr{D} = \hbar. \tag{56}$$

This result provides us with a new definition of $\hbar$, and implies that $\lambda = 2\mathscr{D}/c$ must be the Compton length of the particle:

$$\lambda = \frac{\hbar}{mc}. \tag{57}$$

Once the Compton length obtained, it is easy to get the de Broglie length, that arises from it through a Lorentz transform (see Nottale 1994 for more detail).

The force (Eq. (55)) derives from a complex "fractal potential":

$$\phi_F = -i\,2m\mathscr{D}\,\mathrm{div}\mathscr{V} = -2m\mathscr{D}^2\Delta\ln\psi. \tag{58}$$

The introduction of this potential allows us to derive the Schrödinger equation in a very fast way, by the Hamilton-Jacobi approach (see Pissondes 1996 for a more detailed development of this approach in the scale-relativistic framework). Such a derivation explains the standard quantum mechanical "derivation" via the correspondence principle. We simply write the expression for the total energy, including the fractal potential plus a possible external potential $\Phi$,

$$\mathscr{E} = \frac{\mathscr{P}^2}{2m} + \phi_F + \Phi, \tag{59}$$

then we replace $\mathscr{E}, \mathscr{P}$ and $\phi_F$ by their expressions (47) and (58). This yields (with $2m\mathscr{D} = \hbar$)

$$i\hbar\frac{\partial}{\partial t}\ln\psi = \frac{(-i\hbar\nabla\ln\psi)^2}{2m} - \frac{\hbar^2}{2m}\Delta\ln\psi + \Phi, \tag{60}$$

which is nothing but the standard Schrödinger equation, now obtained in a direct way rather than integrated from the Lagrange equation, i.e.

$$\frac{\hbar^2}{2m}\Delta\psi + i\hbar\frac{\partial}{\partial t}\psi - \Phi\psi = 0. \tag{61}$$

## 4.1.3. Quantization of Newtonian gravitation

A preferential domain of application of our new framework is gravitation. Indeed, gravitation is already understood, in Einstein's theory, as the various manifestations of the geometry of space-time at classical scales. Now our proposal may be summarized by the statement that space-time is not only Riemannian but becomes also fractal at very large scales. The various manifestation of the fractal geometry of space-time could therefore be attributed to new effects of gravitation (this becomes a matter of definition).

We shall give herebelow our system of equation for the motion of particles in a Newtonian gravitational field. Paper II of the present series will be devoted to the study of some of its solutions.

As a first step toward writing a general equation of structure formation by a gravitational potential, we shall consider the special case of an "external" gravitational field that can be considered as unaffected by the evolution of the structure considered. Such a situation corresponds to a structuring field that can be considered as global with respect to the structures that it will contribute to form. Typical examples of such a case are the two-body problem, i.e., test particles in the potential of a central more massive body (e.g., planetary systems, binary systems in terms of reduced mass), and cosmology (particles embedded into a background with uniform density). For this type of problem, the equations of evolution are the classical Poisson equation and the Schrödinger-Newton equation:

$$\Delta\Phi = -4\pi G\rho, \tag{62}$$

$$\mathscr{D}^2\Delta\psi + i\mathscr{D}\frac{\partial\psi}{\partial t} = \frac{\Phi}{2m}\psi. \tag{63}$$

Here the mass density $\rho$ is assumed to remain undisturbed whatever the evolution of the test-particles described in Eq. (63), so that the potential $\Phi$ can be found from Eq. (62) and inserted in Eq. (63). Solving for these equations will yield a probability density $P_\Phi = |\psi|^2(x, t)$ for the test particles subjected to the potential $\Phi$.

Since this probability density is that of all the possible positions of the test-particle, as described by the density of its virtual trajectories (of which the actual trajectory is one particular random achievement), it will be interpreted as a tendency for the system to make structures (Nottale 1996b; Nottale, Schumacher & Gay 1997). To get an understanding of its meaning, one should keep in mind that the above theory holds only at very large time scales, and that at ordinary time scales the classical theory and its predictions must still be used. Such structures may therefore be achieved (and observed) in several different ways.

(i) If there is only one test particle (for example, one planet in the Kepler potential of a star, see Nottale 1996b; Nottale, Schumacher and Gay 1997), the structure will be achieved in a statistical way. While in the standard theory all positions of a planet around a star are equiprobable, some positions, which correspond to the peaks of the probability density distribution, will now be more probable. This effect can be tested by a statistical analysis of several different systems (this can be compared

to a photon by photon Young hole experiment).

(ii) A second way by which the structures can be achieved is when there is a large ensemble of test particles. In this case we expect them to fill the "orbitals" defined by the probability amplitude, i.e. the theory is able to give a basis for morphogenesis. (This case can be compared with a classical Young hole experiment involving a large number of particles). This would be the case for planetesimals at the beginning of the formation of planetary systems, or for asteroid belts in the present epoch. (But one must care that the shape of the observed distribution is also partly determined by the "small" time scale chaos due to the effect of the other bodies, e.g. Kirkwood gaps in the asteroid belts.)

(iii) Once matter is distributed in the orbitals as described by the shape of the PDF, the standard gravitational evolution may go on through accretion and/or collapse, yielding one or several compact bodies in each of the peaks of the orbital. (For example, this allows us to explain the formation of double stars, and more generally of chain and trapeze configurations in zones of star and galaxy formation, as corresponding to the various modes of the quantum 3-dimensionnal isotropic oscillator, which is solution of our Schrödinger-like equation for constant density, see Nottale 1996a).

A detailed treatment of the gravitational case, including an analysis of the main solutions to Eqs. (62)-(63), will be the subject of paper II of this series.

A more general situation can be considered, when the gravitational potential is precisely due to the particles whose evolution is looked for. In this case, the particles can no longer be considered as test-particles. When the particles have equal mass, the mass density in the Poisson equation is proportional to the probability density given by our generalized Schrödinger equation. The equations of evolution of such a system write:

$$\mathscr{D}^2\Delta\psi + i\mathscr{D}\frac{\partial\psi}{\partial t} = \frac{\Phi}{2m}\psi, \tag{64}$$

$$\Delta\Phi = -4\pi G\rho_0|\psi|^2, \tag{65}$$

$$\mathscr{D} = \mathscr{D}(x, t, |\psi|^2, \delta x, \delta \ldots). \tag{66}$$

(See below a still more general fluid-like approach and our quantization of the Euler and Navier-Stokes equations). This is now a "looped", highly non-linear system with feedback. While in microphysical standard quantum mechanics $\mathscr{D}$ is constrained to be $\hbar/2m$ (but see Nottale 1996a for a possible special scale-relativistic generalization to high energy particles), the situation is far more complicated in the macrophysical case. To be fully general, we may also consider the case when the parameter $\mathscr{D}$ becomes a "field", itself dependent on position, time, resolution-scale (as implied by a fractal dimension different from 2, and possibly on the local value of the probability density [see Nottale 1994, 1995, for a first treatment of the case of a variable coefficient $\mathscr{D}$]. In this regard, the theory remains incomplete, since the problem of constructing the equation for this new field remains essentially open.

The field equation and the particle trajectory equation are no longer independent from each other. The gravitational potential

and the probability density are now present in both equations. They can therefore be combined in terms of a unique fourth-order equation for the probability distribution of matter in the Universe, in which the potential $\Phi$ has now disappeared:

$$2m\Delta\left[\frac{\mathscr{D}^2\Delta\psi + i\mathscr{D}\,\partial\psi/\partial t}{\psi}\right] + 4\pi G\rho_0|\psi|^2 = 0. \tag{67}$$

These various systems of equations are too much complicated to be solved in general, so that only simplified situations will be considered when looking for analytical solutions, in Paper II of this series. However, the universal properties of gravitation allows one to reach a general statement about the behavior of these equations and their solutions. The always attractive character of the gravitational potential (except when considering the contribution of a cosmological constant, see Paper II) implies that it acts as a potential well, so that the energy of systems described by Eqs. (64)-(65)-(66) will always be quantized. This equation is then expected to yield definite structures in position and velocity, which are given by the probability densities constructed from its solutions. We therefore suggest that it may stand out as a general equation for the formation and evolution of gravitational structures.

### 4.2. Particle in vectorial field

Our theory can be tentatively generalized to the case when the structuring field is vectorial, as, e.g., in the case of an electromagnetic field (Nottale 1994b, 1996a). Once again, it is easy to make classical mechanics scale-covariant. The generalized momentum and energy of a particle in a vectorial potential $A$ write:

$$\tilde{\mathscr{P}} = \mathscr{P} + qA, \ \ \tilde{\mathscr{E}} = \mathscr{E} + q\Phi, \tag{68}$$

which leads to introduce a $A$-covariant derivative (Nottale 1994b, 1996a, Nottale & Pissondes 1996; Pissondes 1996):

$$2mi\mathscr{D}\tilde{\nabla} = 2mi\mathscr{D}\nabla + qA. \tag{69}$$

The resulting equations have the form of the Schrödinger equation in presence of an electromagnetic field (of vector potential $A$ and scalar potential $\Phi$):

$$\mathscr{D}^2(\nabla - i\frac{q}{2m\mathscr{D}}A)^2\psi + i\mathscr{D}\frac{\partial}{\partial t}\psi - \frac{q\Phi}{2m}\psi = 0. \tag{70}$$

Such an equation may be relevant for a large set of still unsolved astrophysical problems where magnetic fields play an important role (see e.g. Zeldovich et al. 1983). We shall consider its application in subsequent papers of this series.

### 4.3. Particle in tensorial field: Einstein-Schrödinger geodesics equation

The application of our theory to a particle in a gravitational field plays a particular role in its development. Indeed, while Newton's description of gravitation remains in terms of field and

potential, gravitation is identified in the more profound vision of Einstein with the various manifestations of the Riemannian nature of space-time. In this case, our problem corresponds no longer to studying the effect of the fractal geometry of space-time on a particle embedded in an outer field. As recalled above, it now amounts to study the motion of a free particle in a space-time whose geometry would be both fractal (at large scale) and Riemannian (in the mean).

Let us use the general-relativistic and scale-relativistic covariances in order to write the geodesics equations in such a space-time. Einstein's covariant derivative writes:

$$\frac{D}{ds} A^\mu = \frac{d}{ds} A^\mu + \Gamma^\mu_{\nu\rho} v^\nu A^\rho. \tag{71}$$

Using this covariant derivative, Einstein's geodesics equations are written in terms of the free particle equation of motion :

$$\frac{D}{ds} v^\mu = 0 \Rightarrow \frac{d}{ds} v^\mu + \Gamma^\mu_{\nu\rho} v^\nu v^\rho = 0. \tag{72}$$

This equation can now be made scale-covariant, by replacing $d/ds$ by $\acute{d}/ds$ at all levels of the construction. We define a scale+Einstein-covariant derivative:

$$\frac{\tilde{D}}{ds} A^\mu = \frac{d}{ds} A^\mu + \Gamma^\mu_{\nu\rho} \mathscr{V}^\nu A^\rho. \tag{73}$$

The scale-covariant derivative is given in the 4-dimensional relativistic case (see Nottale 1994b, 1996a; Nottale and Pissondes 1996; Pissondes 1996) by

$$\frac{\acute{d}}{ds} = (\mathscr{V}^\mu + i\frac{\lambda}{2}\partial^\mu)\partial_\mu, \tag{74}$$

where $\mathscr{V}^\mu = \acute{d}x^\mu/ds$ is a complex four-velocity. We could then write the equation of motion of a particle in a Riemannian + fractal space-time in terms of the inertial, free particle equation of motion:

$$\frac{\tilde{D}}{ds} \mathscr{V}^\mu = 0 \Rightarrow \frac{\acute{d}}{ds} \mathscr{V}^\mu + \Gamma^\mu_{\nu\rho} \mathscr{V}^\nu \mathscr{V}^\rho = 0. \tag{75}$$

However, such an equation remains incomplete, as shown by Pissondes (1997), in agreement with Dohrn and Guerra (1977). A geodesics correction must be added to the usual parallel displacement, that leads to add to Eq. (75) a term $-i(\lambda/2)R^\mu_\rho \mathscr{V}^\rho$, now involving the Ricci tensor in the new geodesics equation. Moreover, one must be cautious with the interpretation of this equation. It is obtained by assuming that the two (Einstein and scale) covariances do not interact one on each other. This can be only a rough approximation. Indeed, in order to solve the problem of the motion in a general, non flat fractal space-time (which is nothing but the problem of finding a theory of quantum gravity in our framework), one should strictly examine the geometrical effects of curvature and fractality at the level of the construction of the covariant derivatives, not only once they are constructed. This problem reveals to be extraordinarily complicated (Nottale 1997), and will not be considered further in the present paper.

A second problem with Eq. (75) concerns the interpretation of the scale-covariant derivative in the motion-relativistic case. It is obtained by assuming that not only space but space-time is fractal, which implies that the trajectories of particles can go backward in time. This is not a problem in microphysics: on the contrary, it is even needed by the existence of virtual pairs of particle-antiparticles, through Feynman's interpretation of antiparticles as particles going backward in time. (See Ord 1983; Nottale 1989, 1993a for a development of the fractal approach to this question). It is more difficult to make a similar interpretation in the macroscopic case, so that we shall only consider the non(motion)-relativistic limit of Eq. (75) for comparison with actual data (Paper III of this series). This is nothing but the above generalized Newton's equation of dynamics,

$$\frac{d}{dt} \mathscr{V} + \nabla\Phi/m = 0, \tag{76}$$

that can be integrated in terms of the generalized Schrödinger equation (Eq. 50).

### 4.4. Euler-Schrödinger equation

Our approach can be generalized to fluid mechanics in a straightforward way. Actually we have already partly adopted a fluid description when introducing a velocity field $v = v[x(t), t]$. Applying scale-covariance, the Euler equation for a fluid in a gravitational potential $\Phi$,

$$\frac{d}{dt} v = (\partial/\partial t + v \cdot \nabla)v = -\frac{\nabla p}{\rho} - \nabla\Phi, \tag{77}$$

will be transformed into the complex equation:

$$\frac{d}{dt} \mathscr{V} = -\frac{\nabla p}{\rho} - \nabla\Phi. \tag{78}$$

In the general case $\nabla p/\rho$ is not a gradient, and we cannot transform this equation into a Schrödinger-like equation. However, in the case of an *incompressible* fluid ($\rho$ = cst), and more generally in the case of an *isentropic* fluid (including perfect fluids), $\nabla p/\rho$ is the gradient of the enthalpy by unit of mass $w$ (see, e.g., Landau & Lifchitz 1971)

$$\frac{\nabla p}{\rho} = \nabla w. \tag{79}$$

In this approximation Eq. (78) becomes the Euler-Lagrange equation constructed from the Lagrange function $\bar{\mathscr{L}}(x, \mathscr{V}, t) = \frac{1}{2}m\mathscr{V}^2 - \Phi - w$. Therefore it derives from a stationary action principle working with the complex action $\mathscr{S} = \int \bar{\mathscr{L}} dt$. Our whole previous formalism is now recovered. We introduce the probability amplitude $\psi$ (now defined for a unit mass):

$$\mathscr{S} = -2i\mathscr{D} \ln \psi. \tag{80}$$

In terms of $\psi$, $d\mathscr{V}/dt$ is a gradient:

$$\frac{d}{dt} \mathscr{V} = -2\nabla \left[ \frac{\mathscr{D}^2 \Delta\psi + i\mathscr{D} \frac{\partial}{\partial t}\psi}{\psi} \right]. \tag{81}$$

This equation can now be integrated, leading to a generalized Schrödinger-like equation:

$$\mathscr{D}^2 \Delta \psi + i\mathscr{D}\frac{\partial}{\partial t}\psi - \frac{w + \Phi}{2}\psi = 0. \tag{82}$$

### 4.5. Navier-Schrödinger equation

A similar work can be performed with the Navier-Stokes equations, at least formally. Our scale-covariant generalized Navier-Stokes equations write:

$$(\frac{\partial}{\partial t} + \mathscr{V}\cdot\nabla - i\mathscr{D}\Delta)\mathscr{V} = -\frac{\nabla p}{\rho} + \nu\Delta\mathscr{V}. \tag{83}$$

It is quite remarkable that the viscosity term in the Navier-Stokes equation plays a role similar to the coefficient $\mathscr{D}$. This suggest to us to combine them into a new complex parameter

$$\tilde{\mathscr{D}} = \mathscr{D} - i\nu. \tag{84}$$

In terms of $\tilde{\mathscr{D}}$, the complex Navier-Stokes equation recovers the form of the complex Euler equation:

$$(\frac{\partial}{\partial t} + \mathscr{V}\cdot\nabla - i\tilde{\mathscr{D}}\Delta)\mathscr{V} = -\frac{\nabla p}{\rho}. \tag{85}$$

Once again, in the incompressible or isentropic cases, this equation can be integrated to yield a Schrödinger-like equation:

$$\tilde{\mathscr{D}}^2 \Delta\psi + i\tilde{\mathscr{D}}\frac{\partial}{\partial t}\psi - \frac{w}{2}\psi = 0. \tag{86}$$

This equation is also valid in the presence of a gravitational field or in the presence of any field that is the gradient of a potential $\Phi$. It becomes:

$$\tilde{\mathscr{D}}^2\Delta\psi + i\tilde{\mathscr{D}}\frac{\partial}{\partial t}\psi - \frac{w + \Phi}{2}\psi = 0. \tag{87}$$

However, its interpretation is more difficult than in previous calculations. Indeed the complex nature of $\tilde{\mathscr{D}}$ prevents the imaginary part of this equation to be an equation of continuity. We shall no longer consider the viscous case in the present paper. We intend to study this situation in more detail in forthcoming works.

### 4.6. Motion of solids

The equation of the motion of a solid body can be given the form of Euler-Lagrange equations, and therefore comes in a very easy way under our theory. The role of the variables $(x, v, t)$ is now played by the rotational coordinates, $(\phi, \Omega, t)$ where $\phi$ denotes three rotational Euler angles and $\Omega$ is the corresponding rotational velocity. The Euler-Lagrange equation writes (Landau & Lifchitz, 1969):

$$\frac{d}{dt}\frac{\partial L}{\partial \Omega} = \frac{\partial L}{\partial \phi}, \tag{88}$$

in terms of the Lagrange function $L$ of the solid, that writes:

$$L = \frac{1}{2}\mu V^2 + \frac{1}{2}I_{ik}\Omega_i\Omega_k - U, \tag{89}$$

where $I_{ik}$ is the tensor of inertia of the body and $U$ a potential term. We use throughout this section the tensorial notation where a sum is meant on two repeated indices. The right-hand member of Eq. (88) writes:

$$\frac{\partial L}{\partial \phi} = -\frac{\partial U}{\partial \phi} = K = \sum r \times F, \tag{90}$$

which identifies with the total torque, i.e., the sum of the moments of all forces acting on the body. In the left-hand member one recognizes the angular momentum about the center of mass,

$$M = \frac{\partial L}{\partial \Omega}, \tag{91}$$

and we finally recover a rotational equation of dynamics:

$$\frac{d}{dt}M = K. \tag{92}$$

Let us consider the rotational motion of the solid at very large time scales. We are in similar conditions as in the case of translational motion, but now the position angles have replaced the coordinates. In our nondeterministic approach, we definitively give up the hope to make strict predictions on the values of these angles, and we now work in terms of probability amplitude for these values. Once again, by this way we become able to predict (angular) structures, since all values of the angles will no longer be equivalent, but instead some of them will be favored, corresponding to peaks of probability density.

The angular velocity can be decomposed in terms of a backward and forward mean, leading to define a mean complex angular velocity $\tilde{\Omega}_k$ and a fluctuation such that $< W_i W_k > = 2\mathscr{D}_{jk}/dt$, where $\mathscr{D}_{jk}$ is now a tensor. We then build a scale-covariant derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \tilde{\Omega}_k \partial_k - i\mathscr{D}_{jk}\partial_j\partial_k. \tag{93}$$

The quantization of Eq. (92) is straighforward using this scale-covariant derivative. It writes:

$$I_{jk}\frac{d}{dt}\tilde{\Omega}_k = -\partial_j U. \tag{94}$$

We now introduce the wave function as another expression for the action $\mathscr{S}$, $\psi = \exp(i\mathscr{S}/\mathscr{S}_0)$, where $\mathscr{S}_0$ is a constant having the dimension of an angular momentum. Provided this constant is given by $\mathscr{S}_0 = 2I_{jk}\mathscr{D}_{jk}$ (this is a generalization of the previous scalar relation $\mathscr{S}_0 = 2m\mathscr{D}$), Eq. (94) can be integrated in terms of a generalized Schrödinger equation acting on the rotational Euler angles:

$$\mathscr{S}_0(\mathscr{D}_{jk}\partial_j\partial_k\psi + i\frac{\partial}{\partial t}\psi) - U\psi = 0. \tag{95}$$

An example of application of this equation to the Solar System (quantization of the obliquities and inclinations of planets and satellites) has been given in (Nottale 1996c).

### 4.7. Dissipative systems: first hints

One can generalize the Euler-Lagrange equations to dissipative systems thanks to the introduction of a dissipation function $f$ (see e.g. Landau & Lifchitz 1969):

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \mathscr{V}_i} = \frac{\partial \mathscr{L}}{\partial x_i} - \frac{\partial f}{\partial \mathscr{V}_i}, \tag{96}$$

where $f$ is linked to the energy dissipation by the equation $f = -d\mathscr{E}/2dt$. This becomes in the Newtonian case:

$$m\frac{d}{dt}\mathscr{V} = -\nabla\Phi - \frac{\partial f}{\partial \mathscr{V}} = -\nabla\Phi - \sum_j k_{ij}\mathscr{V}_j. \tag{97}$$

We shall only consider here briefly the simplified isotropic case:

$$f = kv, \tag{98}$$

and its complex generalization:

$$\mathscr{F} = k\mathscr{V}. \tag{99}$$

We obtain a new generalized equation (Nottale 1996a):

$$\mathscr{D}^2\Delta\psi + i\mathscr{D}\frac{\partial \psi}{\partial t} - \frac{\Phi}{2m}\psi + i\frac{k}{m}\psi\ln\psi = 0 \tag{100}$$

which is still Schrödinger-like (and remains scale-invariant under the transformation $\psi \to \rho\psi$, up to an arbitrary energy term), since it corresponds to a perturbed Hamiltonian: $H = H_0 + V$, with the operator $V$ such that $V\psi = -i\frac{k}{m}\psi\ln\psi$. The same problem has also been recently considered by Ahmed and Mousa (1996), with equivalent results. The standard methods of perturbation theory in quantum mechanics can then be used to look for the solutions of this equation. This will be presented in a forthcoming work.

### 4.8. Field equations

As is well-known, the profound unity of physics manifests itself by the fact that field equations can also be given the form of Lagrange equations. The potentials play the role of the generalized coordinates, the fields play the role of the time-derivatives of coordinates and the coordinates play the role of time:

$$x \leftrightarrow \Phi, \tag{101}$$

$$\frac{dx}{dt} = v \leftrightarrow F = \frac{d\Phi}{dt}, \tag{102}$$

$$t \leftrightarrow x. \tag{103}$$

Namely, field equations take the same form as the equations of motion of particles, once this substitution is made. For simplicity of the argument we work with only one $x$ variable in what follows (the generalization to any dimension will be given in a forthcoming work). One defines a Lagrange function

$\bar{\mathscr{L}}(\Phi, F, x)$ then an action $\mathscr{S}$ from this Lagrange function. The action principle leads to Euler-Lagrange equations that write:

$$\frac{d}{dx}\frac{\partial \bar{\mathscr{L}}}{\partial F} = \frac{\partial \bar{\mathscr{L}}}{\partial \Phi}. \tag{104}$$

For example, the Lagrange equation constructed from $\bar{\mathscr{L}} = \frac{1}{2}F^2 - k\rho\Phi$ is the Poisson field equation, $d^2\Phi/dx^2 = -k\rho$. This well-known structure of present physical theories allows us to apply our method to fields themselves. This leads to a quantization of classical fields, but in a new way and with an interpretation quite different from that of the second quantization in standard quantum mechanics.

Here we consider a field potential $\Phi(x, t)$ whose evolution with time is known to be chaotic. On a very long time-scale, far larger than its chaos time, it can be described in terms of a long-term, differentiable mean $< \Phi(x, t) >$ and a non-differentiable fluctuation $\xi_\Phi$. We are once again led to the same quantum-like method: we give up the possibility to strictly know the value of the potential $\Phi$ at any point or instant, but we introduce a probability amplitude for it, $\Psi(\Phi)$, such that the probability of a given value of $\Phi$ is given by $P(\Phi) = (\Psi^\dagger\Psi)(\Phi)$. The combined effect of fractal fluctuations and passage to complex numbers due to the breaking of the $(dx \leftrightarrow -dx)$ reflexion invariance leads to defining a complex field $\mathscr{F}$, then a scale-covariant derivative that writes:

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \mathscr{F}\frac{\partial}{\partial \Phi} - i\mathscr{D}_\Phi\frac{\partial^2}{\partial \Phi^2}. \tag{105}$$

Using this covariant derivative, Eq. (104) can be quantized. We obtain:

$$\frac{d}{dx}\frac{\partial \bar{\mathscr{L}}}{\partial \mathscr{F}} = \frac{\partial \bar{\mathscr{L}}}{\partial \Phi}. \tag{106}$$

In the particular case of a scalar field considered above, this equation can be integrated under the form of a generalized Schrödinger equation for the probability amplitude of the potential $\Psi(\Phi)$:

$$\mathscr{D}_\Phi^2\frac{\partial^2}{\partial \Phi^2}\Psi + i\mathscr{D}_\Phi\frac{\partial}{\partial x}\Psi = \frac{1}{2}k\rho\Phi\Psi \tag{107}$$

$$= \frac{1}{2}\Phi\frac{d^2\Phi}{dx^2}\Psi. \tag{108}$$

A study of the solutions of this equation and its writing for any number of dimensions will be presented in a forthcoming work.

## 5. Discussion and conclusion

One of the most important consequences of the quantum-like nature of the world at large time and/or length-scales unveiled by scale-relativity is the profound unity of structures that it implies between very different scales. This unity is a consequence: (i) of the universality of the structuring "force", (i.e. of the fractal geometry of space-time),
(ii) of that of physics, as manifested in the Lagrangian / Hamiltonian formalism, i.e., in the underlying symplectic structure of

physics,
(iii) and of the recovered prevalence of the space-time description.

It was already apparent in standard quantum mechanics, in the fact that several features of the solutions to a given Schrödinger equation can be detailed before the precise form of the potential is specified. The structures come in large part from the quantum terms themselves (which we have interpreted as a manifestation of the fractal space-time geometry) and from the matching and limiting conditions for the wave function.

Now, as can be seen on the various generalized Schrödinger equations written throughout the present paper, these quantum terms are in common to all them. Moreover, these equations describe different systems that must be matched together at different scales in the real world: the matching conditions will then imply a unity and a continuity of the structures observed at these different scales. This prediction has already been fairly well verified for various gravitational potentials from the scale of star radii ($< 10^6$ km) to extragalactic scales ($\approx 100$ Mpc) (Nottale 1996a,b, Nottale, Schumacher & Gay 1996, Nottale, Schumacher & Lefèvre 1997). We shall in future work investigate whether it also applies to systems whose structure do not depend on the gravitational field only, but also on magnetic fields, pressure and dissipative terms (e.g., stellar interiors, stellar atmospheres...).

Before concluding, we want to stress once again the difference between the application of our theory to standard quantum mechanics (at small scales) and to very large time-scale phenomena as studied in the present series of papers. In the case of quantum mechanics (Nottale 1993a), our fundamental assumption is that space-time itself is continuous but non-differentiable, then fractal *without any lower limit*. The complete withdrawal of the hypothesis of differentiability is necessary if we want the theory not to be a hidden parameter one and to agree with Bell's theorem and the undeterminism of quantum paths. Moreover, we also give up the concept of elementary particle as being something which would own internal properties such as mass, charge or spin, since we are able to recover these properties from the geometric structures of fractal geodesics of the non-differentiable space-time. Particles, with their wave-corpuscle duality, are identified with the geodesics themselves (Nottale 1989, 1993a, 1996a), i.e., with the shortest lines of topological dimension 1 (singularizing also the topological dimension 2 would lead to string theories, see Castro 1996).

On the contrary, in the application to chaos and fractal space beyond the predictability horizon, we know from the beginning (i) that non-differentiability is only a large time-scale approximation ($t \gg \tau$), and that when going back to small time-resolution we recover differentiable, predictable classical trajectories; and (ii) that the geodesics are indeed trajectories followed by extended bodies. This motivates the use of some of the quantum mechanical tools (probability amplitude, Schrödinger-like equations) but not its whole interpretation, concerning in particular measurement theory, in agreement with the recent construction by Ord (1996a,b) of a microscopic model of the Schrödinger equation in the fractal space-time / random walk framework.

Recall also that the application of the scale-relativity theory to the macrophysical domain implies a different interpretation of our construction respectively to the microphysical domain for yet another reason. In the macroscopic case indeed, the transition to classical physics is toward the small scales, while no upper limit is expected to the scaling behavior at large scales. This can be achieved provided our theory applies only to a "fully quantum" system, i.e., a system for which the mean classical velocity $< V >$ is zero (such as the hydrogen atom in microphysics). Indeed, the upper transition from quantum (fractal) laws to classical (non fractal) laws is given by the equivalent de Broglie length, $\lambda = 2\mathscr{D} / < v >$, which is sent to infinity when $< V >= 0$.

The scale relativity theory shares some common features with other approaches, even if it also differs from them on essential points. A first related approach is Nelson's (1966, 1985) stochastic mechanics, in which particles are described in terms of a diffusion, Brownian-like process, but with a Newtonian rather than Langevin dynamics. Nelson obtains a complex Schrödinger equation as a combination of real equations, namely a Newton equation of dynamics in which the form of the acceleration is postulated, and two backward and forward Fokker-Planck equations. (Note that this implies that Nelson's diffusion is not a standard diffusion process, since his backward Fokker-Planck equation is a time-reversed forward Kolmogorov equation, which is therefore incompatible with the standard backward Kolmogorov equation, see e.g. Welch 1970). Nelson's theory has been used by Albeverio et al. (1983) and Blanchard (1984) to obtain models of the protoplanetary nebula.

Contrarily to such diffusion approaches and to standard quantum mechanics itself, the scale-relativity theory is not statistical in its essence. In scale relativity, the fractal space-time could be completely "determined", so that the undeterminism of trajectories is not set as a founding stone of the theory, but as a consequence of the nondifferentiability of space-time. This is clear from the fact that we do not use Fokker-Planck equations, but only the equation of dynamics, properly made scale-covariant.

The implications of this difference between the two approaches are very important. The diffusion approach is expected to apply only in fluid-like or many-body environments. On the contrary, the structuring "field" in our theory being the underlying fractal geometry of space-time itself, we predict that there is a universal tendency of nature to make structures, even for two-body problems, and that these structures must be themselves related together in a universal way. This prediction has been already verified in a remarkable way for gravitational structures (Nottale 1993a,b, 1995b, 1996a,b; Nottale, Schumacher & Gay 1997).

Recall also that it is now known that Nelson's stochastic mechanics is in contradiction with standard quantum mechanics concerning multitime correlations (Grabert et al 1979, Wang & Liang 1993). The source of the disagreement comes precisely from the Brownian motion interpretation of Nelson's theory,

leading to the use of the Fokker-Planck equations, and from the wave function reduction. Once again the fact that we do not use the Fokker-Planck equations reveals itself as an essential features of our theory, since it allows our theory not to come under the Wang & Liang argument. Once we have jumped to the quantum tool (i.e., when we pass from our representation in terms of $\mathscr{V}$ and $\mathscr{S}$ to the equivalent representation in terms of $\psi$) we know by construction that the representation is complete, (i.e. we recover the quantum equations without any additional constraint) so that the identity of predictions of standard quantum mechanics and of scale relativity is ensured in the microphysical domain (at energies where Galilean scale-laws hold). Another related approach is that of Petrovsky and Prigogine (1996), who attempt to extend classical dynamics by formulating it on the statistical level. They also give up individual trajectories and jump to a non-deterministic and irreversible description. The difference with our own approach is that they keep classical probabilities and irreversibility central to the theory without invoking an explicit scale dependence. In contrast, in scale relativity, the description is fundamentally irreversible (in terms of the elementary displacements on fractal geodesics), but this is not an axiom so much as a consequence of giving up differentiability. Moreover, we recover a reversible description in terms of our complex representation (i.e., of the quantum mechanical tool) which combines the forward and backward process: in other words, irreversibility is at the origin of our complex formulation, but it becomes hidden in the formalism, even though it reappears through the wave function collapse.

Finally, an important point to emphasize once again is also that, in scale-relativity, we really deal with a *fractal space*, not only with fractal trajectories in a space that could remain flat or curved. This is apparent in our trajectory equation, which is written (in the absence of an external field) in terms of a scale-covariant *geodesics* equation, which takes the form of the free, Galilean equation of motion $d\mathscr{V}/dt = 0$. This is the equation for rectilinear uniform motion. It means that the particle goes straight ahead in its proper coordinate system swept along in the fractal space-time, and that its structure, which looks fractal when seen from an exterior reference frame, comes from the very fractal geometry of space-time itself.

In the papers of this series to follow, we shall enter in more detail into our theoretical predictions by looking at the solutions of our equations for different fields, with particular attention given in paper II on gravitational structures, then we shall compare these predictions to observational data (paper III). We shall see that the theory allows us to explain several misunderstood facts concerning gravitational structures at all scales, it allows us to make new predictions (see already Nottale 1996a,b,c, 1997, Nottale et al 1997), and it also opens new domains of investigation, concerning in particular the open question of a more complete description of the "field" of fractal fluctuations.

## Appendix A: continuity and nondifferentiability implies scale-divergence

One can demonstrate (Nottale, 1993a, 1996a) that the length of a continuous and nowhere differentiable curve is dependent on resolution $\epsilon$, and, further, that $\mathscr{L}(\epsilon) \to \infty$ when $\epsilon \to 0$, i.e. that this curve is fractal (we used the word "fractal" in this extended meaning throughout this paper). The scale divergence of continuous and almost nowhere differentiable curves is a direct consequence of Lebesgue's theorem, which states that *a curve of finite length is almost everywhere differentiable*.

Consider indeed a continuous but nondifferentiable function $f(x)$ between two points $A_0\{x_0, f(x_0)\}$ and $A_\Omega\{x_\Omega, f(x_\Omega)\}$ in the Euclidean plane. Since $f$ is non-differentiable, there exists a point $A_1$ of coordinates $\{x_1, f(x_1)\}$ with $x_0 < x_1 < x_\Omega$, such that $A_1$ is not on the segment $A_0A_\Omega$. Then the total length $\mathscr{L}_1 = \mathscr{L}(A_0A_1) + \mathscr{L}(A_1A_\Omega) > \mathscr{L}_0 = \mathscr{L}(A_1A_\Omega)$. We can now iterate the argument and find two coordinates $x_{01}$ and $x_{11}$ with $x_0 < x_{01} < x_1$ and $x_1 < x_{11} < x_\Omega$, such that $\mathscr{L}_2 = \mathscr{L}(A_0A_{01}) + \mathscr{L}(A_{01}A_1) + \mathscr{L}(A_1A_{11}) + \mathscr{L}(A_{11}A_\Omega) > \mathscr{L}_1 > \mathscr{L}_0$. By iteration we finally construct successive approximations $f_0, f_1, \cdots f_n$ of $f(x)$ whose lengths $\mathscr{L}_0, \mathscr{L}_1, \cdots \mathscr{L}_n$ increase monotonically when the "resolution" $\epsilon \approx (x_\Omega - x_0) \times 2^{-n}$ tends to zero. In other words, continuity and nondifferentiability implies a monotonous scale dependence of $f$ on resolution $\epsilon$. Now, Lebesgue's theorem states that *a curve of finite length is almost everywhere differentiable* (see e.g. Tricot 1993).

Therefore, if $f$ is continuous and almost everywhere non-differentiable, then $\mathscr{L}(\epsilon) \to \infty$ when the resolution $\epsilon \to 0$; namely $f$ is not only scale-dependent, but even *scale-divergent*. This theorem is also demonstrated in (Nottale 1993a, p.82) by using non-standard analysis.

What about the reverse proposition: Is a continuous function whose length is scale-divergent between any couple of points (such that $x_A - x_B$ finite), i.e., $\mathscr{L}(\epsilon) \to \infty$ when $\epsilon \to 0$, non-differentiable? The answer is as follows:
(i) If the length diverges as fast as a power law, i.e. $\mathscr{L}(\epsilon) \propto (\lambda/\epsilon)^\delta$, or faster than a power law (e.g., exponential divergence $\mathscr{L}(\epsilon) \propto \exp(\lambda/\epsilon)$, etc...), then the function is certainly nondifferentiable. It is interesting to see that the standard (self-similar, power-law) fractal behavior plays a critical role in this theorem: it gives the limiting behavior beyond which non-differentiability is ensured.
(ii) In the intermediate domain of slower divergences (for example, logarithmic divergence, $\mathscr{L}(\epsilon) \propto \ln(\lambda/\epsilon), \ln(\ln(\lambda/\epsilon))$, etc...), the function may be either differentiable or nondifferentiable.

This can be demonstrated by looking at the way the length increases and the slope changes under successive zooms of a constant factor $\rho$. There are two ways by which the divergence can occur: either by a regular increase of the length (due to the regular appearance of new structures at all scales that continuously change the slope), or by the existence of jumps (in this case, whatever the scale, there will always exist a smaller scale at which the slope will change). The power law corresponds to a continuous length increase, $\mathscr{L}(\rho\epsilon) = \mu\mathscr{L}(\epsilon)$, then

to a continuous and regular change of slope when $\epsilon \to 0$: therefore the function is nondifferentiable in this case. Divergences slower than power laws may correspond to a regular length increase, but with a factor $\mu$ which becomes itself scale-dependent: $\mathscr{L}(\rho\epsilon) = \mu(\epsilon)\mathscr{L}(\epsilon)$ with $\mu(\epsilon) \to 0$ when $\epsilon \to 0$. In this case, some functions can be differentiable, if they are such that new structures indeed appear at all scales (and could then be named "fractal" under the general definition initially given by Mandelbrot 1975 to this word), but these structures become smaller and smaller with decreasing scale, so that a slope can finally be defined in the limit $\epsilon \to 0$. Some other functions diverging slower than power laws are not differentiable, e.g. if there always exists a scale smaller than any given scale such that an important change of slope occurs: in this case, the slope limit may not exist in the end.

## Appendix B: special and generalized scale-relativity

### B.1. Special scale relativity

It is well known that the Galileo group of motion is only a degeneration of the more general Lorentz group. The same is true for scale laws. Indeed, if one looks for the general linear scale laws that come under the principle of scale relativity, one finds that they have the structure of the Lorentz group (Nottale 1992). Therefore, in special scale relativity, we have suggested to substitute to the Galilean laws of dilation $\ln \rho'' = \ln \rho + \ln \rho'$ the more general Lorentzian law (Nottale 1992, 1993a):

$$\ln \rho'' = \frac{\ln \rho + \ln \rho'}{1 + \frac{\ln \rho \ln \rho'}{\mathbb{C}^2}}. \tag{B1}$$

This expression is yet uncomplete, since under this form the scale relativity symmetry remains unbroken. Such a law corresponds, at the present epoch, only to the null mass limit. It is expected to apply in a universal way during the very first instants of the Universe. This law assumes that, at very high energy, no static scale and no space or time unit can be defined, so that only pure contractions and dilations have physical meaning. The corresponding physics is a physics of pure numbers. In Eq. (B1), there appears a universal, purely numerical constant $\mathbb{C} = \ln \mathbb{K}$, which plays the role of a maximal possible *dilation*. We have found that the value of $\mathbb{K}$ is about $5 \times 10^{60}$ (Nottale 1993a, 1995d, 1996a): its existence yields an explanation to the Eddington-Dirac large number hypothesis, and connects the cosmological constant to the Planck scale. A more detailed study of these questions will be presented in a forthcoming work (Nottale 1997).

However, the effect of the spontaneous scale symmetry breaking which arises at some scale $\lambda_0$ is to yield a new law in which the invariant is no longer a dilation $\mathbb{K}$, but becomes a *length-time scale* $\bar{\Lambda}$. In other words, there appears in the theory a fundamental scale that plays the role of an unpassable resolution, invariant under dilations (Nottale 1992). Such a scale of length and time is an horizon for scale laws, in a way similar to the status of the velocity of light for motion laws. The new law of composition of dilations and the scale-dimension now
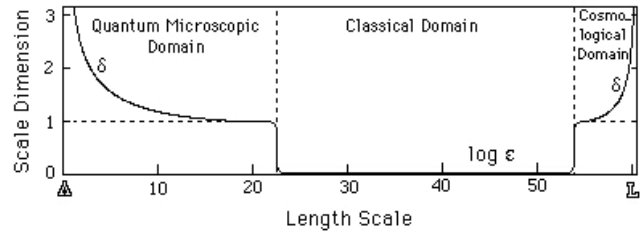


**Fig. 5.** Schematic representation of the three domains of the present era, (quantum microscopic, classical and cosmological) in the case of special scale-relativistic (Lorentzian) laws. The variation of the *effective* fractal dimension ($D = 1 + \delta$) is given in terms of the logarithm of resolution. It is constant and equal to the topological dimension in the classical, scale-independent domain. It jumps fastly to $D = 2$ towards small and large scales (Galilean regime), then it increases continuously in the Lorentzian regime (Eq. (A.2.2)). The (relative) transitions are given by the Compton length at small scale and (presumably) by the Emden radius at large scale.

write respectively (in the scale-dependent domains, i.e. only below the transition scale in microphysics and beyond it in the cosmological case):

$$\ln \frac{\epsilon'}{\lambda_0} = \frac{\ln(\epsilon/\lambda_0) + \ln \rho}{1 + \frac{\ln \rho \ln(\epsilon/\lambda_0)}{\ln^2(\bar{\Lambda}/\lambda_0)}}, \tag{B2}$$

$$\delta(\epsilon) = \frac{1}{\sqrt{1 - \frac{\ln^2(\lambda_0/\epsilon)}{\ln^2(\lambda_0/\bar{\Lambda})}}}. \tag{B3}$$

A fractal curvilinear coordinate becomes now scale-dependent in a covariant way, namely $\mathscr{L} = \mathscr{L}_0 [1 + (\lambda_0/\epsilon)^{\delta(\epsilon)}]$. One of the main new feature of special scale relativity with respect to the previous fractal or scale-invariant approaches is that the scale-dimension $\delta$, which was previously constant, is now explicitly varying with scale (see Fig. 5) and even diverges when resolution tends to the new invariant scales. In the microphysical domain, the invariant length-scale is naturally identified with the Planck scale, $\Lambda_P = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point (Nottale 1992, 1993a). The same is true in the cosmological domain, with once again an inversion of the scale laws. We have identified the invariant maximal scale with the scale of the cosmological constant, $\mathbb{L} = \Lambda^{-1/2}$. The consequences of this new interpretation of the cosmological constant have been considered in (Nottale 1993a, 1995d, 1996a) and will be developed further in a forthcoming work (Nottale 1997).

Note that special scale-relativistic laws (Nottale 1992) have also recently been considered by Dubrulle (1994) and Dubrulle and Graner (1996) for the description of turbulence, with a different interpretation of the variables.

It is also noticeable that recent developments in string theories (Witten 1996) have reached conclusions that are extraordinarily similar to those of scale relativity. One finds that there is a smallest circle in string theory (whose radius is about the Planck length), and that strings are characterized by duality symmetries.

Two of these dualities are especially relevant to our approach, since they make already part of it in a natural way. The first is the quantum / classical duality, which we recover in terms of our scale / motion duality. The second is a microscopic / macroscopic duality: it has been found that strings do not distinguish small spacetime scales from large ones, relating them through an inversion. But scale inversion is a transformation which is naturally included in the scale-relativistic framework (see Sect. 2.5), since this is nothing but the symmetric element of the scale group ($\bar{V}' = -\bar{V} \Leftrightarrow \ln(\lambda/\epsilon') = -\ln(\lambda/\epsilon) \Leftrightarrow \epsilon' = \lambda^2/\epsilon$ in the Galilean case). Therefore it has recently been claimed by Castro (1996) that scale relativity is the right framework in which the newly discovered string structures will take their full physical meaning. The string duality between the small and large scales adds a new argument to our main conclusion: namely, that the laws of physics take again a quantum-like form at very large spacetime scales.

### B.2. From scale dynamics to general scale relativity

The whole of our previous discussion indicates to us that the scale invariant behavior corresponds to freedom in the framework of a scale physics. However, in the same way as there exists forces in nature that imply departure from inertial, rectilinear uniform motion, we expect most natural fractal systems to also present distorsions in their scale behavior respectively to pure scale invariance. Such distorsions may be, as a first step, attributed to the effect of a scale "dynamics", i.e. to "scale-forces". (Caution: this is only an *analog* of "dynamics" which acts on the scale axis, on the internal structures *of a given point* at this level of description, not in space-time. See Sect. B.3 for first hints about the effects of coupling with space-time displacements). In this case the Lagrange scale-equation takes the form of Newton's equation of dynamics:

$$\bar{F} = \mu \frac{d^2 \ln \mathscr{L}}{d\delta^2},\qquad\text{(B4)}$$

where $\mu$ is a "scale-mass", which measures the way the system resists to the scale-force.

### B.2.1. Constant scale-force

Let us first consider the case of a constant scale-force. Eq. (B4) writes

$$\mu \frac{d^2 \ln \mathscr{L}}{d\delta^2} = \bar{G},\qquad\text{(B5)}$$

where $\bar{G} = \bar{F}/\mu$ = constant. It is easily integrated in terms of the usual parabolic solution (where $\bar{V} = \ln \frac{\lambda}{\epsilon}$:

$$\bar{V} = \bar{V}_0 + \bar{G}\delta,\qquad\text{(B6)}$$

$$\ln \mathscr{L} = \ln \mathscr{L}_0 + \bar{V}_0\delta + \frac{1}{2}\bar{G}\delta^2.\qquad\text{(B7)}$$

However the physical meaning of this result is not clear under this form. This is due to the fact that, while in the case of motion

laws we search for the evolution of the system with time, in the case of scale laws we search for the dependence of the system on resolution, which is the directly measured observable. We find, after redefinition of the integration constants:

$$\delta = \delta_0 + \frac{1}{\bar{G}} \ln\left(\frac{\lambda}{\epsilon}\right),\qquad\text{(B8)}$$

$$\ln \frac{\mathscr{L}}{\mathscr{L}_o} = \frac{1}{2\bar{G}} \ln^2\left(\frac{\lambda}{\epsilon}\right).\qquad\text{(B9)}$$

The scale dimension $\delta$ becomes a linear function of resolution (the same being then true of the fractal dimension $1+\delta$), and the $(\ln \mathscr{L}, \ln \epsilon)$ relation is now parabolic rather than linear as in the standard power-law case. There are several physical situations where, after careful examination of the data, the power-law models were clearly rejected since no constant slope could be defined in the $(\ln \mathscr{L}, \ln \epsilon)$ plane. In the several cases where a clear curvature appears in this plane (e.g., turbulence, sand piles,...), the physics could come under such a "scale-dynamical" description. In these cases it might be of the highest interest to identify and study the scale-force responsible for the scale distorsion (i.e., for the deviation to standard scaling).

### B.2.2. Harmonic oscillator

Another interesting case of scale-potential is that of a repulsive harmonic oscillator. It is solved as

$$\ln \frac{\mathscr{L}}{\mathscr{L}_0} = \alpha \sqrt{\ln^2\left(\frac{\lambda}{\epsilon}\right) - \frac{1}{\alpha^2}}.\qquad\text{(B10)}$$

For $\epsilon \ll \lambda$ it gives the standard Galilean case $\mathscr{L} = \mathscr{L}_0(\lambda/\epsilon)^\alpha$, but its large-scale behavior is particularly interesting, since it does not permit the existence of resolutions larger than a scale $\lambda_{max} = \lambda e^{1/\alpha}$. Such a behavior could provide a model of confinement in QCD (Nottale 1997).

More generally, we shall be led to look for the general nonlinear scale laws that satisfy the principle of scale relativity (see also Dubrulle and Graner 1997). As remarked in (Nottale 1994b, 1996a), such a generalized framework implies working in a five-dimensional fractal space-time. The development of such a "general scale-relativity" lies outside the scope of the present paper and will be considered elsewhere (Nottale 1997).

### B.3. Scale-motion coupling and gauge invariance

The theory of scale relativity also allows to get new insights about the physical meaning of gauge invariance (Nottale 1994b, 1996a). In the previous scale laws, only scale transformations at a given point were considered. But we must also wonder about what happens to the structures in scale of a scale-dependent object when it is displaced. Consider anyone of these structures, lying at some (relative) resolution $\epsilon$ (such that $\epsilon < \lambda$, where $\lambda$ is the fractal/nonfractal transition) for a given position of the particle. In a displacement of the object, the relativity of scales implies that the resolution at which this given structure appears

in the new position will a priori be different from the initial one. In other words, $\epsilon$ is now a function of the space-time coordinates, $\epsilon = \epsilon(x,t)$, and we expect the occurrence of *dilatations of resolutions induced by translations*, which read:

$$e\frac{d\epsilon}{\epsilon} = -A_\mu dx^\mu, \tag{B11}$$

where a four-vector $A_\mu$ must be introduced since $dx^\mu$ is itself a four-vector and $d\ln\epsilon$ a scalar (in the case of a global dilation). This behavior can be expressed in terms of a new scale-covariant derivative:

$$eD_\mu \ln\left(\frac{\lambda}{\epsilon}\right) = e\partial_\mu \ln\left(\frac{\lambda}{\epsilon}\right) + A_\mu. \tag{B12}$$

However, if one wants such a "field" $A_\mu$ to be physical, it must be defined whatever the initial scale from which we started. Starting from another scale $\epsilon' = \rho\epsilon$ (we consider only Galilean scale-relativity here, see Nottale 1994b, 1996a for the additional implications of special scale-relativity), we get

$$e\frac{d\epsilon'}{\epsilon'} = -A'_\mu dx^\mu, \tag{B13}$$

so that we obtain:

$$A'_\mu = A_\mu + e\partial_\mu \ln\rho, \tag{B14}$$

which depends on the relative "state of scale", $\bar{V} = \ln\rho = \ln(\epsilon/\epsilon')$. However, if one now considers translation along two different coordinates (or, in an equivalent way, displacement on a closed loop), one may write a commutator relation:

$$e(\partial_\mu D_\nu - \partial_\nu D_\mu)\ln(\lambda/\epsilon) = (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{B15}$$

This relation defines a tensor field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which, contrarily to $A_\mu$, is independent of the initial scale. One recognizes in $F_{\mu\nu}$ the analog of an electromagnetic field, in $A_\mu$, that of an electromagnetic potential, in $e$ that of the electric charge, and in Eq. (B14) the property of gauge invariance which, in accordance with Weyl's initial ideas (Weyl 1918), recovers its initial status of scale invariance. However, Eq. (B14) represents a progress compared with these early attempts and with the status of gauge invariance in today's physics. Indeed the gauge function, which has, up to now, been considered as arbitrary and devoid of physical meaning, is now identified with the logarithm of internal resolutions. In Weyl's theory, and in its formulation by Dirac (1973), the metric element $ds$ (and consequently the length of any vector) is no longer invariant and can vary from place to place in terms of some (arbitrary) scale factor. Such a theory was excluded by experiment, namely by the existence of universal and unvarying lengths such as the electron Compton length (i.e., by the existence of particle masses). In scale relativity, we are naturally led to introduce two "proper times", the classical one $ds$ which remains invariant, and the fractal one $d\mathscr{S}$, which is scale-divergent and can then vary from place to place (its variation amounting to a scale transformation of resolution). In Galilean scale-relativity,

the fractal dimension of geodesics is $D = 2$, so that the scale-dependence of $d\mathscr{S}$ writes $d\mathscr{S} = d\sigma(\lambda/\epsilon)$. Therefore we have $\delta(d\mathscr{S})/d\mathscr{S} = -\delta\epsilon/\epsilon \propto A_\mu\delta x^\mu$, and we recover the basic relation of the Weyl-Dirac theory, in the asymptotic high energy domain ($\epsilon < \lambda$). Another advantage with respect to Weyl's theory is that we are now allowed to define four different and independent dilations along the four space-time resolutions instead of only one global dilation. The above U(1) field is then expected to be embedded into a larger field, in agreement with the electroweak theory, and the charge $e$ to be one element of a more complicated, "vectorial" charge (Nottale 1997). Moreover, when combined with the Lorentzian structure of dilations of special scale relativity, our interpretation of gauge invariance yields new relations between the charges and the masses of elementary particles (Nottale 1994b, 1996a).

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