Theory of Electron in Scale Relativity*

L. NOTTALE
CNRS, LUTH, Observatoire de Paris-Meudon, F-92195 Meudon Cedex, France

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Abstract

The question of the physical nature of the electron, of the electromagnetic field and of gauge invariance is revisited in the framework of the theory of scale relativity. Space-time is described as a non-derivable "manifold", which implies that its geometry is fractal (i.e., it is explicitly dependent on the resolution scale). The electromagnetic field can then be re-interpreted as a field of dilations induced by displacements, while the electric charge is the conservative quantity that originates from the scale symmetry. This means that gauge transformations are identified with scale transformations on the resolution variables (which are internal to the electron, i.e., relevant at scales smaller than its Compton length). In this general framework, the Lorentz force and the Maxwell equations are derived from first principles, and the QED-covariant derivative naturally emerges instead of being merely postulated. Now, in the framework of special scale relativity, the Planck length-scale becomes a minimal, impassable scale, invariant under dilations (that replaces the zero point). As a consequence scale ratios are limited and therefore one demonstrates charge quantization and the existence of a relation between the mass and the charge of the electron. From this relation, the mass of the electron explicitly depends on the number of Higgs doublets : one finds that the theoretical expectation agrees with the experimental mass for one Higgs doublet.

1 Introduction

In the present physical theory, one still does not really understand the nature of the electric charge and of the electromagnetic field. As recalled by Landau ([8], Chap. 16), in the classical theory the very existence of the charge e and of the electromagnetic 4-potential A_μ are ultimately derived from experimental data. Moreover, the form of the action for a particle in an electromagnetic field,

$$S_{\rm pf} = -\frac{e}{c} \int A_\mu dx^\mu \tag{1}$$

can not be chosen only from general considerations, and it is therefore merely postulated.

Now, once these three points are set, most of the classical theory of electromagnetism can be constructed. Indeed (see e.g. [8]), the total action of a charge in an electromagnetic field is the sum of the free particle action and of the charge-potential coupling term:

$$S = S_{\rm p} + S_{\rm pf} = -mc \int ds \; - \frac{e}{c} \int A_\mu dx^\mu . \tag{2}$$

The least-action principle subsequently yields motion equations that write:

$$mc \, \frac{du_\mu}{ds} = \frac{e}{c} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) u^\nu . \tag{3}$$

This establishes the 4-dimensional expression for the Lorentz force and allows one to identify the 4-tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4}$$

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as the relativistic expression for the electromagnetic field.

As concerns Maxwell equations, no additional information is needed. Indeed, the action for the field must be a scalar quantity that can not depend on the potential because of its gauge dependence. The only possible remaining scalar is therefore $F_{\mu\nu}F^{\mu\nu}$, and the field action term reads [8]:

$$S_{\rm f} = -\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}d\Omega. \tag{5}$$

Finally the variational principle applied on the total action $S_{\rm pf} + S_{\rm f}$ allows one to demonstrate Maxwell's field equations in their relativistic form, namely,

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c}j^\nu. \tag{6}$$

These well-known results are reminded here only in order to point out clearly that the key point to a genuine understanding of the nature of electromagnetism would be a demonstration from first principle of the particle-field coupling action term $S_{\rm pf}$ (Eq. 1), itself based on physical definitions of the charge and of the potential. The aim of the present paper is precisely to make a proposal in this direction, which involves a new interpretation of gauge transformations, and then to analyse some of its possible consequences.

2 The theory of the relativity of scales: a brief reminder

2.1 Principle of scale relativity

The theory of scale relativity [12] studies the consequences of giving up the hypothesis of space-time derivability. One can show [12, 14, 1] that a continuous but nondifferentiable space-time is necessarily fractal. Here the word fractal [9] is taken in a general meaning, as defining a set, object or space that shows structures at all scales, or on a wide range of scales. More precisely, one can demonstrate [15] that a continuous but nondifferentiable function is explicitly resolution-dependent, and that its length $\mathcal{L}$ is strictly increasing and tends to infinity when the resolution interval tends to zero, i.e. $\mathcal{L} = \mathcal{L}(\varepsilon)_{\varepsilon\to 0} \to \infty$. This theorem naturally leads to the proposal that the concept of fractal space-time [19, 20, 10, 12, 4] is the geometric tool adapted to the research of such a new description.

Since a nondifferentiable, fractal space-time is explicitly resolution-dependent, the same is a priori true of all physical quantities that one can define in its framework. We thus need to complete the standard laws of physics (which are essentially laws of motion in classical physics) by laws of scale, intended to describe the new resolution dependence. We have suggested [11] that the principle of relativity can be extended to constrain also these new scale laws.

Namely, we generalize Einstein's formulation of the principle of relativity, by requiring that the laws of nature be valid in any reference system, whatever its state. Up to now, this principle has been applied to changes of state of the coordinate system that concerned the origin, the axes orientation, and the motion (measured in terms of velocity and acceleration).

In scale relativity, we assume that the space-time resolutions are not only a characteristic of the measurement apparatus, but acquire a universal status. They are considered as essential variables, inherent to the physical description. We define them as characterizing the "state of scale" of the reference system, in the same way as the velocity characterizes its state of motion. The principle of scale relativity consists of applying the principle of relativity to such a scale-state. Then we set a principle of scale-covariance, requiring that the equations of physics keep their form under resolution transformations.

Among the consequences of this approach, one is able to recover the main axioms of quantum mechanics [12, 14, 5, 21, 22] from a geometric description using the concepts of general relativity (metric element, geodesics). In the present paper, we shall first briefly review the fundamental results of the theory upon which the new gauge theory relies, namely, (i) identification of a particle mass scale with a fractal / non-fractal transition in scale space; (ii) new interpretation of the Planck-length as a scale invariant under dilations, (iii) derivation of the Schrödinger, Klein-Gordon and Dirac equations from non-differentiable geometry. Then we shall consider some of their consequences in the domains of gauge field theories.

2.2 Galilean scale-relativity

Simple fractal scale-invariant laws can be identified with a "Galilean" version of scale-relativistic laws. Indeed, let us consider a non-differentiable coordinate $\mathcal{L}$. Our basic theorem that links non-differentiability to fractality implies that $\mathcal{L}$ is an explicit function $\mathcal{L}(\varepsilon)$ of the resolution interval ε. As a first step, one can assume that $\mathcal{L}(\varepsilon)$ satisfies the simplest possible scale differential equation one may write, namely, a first order equation where the scale variation of $\mathcal{L}$ depends on $\mathcal{L}$ only, $d\mathcal{L}/d\ln\varepsilon = \beta(\mathcal{L})$. The function $\beta(\mathcal{L})$ is a priori unknown but, still taking the simplest case, we may consider a perturbative approach and take its Taylor expansion. We obtain the equation:

$$\frac{d\mathcal{L}}{d\ln\varepsilon} = a + b\mathcal{L} + \dots \tag{7}$$

This equation is solved in terms of a standard power law of power $\delta = -b$, broken at some relative scale λ (which is a constant of integration):

$$\mathcal{L} = \mathcal{L}_0 \, [1 + \left(\frac{\lambda}{\varepsilon}\right)^{\delta}]. \tag{8}$$

Here δ is the scale dimension, i.e., $\delta = D - D_T$, the fractal dimension minus the topological dimension. The scale symmetry breaking at the transition scale λ plays an important role in the theory, since this scale is subsequently identified with the Einstein-de Broglie scale, so that it ultimately provides the mass of the particle under consideration.

The Galilean structure of the group of scale transformation that corresponds to the scaling law in the asymptotic small scale domain is verified in a straightforward manner [11, 12].

2.3 Special scale-relativity

We have suggested that, in the fractal asymptotic domain, (i.e. beyond the fractal / non-fractal transition λ, that is identified in rest frame with the Compton length of the particle), the Galilean law of composition of dilations $\ln(\varepsilon'/\lambda) = \ln\rho + \ln(\varepsilon/\lambda)$ is only a low energy approximation, and should be replaced by the more general log-Lorentzian law [11]:

$$\ln\frac{\varepsilon'}{\lambda} = \frac{\ln\rho + \ln(\varepsilon/\lambda)}{1 + \ln\rho\ln(\varepsilon/\lambda)/\ln^2(\lambda_P/\lambda)}\,. \tag{9}$$

In the framework of such a 'special scale-relativistic' law, the length-time scale λ_P is a minimal scale of space-time resolution which is invariant under dilations and contractions, and plays the same role for scales as that played by the velocity of light for motion.

Toward the small scales, this invariant length-scale is naturally identified with the Planck scale, $\lambda_P = (\hbar G/c^3)^{1/2}$, that now becomes impassable and plays the physical role that was previously devoted to the zero point. Some consequences of this new interpretation of the Planck length-time-scale have been considered elsewhere [11, 12, 14], concerning in particular the unification of fundamental fields. We shall point out here its consequences for the quantization of the electric charge and for a theoretical prediction of a relation between the electric charge and the electron mass.

2.4 Quantum laws

One of the main consequences of the scale-relativity / fractal space-time approach is its ability to build from geometric structures some of the fundamental rules of quantum mechanics, that were up to now set as mere axioms.

In analogy with Einstein's general relativity, trajectories are identified with the geodesics of space-time, which is now fractal. The giving up of the hypothesis of derivability of space-time coordinates implies at least three conditions for these geodesics [12]:

(i) They are in infinite number. This leads one to give up determinism (for trajectories) and to introduce a velocity field $v = v(x(t), t)$;

(ii) The geometry of each trajectory is of fractal dimension $D = 2$ (which corresponds to a Markov process, i.e. to a loss of information between each elementary displacement). These displacements $dX = dx + d\xi$ are decomposed in terms of the sum of two terms, $dx = v\,dt$ being a derivable 'classical part' which dominates at large scales, and $d\xi$ a 'fractal part' (see e.g. [6]), which describes the asymptotic non-derivable behavior at small scale. It is such that $< d\xi >= 0$ and $< d\xi^2 >= 2\mathcal{D}dt$ for $D = 2$.

The information about the non-differentiability of space is contained in the variables $d\xi$. Indeed, since $d\xi \sim dt^{1/2}$, their derivatives are such that $d\xi/dt \sim dt^{-1/2}$ and they are therefore formally infinite when $dt \to 0$.

One can check that the scale dependence of the fractal fluctuation comes under the principle of scale relativity (in its simplest "Galilean" version). The existence of this fluctuation implies introducing new second order terms in the differential equations of motion.

(iii) Time reversibility is broken at the level of differentials, i.e., the velocity field is no longer invariant under the reflection ($dt \leftrightarrow -dt$). This leads to a two-valuedness of the velocity vector that we represent in terms of a complex velocity,

$$\mathcal{V} = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2}. \tag{10}$$

These three effects can be combined to construct a complex time-derivative operator that writes

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V}\cdot\nabla - i\,\mathcal{D}\,\triangle \tag{11}$$

where $\mathcal{V} = \acute{d}\,x/dt$ is the complex velocity and $\mathcal{D}$ is a re-expression of the previously defined fractal-nonfractal transition, namely $\mathcal{D} = \lambda c/2$). This operator plays the role of a 'quantum-covariant' derivative, that describes the effects induced by the internal fractal structures on the dynamics.

Since the velocity $\mathcal{V}$ is complex, the same is true of the Lagrange function, then of the generalized action $\mathcal{S}$. Setting $\psi = e^{i\mathcal{S}/2m\mathcal{D}}$, Newton's equation of dynamics becomes $m\,\acute{d}\,\mathcal{V}/dt = -\nabla\phi$, and can be integrated in terms of a generalized Schrödinger equation [12]:

$$\mathcal{D}^2\,\triangle\,\psi + i\mathcal{D}\frac{\partial}{\partial t}\psi = \frac{\phi}{2m}\,\psi. \tag{12}$$

In the free case the strong covariance principle (i.e. generalized equivalence principle) is implemented, since the equation of motion is written using the covariant derivative in the form of a free motion / geodesics equation, namely $\acute{d}^{\,2}x/dt^2 = 0$, then it becomes the free Schrödinger equation. This result has been generalized to the Klein-Gordon equation [13, 14] (by jumping to a fractal space-time instead of only a fractal space), then to the Dirac equation [2] (by introducing also the discrete symmetry breaking on the reflection $dx^\mu \leftrightarrow -dx^\mu$).

This result plays an important role in what follows when one jumps from classical to quantum electrodynamics. Indeed, it means that the quantum behavior and the appearance of the gauge field are both derived from the fractality and non-differentiability of space-time.

3 Nature of the electromagnetic field (classical theory)

3.1 Electromagnetic potential as dilation field

The theory of scale relativity allows one to get new insights about the nature of the electromagnetic field, of the electric charge, and about the physical meaning of gauge invariance [12]. Consider an electron (or any other charged particle). In scale relativity, we identify the particle with a family of fractal trajectories, described as the geodesics of a nondifferentiable space-time. These trajectories are characterized by internal (fractal) structures.

Now consider anyone of these structures, lying at some (relative) resolution ε smaller that the Compton length of the particle (i.e. such that $\varepsilon < \lambda_c$) for a given relative position of the particle. In a displacement of the particle, the relativity of scales implies that the resolution at which this given structure appears in the new position will a priori be different from the initial one. Indeed, if the whole internal fractal structure of the electron was rigidly fixed, this would mean an absolute character of the scale space and a description of the fractal set of trajectories in terms of fractal rigid objects, which would be clearly irrelevant.

Therefore we expect the occurrence of dilatations of resolutions induced by translations, which read:

$$Q\frac{\delta\varepsilon}{\varepsilon} = -A_\mu \delta x^\mu. \tag{13}$$

In this expression, the elementary dilation is written as $\delta\varepsilon/\varepsilon = \delta \ln(\varepsilon/\lambda)$: this is justified by the Gell-Mann-Levy method, from which the dilation operator is found to take the form $\tilde{D} = \varepsilon\partial/\partial\varepsilon = \partial/\partial \ln \varepsilon$. Since the elementary displacement in space-time δx^μ is a four-vector and since $\delta\varepsilon/\varepsilon$ is a scalar, one must introduce a four-vector A_μ in order to ensure covariance. The constant Q measures the amplitude of the scale-motion coupling; it will be subsequently identified with the active electric charge that intervenes in the potential. This form ensures that the dimensionality of A_μ be CL^{-1}, where C is the electric charge unit (e.g., $\varphi = Q/r$ for a Coulomb potential).

This behaviour can be expressed in terms of a scale-covariant derivative:

$$Q\, D_\mu \ln(\lambda/\varepsilon) = Q\, \partial_\mu \ln(\lambda/\varepsilon) + A_\mu. \tag{14}$$

This is reminiscent of Einstein's contruction of generalized relativity of motion, in which the Christoffel components $\Gamma^\mu_{\nu\varrho}$ can be introduced directly from the mere principle of relativity of motion by the following reasoning (see e.g. ref. [8]). Since space-time is relative, a vector can not stay identical to itself in a space-time displacement (the reverse would mean absolute space-time). Therefore in a displacement δx^ρ, a vector V^μ is subjected to a rotation $\delta V^\mu = \Gamma^\mu_{\nu\rho} V^\nu \delta x^\rho$, where the three-component Christoffel symbol should be introduced under the constraint of the Einstein convention upon indices. The subsequent developments of the theory allow to identify the Christoffels with the gravitational field components, since they are expressed in terms of derivatives of the gravitational metric potentials.

3.2 Nature of gauge invariance

Let us go one with the dilation field A_μ. If one wants such a "field" to be physical, it must be defined whatever the initial scale from which we started. Moreover, the principle of scale relativity also means that, a scale being always relative to another reference scale (that defines the state of scale of the reference system) the scale can change for two equivalent reasons (which are in the end undistinguishable): either because of a scale change while the scale of reference is kept fixed, or because of a change of the reference scale itself. This is the same situation as in the case of motion laws: namely, it is equivalent to move an object 1 with respect to an object 0 (that serves as reference) or to make the reverse motion of object 0 relatively to object 1.

Therefore, starting from another relative scale $\varepsilon' = \varrho\varepsilon$ (we consider Galilean scale-relativity for the moment), where the scale ratio ϱ may be any function of coordinates, $\varrho = \varrho(x, y, z, t)$, we get

$$Q\, \frac{\delta\varepsilon'}{\varepsilon'} = -A'_\mu\, \delta x^\mu, \tag{15}$$

so that we obtain:

$$A'_\mu = A_\mu + Q\, \partial_\mu \ln \varrho(x, y, z, t). \tag{16}$$

Therefore the 4-vector A_μ depends on the relative "state of scale", or "scale velocity", $\ln \varrho = \ln(\varepsilon/\varepsilon')$.

We have suggested [13, 14] to identify A_μ with an electromagnetic 4-potential and Eq. (16) with the gauge invariance relation for the electromagnetic field, that writes in the standard way:

$$A'_\mu = A_\mu + e\, \partial_\mu \chi(x, y, z, t), \tag{17}$$

where χ is usually considered as a function of coordinates devoid of physical meaning. This is no longer the case here, since it is now identified with a scale ratio $\chi = \ln \varrho$ between internal structures of the electron geodesics (at scales smaller than its Compton length). Our interpretation of the nature of the gauge function is compatible with its inobservability. Indeed, such a scale ratio is impossible to measure explicitly, since it would mean to make two measurements of two different relative scales smaller than the electron Compton length. But the very first measurement with resolution ε would change the state of the electron: namely, just after the measurement, its de Broglie length would become of order $\lambda_{\rm dB} \approx \varepsilon$ (see e.g. [12]), so that the second scale ε' would not be measured on the same electron. Therefore the ratio $\varrho = \varepsilon'/\varepsilon$ is destined to remain a virtual quantity. However, as we shall see in what follows, even whether it can not be directly measured, it has indirect consequences, so that the knowledge of its nature finally plays an important role: it allows one to demonstrate the quantization of the electron charge and to relate its value to that of its mass.

Note that, from the viewpoint of the scale-relativity theory, the scale velocity has now become a field, and the gradient of this field $\partial_\mu \ln \varrho$ now intervene in an explicit way in the equations of physics. Remembering that it is ultimately defined as a first order derivative in terms of the fifth dimension or 'djinn" δ, namely,

$$\ln \varrho = \frac{d \ln \mathcal{L}}{d\delta}, \tag{18}$$

this means that this gradient corresponds to a second order derivative that mixes the space-time and scale variables:

$$\partial_\mu \ln \varrho = \frac{d^2 \ln \mathcal{L}}{\partial X^\mu d\delta}. \tag{19}$$

This justifies our above claim that the understanding of the gauge fields in the scale-relativity framework comes under the question of scale-motion coupling in generalized scale-relativity, that indeed involves non-linear, second-order scale transformations.

3.3 Electromagnetic field

Let us now show that the subsequent developments of the properties of this dilation field support its interpretation in terms of electromagnetic potentials.

If one considers translation along two different coordinates (or, in an equivalent way, displacement on a closed loop), one may write a commutator relation (once again, in analogy with the definition of the Riemann tensor in Einstein's general relativity):

$$e\,(\partial_\mu D_\nu - \partial_\nu D_\mu) \ln(\lambda/\varepsilon) = (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{20}$$

This relation defines a tensor field:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{21}$$

which, contrarily to A_μ, is independent of the initial relative scale (i.e., of the gauge). One recognizes in $F_{\mu\nu}$ the expression for the electromagnetic field.

Before continuing the development of this new approach to the nature of electromagnetism, let us remark that in this interpretation, the property of gauge invariance recovers its initial status of scale invariance, in accordance with Weyl's initial ideas [25]. However, equation (16) represents a progress compared with these early attempts and with the status of gauge invariance in today's physics. Indeed the gauge function, which has, up to now, been considered as arbitrary and devoid of physical meaning, is now identified with the logarithm of internal resolutions. In Weyl's theory [25], and in its formulation by Dirac [3], the metric element ds (and consequently the length of any vector) is no longer invariant and can vary from place to place in terms of some (arbitrary) scale factor. Such a theory was excluded by experiment, namely by the existence of universal and unvarying lengths such as the electron Compton length. In scale relativity, we are naturally led to introduce two "proper times", the classical one ds which remains invariant, and the fractal one $d\mathcal{S}$, which is scale-divergent and can then vary from place to place. In Galilean scale-relativity, the fractal dimension of geodesics is $D = 2$, so that the scale-dependence of $d\mathcal{S}$ writes $d\mathcal{S} = d\sigma(\lambda/\varepsilon)$ (see [14]). Therefore we have:

$$\delta(d\mathcal{S})/d\mathcal{S} = -\delta\varepsilon/\varepsilon = \frac{1}{Q}\, A_\mu \delta x^\mu, \tag{22}$$

and we recover the basic relation of the Weyl-Dirac theory. But the problem encountered by the Weyl approach is now solved, since the explicit dependence on resolutions exists only in the asymptotic high energy domain, i.e. below the Einstein-Compton scale ($\varepsilon_t < \tau = \hbar/mc^2$). Therefore the Compton scale remains an invariant scale, while the fractal structures which take place at smaller scales are subjected to differential expansion and contraction.

3.4 Nature of electric charge (classical theory)

The fundamental new point of scale relativity with respect to the standard view is the fractal nature of space-time, i.e. its explicit dependence on the relative resolution scale, which is characterized by $\ln \varrho = \ln(\lambda/\varepsilon)$ in the simplified case of a global dilation. In other word, the space of positions and instants must be completed by a space of scales. Therefore we expect the action to be an explicit function of $\ln \varrho$, namely,

$$S = S(x^\mu,\ u^\mu,\ \ln \varrho), \tag{23}$$

where $\ln \varrho = \ln \varrho(x, y, z, t)$ is now a field. Therefore we may write:

$$S = \int \left(\frac{\partial S}{\partial x^\mu} dx^\mu + \frac{\partial S}{\partial \ln \varrho} \, \frac{D \ln \varrho}{\partial x^\mu} \, dx^\mu \right) \tag{24}$$

The first term in the integral is the classical free particle action, i.e. $-mc \int ds$. In the second term, the geometric effect of the fractality of space-time leads one to replace the standard derivative $\partial_\mu(\ln \varrho)$ by the covariant derivative (Eq. 14) $D_\mu \ln \varrho = \partial_\mu \ln \varrho + (1/Q)A_\mu$. Now, by a scale transformation on internal resolutions (i.e., a gauge transformation in the framework of the present new interpretation), it reduces to $(1/Q)A_\mu$, so that the second term in the action becomes:

$$\frac{1}{Q} \, \frac{\partial S}{\partial \ln \varrho} \, A_\mu dx^\mu \tag{25}$$

What is the meaning of the derivative $\partial S/\partial(Q \ln \varrho)$? Noether's theorem tells us that universal conservative quantities must emerge from the symmetries of the underlying space variables x^i; moreover, when considering the action as a function of coordinates at the upper limit of integration in the action integral, one finds that the conservative quantities are given by $p_i = \partial_i S$ [7]. Now, space-time is completed in the scale-relativity framework by a scale space. Therefore, from the uniformity of the new scale variable $\ln \varrho$, a new conservative quantity can be constructed, so that we may write:

$$\frac{1}{Q} \, \frac{\partial S}{\partial \ln \varrho} = \frac{-q}{c}, \tag{26}$$

which defines the 'passive' charge of the particle subjected to the potential A_μ. In other words, the electric charge is defined in the new approach as the conservative quantity that arises from the uniformity of the scale space. The above choice ($\partial S/\partial \ln \varrho = -Qq/c$) is motivated by the expected symmetry of the 'active' and 'passive' charges in the final potential energy, and by the fact that the action has the dimensionality of an angular momentum [$\mathrm{ML}^2\mathrm{T}^{-1}$], while the squared charge dimensionality is [$\mathrm{ML}^3\mathrm{T}^{-2}$].

Finally, the known form of the coupling term in the action is now demonstrated, while it was merely postulated in the standard theory:

$$S_{\mathrm{pf}} = \int -\frac{q}{c} A_\mu dx^\mu. \tag{27}$$

3.5 Lorentz force and Maxwell equations

We have now at our disposal all the foundations needed to construct the known form of the classical theory of electromagnetism. Indeed, the standard form of the particle-field coupling term in the action has now been established from first principles. The field term of the action, as recalled in the introduction, is forced to be given by the square of the electromagnetic tensor. Therefore, the total action writes:

$$S = S_{\mathrm{p}} + S_{\mathrm{pf}} + S_{\mathrm{f}} = -\int mc \, ds - \int \frac{q}{c} A_\mu dx^\mu - \frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} d\Omega. \tag{28}$$

The variational principle applied on the two first terms of this action yields the motion equation and therefore the known expression for the Lorentz force acting on charge q:

$$mc \, \frac{du_\mu}{ds} = \frac{q}{c} F_{\mu\nu} u^\nu. \tag{29}$$

The variational principle applied on the two last terms (after their generalization to the current of several charges) yields Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} j^\nu. \tag{30}$$

In conclusion of this section, the progress here respectively to the standard classical electromagnetic theory is that, instead of being independently constructed, the Lorentz force and the Maxwell equations are derived in the scale relativity theory as being both manifestations of the fractal geometry of space-time. Moreover, a new physical meaning can be given to the electric charge and to gauge transformations in this framework.

We shall now consider its consequences for the quantum theory of electrodynamics.

4 Scale-relativistic quantum electrodynamics

4.1 Analysis of the problem

It is well-known that the quantum theory of electromagnetism and of the electron has added a new and essential stone in our understanding of the nature of charge. Indeed, in its framework, gauge invariance becomes deeply related to phase invariance of the wave function. The electric charge conservation is therefore directly related to the gauge symmetry. However, despite the huge progress that such a success has been (in particular, the extension of the approach to non-Abelian gauge theories has allowed to incorporate the weak and strong field into the same scheme) the lack of a fundamental understanding of the nature of gauge transformations has up to now prevented from reaching the final goal of gauge theories: namely, understand why charge is quantized and, as a consequence, theoretically calculate its quantized value.

Let us indeed consider the wave function of an electron. It writes:

$$\psi = \psi_0 \, \exp\left\{ \frac{i}{\hbar}(px - Et + \sigma\varphi + e\chi) \right\} \tag{31}$$

Its phase contains the usual products of fundamental quantities (space position, time, angle) and of their conjugate quantities (momentum, energy, angular momentum). They are related through Noether's theorem. Namely, the conjugate variables are the conservative quantities that originate from the space-time symmetries. This means that our knowledge of what are the energy, the momentum and the angular momentum and of their physical properties is founded on our knowledge of the nature of space, time and its transformations (translations and rotations).

This is true already in the classical theory, but there is something more in the quantum theory. In its framework, the conservative quantities are quantized when the basic variables are limited. Concerning energy-momentum, this means that it is quantized only in some specific circumstances (e.g., bound states in atoms for which $r > 0$ in spherical coordinates). The case of the angular momentum is instructive: its differences are quantized in an universal way in units of $\hbar$ because angles differences can not exceed 2π.

In comparison, the last term in the phase of Eq. (31) keeps a special status in today's standard theory. The gauge function χ remains arbitrary, while it is clear from a comparison with the other terms that the meaning of charge e and the reason for its universal quantization can be obtained only from understanding the physical meaning of χ and why it is universally limited, since it is the quantity conjugate to the charge. As we shall now see, the identification of χ with the resolution scale factor $\ln \varrho$ that we have developped in the previous sections in the classical framework, can be transported to the quantum theory and allows one to suggest solutions to these problems in the special scale-relativity framework.

4.2 QED covariant derivative

Let us first show how one can recover the standard QED quantum derivative in the scale-relativity approach. Let us consider again the generalized action introduced in the previous section, which depends on motion and on scale variables. In the scale-relativistic quantum description, the 4-velocity is now complex (see Sec. 2.4), so that the action writes, $\$ = \$(x^\mu, \mathcal{V}^\mu, \ln \varrho)$. This action gives the fundamental meaning of the wave function, namely, ψ is defined as:

$$\psi = e^{i\$/\hbar}. \tag{32}$$

Since the action is a complex number (and becomes a complex quaternion in the generalized case that leads to the demonstration of the Dirac equation [2]), this expression contains a phase and a modulus (that becomes in the end a square-root of probability density).

The decomposition performed in the framework of the classical theory (Eq. 24) still holds and now becomes (we now take for q an electron charge e):

$$d\$ = -i\hbar \, d\ln\psi = -mc\mathcal{V}_\mu dx^\mu - \frac{e}{c} A_\mu dx^\mu. \tag{33}$$

Equation (33) allows one to define a new generalized complex four-momentum,

$$\tilde{\mathcal{P}}^\mu = \mathcal{P}^\mu + \Delta P^\mu = mc\mathcal{V}^\mu + \frac{e}{c}\, A^\mu. \tag{34}$$

This leads to a new expression for the velocity:

$$\mathcal{V}_\mu = i\lambda\, \partial_\mu(\ln\psi) - \frac{e}{mc^2}A_\mu, \tag{35}$$

where $\lambda = \hbar/mc$ is the Compton length of the electron.

Therefore we can generalize the identity $\mathcal{V}_\mu = i\lambda\, \partial_\mu(\ln\psi)$ to its covariant form by introducing a covariant derivative:

$$\mathcal{V}_\mu = i\lambda\, D_\mu(\ln\psi). \tag{36}$$

We recognize in this derivative the standard QED-covariant derivative operator acting on the wave function ψ:

$$-i\hbar D_\mu = -i\hbar\partial_\mu + \frac{e}{c}\,A_\mu, \tag{37}$$

since we can write Eq. (35) as $mc\mathcal{V}_\mu\psi = [i\hbar\partial_\mu - (e/c)A_\mu]\;\psi$.

We have therefore reached an understanding from first principles of the nature and origin of the QED covariant derivative, while it was merely set as a rule devoid of geometric meaning in the standard quantum field theory.

This covariant derivative is directly related to the previous one introduced in the classical framework. Indeed, the classical covariant derivative was written $D_\mu = \partial_\mu + (1/Q)A_\mu$ acting on ϱ, while $\psi = \psi_0\; \exp[(i/\hbar)(eQ/c)\ln\varrho]$. We therefore recover expression (37) acting on ψ.

4.3 Klein-Gordon equation in electromagnetic field

Let us consider the free particle / geodesics equation written in terms of the covariant derivative built from a fractal space-time (i.e., in the motion-relativistic case). It writes:

$$\hat{d}^{\,2}x^\nu/ds^2 = \left(\mathcal{V}_\mu + i\frac{\lambda}{2}\partial_\mu\right)\partial^\mu\mathcal{V}^\nu = 0. \tag{38}$$

After a change of variable ($\mathcal{V}_\mu = i\lambda\;\partial_\mu\ln\psi$), it can be integrated to give the Klein-Gordon equation:

$$[i\hbar\partial_\mu][i\hbar\partial^\mu]\psi = m^2c^2\psi. \tag{39}$$

In order to obtain the Klein-Gordon equation with electromagnetic field, it is not sufficient to replace $\mathcal{V}$ by its new expression directly in Eq. (38), because, while $\partial^\mu\mathcal{V}^\nu - \partial^\nu\mathcal{V}^\mu = 0$, we find from Eq. (35) that it no longer commutes in the electromagnetic case (see also [24]):

$$\partial^\mu\mathcal{V}^\nu - \partial^\nu\mathcal{V}^\mu = -\frac{e}{mc^2}F^{\mu\nu}. \tag{40}$$

Now, in the free equation (38) the indices can be exchanged. It becomes

$$\left(\mathcal{V}_\mu + i\frac{\lambda}{2}\partial_\mu\right)\partial^\nu\mathcal{V}^\mu = 0. \tag{41}$$

Starting from this new form of the free equation, we can now make QED-covariant the geodesics equation. We replace $\mathcal{V}^\mu$ by its covariant form and we obtain:

$$\left(i\lambda\;\partial_\mu(\ln\psi) - \frac{e}{mc^2}A_\mu + i\frac{\lambda}{2}\partial_\mu\right)\partial^\nu\left(i\lambda\;\partial^\mu(\ln\psi) - \frac{e}{mc^2}A^\mu\right) = 0. \tag{42}$$

After integration, this equation takes the form of the Klein-Gordon equation for a particle in an electromagnetic field:

$$[i\hbar\partial_\mu - (e/c)A_\mu][i\hbar\partial^\mu - (e/c)A^\mu]\psi = m^2c^2\psi. \tag{43}$$

Note that this process does not work when inserting the covariant velocity directly in Eq. 38, which means that the covariance is not yet fully implemented at this stage. This problem has been solved by Pissondes [24], thanks to the introduction of a symmetric product that accounts for the second order term in the quantum-covariant derivative.

4.4 Nature of the electric charge (quantum theory)

In a gauge transformation $A'_\mu = A_\mu + e\partial_\mu\chi$ the wave function of an electron of charge e becomes:

$$\psi' = \psi \, \exp\left\{ \frac{i}{\hbar} \times \frac{e}{c} \times e\chi \right\}. \tag{44}$$

We have reinterpreted in the previous sections the gauge transformation as a scale transformation of resolution, $\varepsilon \to \varepsilon'$, yielding an identification of the gauge function with a scale ratio, $\chi = \ln \varrho = \ln(\varepsilon/\varepsilon')$, which is a function of space-time coordinates. In such an interpretation, the specific property that characterizes a charged particle is the explicit scale-dependence on resolution of its action, then of its wave function. The net result is that the electron wave function writes

$$\psi' = \psi \exp\left\{ i\frac{e^2}{\hbar c} \ln \varrho \right\}. \tag{45}$$

Since, by definition (in the system of units where the permittivity of vacuum is 1),

$$e^2 = 4\pi\alpha\hbar c, \tag{46}$$

where α is the fine structure constant, equation 45 becomes

$$\psi' = \psi \, e^{i4\pi\alpha \ln \varrho}. \tag{47}$$

This result supports the previous solution brought to the problem of the nature of the electric charge in the classical theory. Indeed, considering now the wave function of the electron as an explicitly resolution-dependent function, we can write the scale differential equation of which it is solution as:

$$-i\hbar \frac{\partial\psi}{\partial\left(\frac{e}{c}\ln\varrho\right)} = e\psi. \tag{48}$$

We recognize in $\tilde{D} = -i(\hbar c/e)\partial/\partial \ln \varrho$ a dilatation operator similar to that introduced in Section 3. Equation 48 can then be read as an eigenvalue equation issued from an extension of the correspondence principle (but here, demonstrated),

$$\tilde{D}\psi = e\psi. \tag{49}$$

This is the quantum expression of the above classical suggestion, according to which the electric charge is understood as the conservative quantity that comes from the new scale symmetry, namely, from the uniformity of the resolution variable $\ln \varepsilon$.

4.5 Charge quantization and mass-coupling relations

While the results of the scale relativity theory described in the previous sections mainly deal with a new interpretation of the nature of the electromagnetic field, of the electric charge and of gauge invariance, we now arrive at the principal consequences of this approach: as we shall see, it allows one to establish the universality of the quantization of charges (for any gauge field) and to theoretically predict the existence of fundamental relations between mass scales and coupling constants.

In the previous section, we have recalled our suggestion [13, 14] to elucidate the nature of the electric charge as being the eigenvalue of the dilation operator corresponding to resolution transformations. We have written the wave function of a charged particle under the form Eq. (48).

Let us now consider in more detail the nature of the scale factor $\ln \varrho$ in this expression. This factor describes the ratio of two relative resolution scales ε and ε' that correspond to structures of the fractal geodesical trajectories that we identify with the electron. However the electron is not structured at all scales, but only at scales smaller than its Compton length $\lambda = \hbar/m_e c$. We can therefore take this upper limit as one of the two scales and write:

$$\psi' = \exp\left\{ i \; 4\pi\alpha \; \ln\left(\frac{\lambda}{\varepsilon}\right) \right\} \; \psi. \tag{50}$$

In the case of Galilean scale-relativity, such a relation leads to no new result, since ε can go to zero, so that $\ln(\lambda/\varepsilon)$ is unlimited. But in the framework of special scale-relativity, scale laws take a log-Lorentzian form below the scale λ (see Section 2). The Planck length l_P becomes a minimal, unreachable scale, invariant under dilations, so that $\ln(\lambda/\varepsilon)$ becomes limited by $\mathbb{C} = \ln(\lambda/l_P)$. This implies a quantization of the charge which amounts to the relation $4\pi\alpha\mathbb{C} = 2k\pi$, i.e.:

$$\alpha\,\mathbb{C} = \frac{1}{2}k, \tag{51}$$

where k is integer. Since $\mathbb{C} = \ln(\lambda/l_p)$ and is equal to $\ln(m_P/m_e)$ for the electron (where m_P is the Planck mass), equation (51) amounts to a general relation between mass scales and coupling constants..

In order to explicitly apply such a relation to the electron, we must account for the fact that we now know from the electroweak theory that the electric charge is only a residual of a more general, high energy electroweak coupling. This coupling can be defined from the U(1) and SU(2) couplings as:

$$\alpha_0^{-1} = \frac{3}{8}\alpha_2^{-1} + \frac{5}{8}\alpha_1^{-1}. \tag{52}$$

It is such that $\alpha_0 = \alpha_1 = \alpha_2$ at unification scale and it is related to the fine structure constant at Z scale by the relation $\alpha = 3\alpha_0/8$. This means that, because the weak gauge bosons acquire mass through the Higgs mechanism, the interaction becomes transported at low energy only by the residual null mass photon. As a consequence the amplitude of the electromagnetic force abruptly falls by a factor 3/8 at the WZ scale. Therefore it is α_0 instead of α which must be used in Eq. (51) for relating the electron mass to its charge.

Finally, disregarding as a first step threshold effects (that occur at the Compton scale), we get a mass-charge relation for the electron [13, 14]:

$$\ln\frac{m_P}{m_e} = \frac{3}{8}\alpha^{-1}. \tag{53}$$

The existence of such a relation between the mass and the charge of the electron is supported by the experimental data. Indeed, using the known experimental values, the two members of this equation agree to 0.2 %: $\mathbb{C}_e = \ln(m_P/m_e) = 51.528(1)$ while $(3/8)\alpha^{-1} = 51.388$. The agreement is made even better if one accounts from the fact that the measured fine structure constant (at Bohr scale) differs from the limit of its asymptotic behavior (that includes radiative corrections). One finds that the asymptotic inverse running coupling at the scale where the asymptotic running mass reaches the observed mass m_e is $\alpha_0^{-1}\{r(m = m_e)\} = 51.521$, which lies within 10^{-4} of the value of $\mathbb{C}_e$.

4.6 Possible implications for the electron mass

One of the possible ways to interpret the above mass-charge relation consists of concluding that most of the electron mass is of electroweak origin, i.e., to write it under the form:

$$m_e = m_P \times e^{-\frac{3}{8}\alpha^{-1}}. \tag{54}$$

This relation can also be viewed as an equation for the running mass and the running coupling, whose solution is precisely given by the observed mass and the observed charge of the electron. Namely, using the solution to its renormalization group equation, one runs the electromagnetic inverse coupling $\alpha_0^{-1}(m)$ from infinite mass-energy to Z mass scale, then at lower energies, accounting for the various particle-antiparticle pairs that contribute in the radiative corrections, i.e., in the scale variation of the charge (see e.g. [14] for a detailed calculation to order two). The physical electron mass and charge are finally given by the point in the (mass scale, inverse coupling) diagram where the line $\ln(m/m_P)$ crosses the function $\alpha_0^{-1}(m)$ (up to small threshhold effect corrections). Such a calculation is made possible in special scale relativity because, thanks to the new status of the Planck length, quantum electrodynamics is no longer divergent in its framework. Indeed, in terms of the log-Lorentzian dilation laws, the Planck mass scale corresponds to the grand unification scale, while the infinite energy scale now corresponds to the Planck length-scale. A finite value of the bare (infinite energy) charge is therefore expected.

Reversely, from the renormalization group equations and the conjecture that the bare inverse coupling is $\alpha_0^{-1}(\infty) = 4\pi^2$, and using recent precision experimental data [23], one finds a value of the fine structure constant at the Compton scale of the electron given by [11, 12, 14, 16, 17]:

$$\alpha^{-1} = 137.04 \pm 0.03 + 2.11(N_H - 1), \tag{55}$$

where N_H is the number of Higgs doublets. Since the experimental value is 137.036, this establishes the number of real Higgs doublets as only one with a high confidence level.

However such a result may have experimental consequences. One may contemplate the possibility that the future particle physics be able to construct new boson fields in some specific situations, e.g. from fermion pairs (like the Cooper pairs in superconductivity), which could play the role of additional effective Higgs fields. If such a 'Higgs / vacuum' technology is one day possible, its consequences would be (i) to change the fine structure constant following the above relation, i.e. by about 2% for each additional effective Higgs doublet; (ii) to change the mass of the electron following Eq. (54), i.e. by an important amount:

$$m'_e = m_e \times 2.20^{-N_H+1}. \tag{56}$$

This result implements (for the moment in terms of a thought experiment) the scale relativity of mass (first stressed by Ernst Mach), since even the mass of a very elementary stable particle like the electron is here shown to be defined only through a ratio, and, moreover, may vary relatively to the organization of other particles and their associated vacuum.

5 Discussion and conclusion

Despite the results already obtained in this approach, including (i) a geometric understanding from first principles of the nature of electric charge, electromagnetism and gauge invariance, (ii) a demonstration of the quantization of charge and of the existence of a general relation between mass scales and coupling constants, (iii) a proposal of application to a possible experimental variation of the mass of the electron (although it is yet, for the time being, a thought experiment), there is still a huge work to be done.

Even though the Maxwell equations and the Lorentz force are derived, in the scale relativity framework, from the same geometric background (namely, the particle-field coupling term in the action is now demonstrated rather than postulated), this is not the last word on the subject. Indeed, a complete success concerning a motion equation coming under a relativity theory would be to write it in terms of a free motion / geodesics equation of the form $D^2x^\mu/ds^2 = 0$ (according to a generalized equivalence principle / strong covariance principle), which is not yet achieved in the present work.

Another fundamental uncompleteness of the theory in its present status deals with the fact that the electromagnetic field is now known to be only a subpart of a larger electroweak, maybe even electroweakstrong, field.

However, only global dilations of resolutions $\varepsilon'_\mu = \varrho\varepsilon_\mu$ have been considered here as a simplifying first step. But the theory allows one to work with four different and independent dilations along the four space-time resolutions. It is then clear that the electromagnetic field is indeed expected in the scale relativity approach to be embedded into a larger field and the electric charge to be one element of a more complicated, "vectorial" charge. We shall consider in forthcoming works the possibility to recover in this way the electroweak theory or a generalization of it (see [16, 17, 18] for first tentative attempts in this direction, including a theoretical prediction of the Higgs mass at $\sqrt{2}m_W = 113.7$ GeV).

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