Structuring of the semi-major axis and eccentricity distributions of exoplanets*

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February 17, 2004

Abstract

We describe the motion of planetesimals in the protoplanetary nebula in terms of a fractal and irreversible process. As a consequence the equation of dynamics can be transformed to take a Schrödinger-like form. Its solutions yield a planetesimal distribution showing peaks of probability for particular values of conservative quantities such as the energy and the Runge-Lenz vector. After accretion, this results in expected probability peaks of the semi-major axis distribution at $a_n = (GM/w^2)n^2$, and of the eccentricity distribution at $e = k/n$, where k and n are integer numbers, M is the star mass and w is a constant having the dimension of a velocity. The current observational data support these predictions in a statistically significant way, in terms of a constant $w = 144.7 \pm 0.5$ km/s which is common to the inner solar system and to the presently known extrasolar systems.

1 Fractal description of protoplanetary nebulae

The standard model of formation of planetary systems is reconsidered in terms of a fractal description of the motion of planetesimals in the protoplanetary nebula. On length-scales much larger than that their mean free path, we assume that their highly chaotic motion fulfills the three following conditions:
(i) Infinite (or very large) number of possible trajectories (loss of determinism);
(ii) Each potential trajectory is a fractal curve of fractal dimension $D_F = 2$ (Markovian, Brownian-like motion);
(iii) "Microscopic" irreversibilty (breaking of the reflection invariance under the

*Poster shown at the XIXth IAP colloquium "Extrasolar Planets: Today and Tomorrow", 30 June-4 July 2003

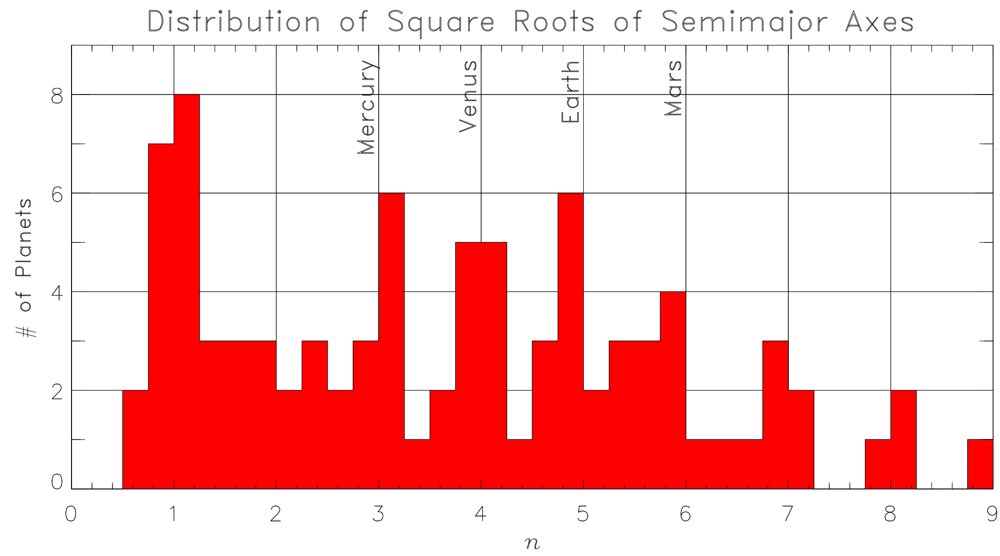

Figure 1: 89 planets such that $\sigma_n < 0.25$: $n = w_0(P/2\pi GM_*)^{1/3}$ with $w_0 = 144$ km/s.

transformation $dt \leftrightarrow -dt$ (consequence of nondifferentiability).
As a consequence of these three conditions, the elementary displacements dX along fractal trajectories during the time interval dt are decomposed as the sum of two terms: $dX = dx + d\xi$, where x is a differentiable position coordinate such as $dx = vdt$, and $d\xi$ is a fluctuation of fractal dimension 2 that fulfills the conditions: $< dx >= 0$ and $< d\xi^2 >= 2\mathcal{D}dt$ (note that such a law comes under the principle of scale-relativity, in the 'Galilean' case [1]). The parameter $\mathcal{D}$, which is introduced fro dimensional reasons, represents a charateristic length-scale for the system. As the motion is supposed to be locally irreversible, the time-derivative becomes two-valued. One defines (d_+/dt) and (d_-/dt), then they are combined to form a complex derivative operator [1]:

$$\frac{\acute{d}}{dt} = \frac{(d_+ + d_-) - i(d_+ - d_-)}{2\,dt}, \tag{1}$$

which allows one to define a new complex velocity:

$$\mathcal{V} = \frac{\acute{d}\,x}{dt} = \frac{v_+ + v_-}{2} - i\,\frac{v_+ - v_-}{2} = V - iU. \tag{2}$$

One can prove that the complex total derivative along fractal trajectories becomes [1,2]:

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V} \cdot \nabla - i\,\mathcal{D}\,\triangle\,. \tag{3}$$

The effects of the fractality and nondifferentiability of trajectories are now included in this new total derivative. Now replacing the standard time derivative d/dt by the complex operator $\acute{d}/dt$ in the equations of dynamics, the latter can be integrated in terms of a generalized Schrödinger equation that reads:

$$\mathcal{D}^2 \triangle \psi + i\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\Phi}{2}\,\psi = 0, \tag{4}$$

where $\psi = \exp(iS/2m\mathcal{D})$ is a mere re-expression of the action S. The standard description is recovered in the limit $\mathcal{D} \to 0$, since this equation is equivalent to an (Euler-Newton + continuity) equation system, namely,

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) V = -\nabla \left(\Phi - \mathcal{D}^2 \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \quad \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho V) = 0, \tag{5}$$

where $\rho = |\psi|^2$ is a density of probability, which is proportional to the density of matter provided it is applied to a large ensemble of planetesimals.

We assume that such a method can be applied to the distribution of planetesimals in a protoplanetary nebula which has formed in the potential $\Phi = -GM/r$ of a star of mass M. During the planetesimal era, there is no defined orbital parameter such as semi-major axis a or eccentricity e. But the solutions of the Schrödinger equation describe stationary states for which conservative quantities, such as the energy E, the projection on a given axis of the angular momentum, L_z, and of the Runge-Lenz vector, A_z, can have determined values. Once the planet formed from a distribution of planetesimals described by such a state and once the system stabilized, the planet recovers classical orbital parameters. Concerning for example semi-major axes, the planetesimals are expected to fill the 'orbital' characterized by a conservative energy E, then they finally accrete to yield a planet whose semi-major axis will be, with highest probability, given by $a = -GMm/2E$, according to conservation laws.

In other words, the Schrödinger regime applies to the planetesimals (for which there is no determined orbit, because of strong chaos), while the classical variables (i.e. the finally observed ones) concern the planets as they are now observed, i.e. after the end of the accretion process. But the conservative quantities are common to the two regimes, and they therefore allow us to do statistical theoretical predictions about the most probable values of the orbital parameters of planets.

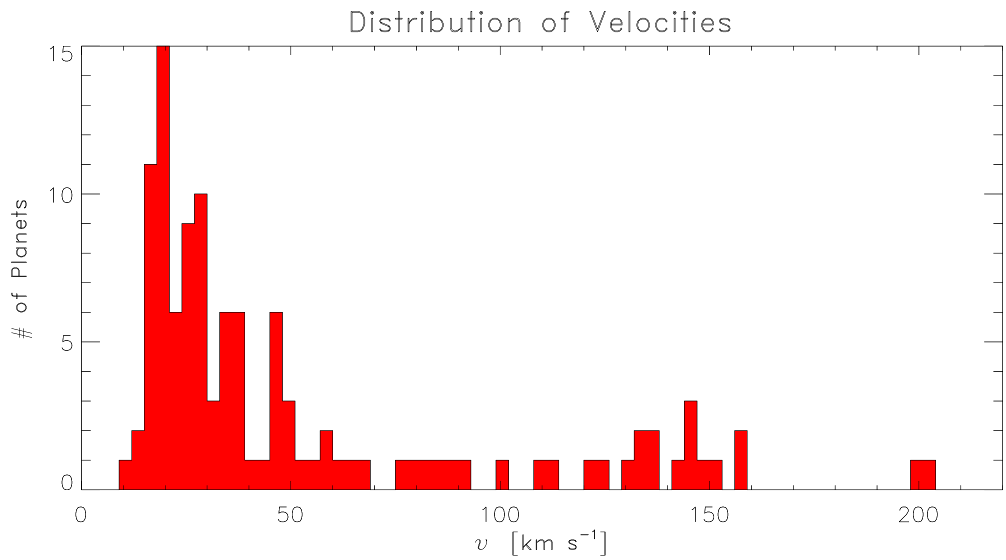

Figure 2: 113 inner solar system planets and exoplanets (all data): $v = (2GM_*/P)^{1/3}$

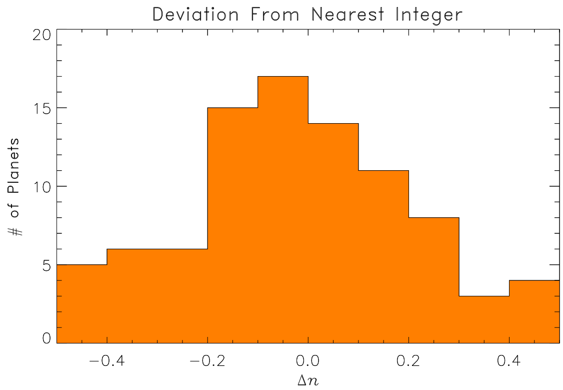

Figure 3: Histogram of the deviation from nearest integer of data in Figure 1. The probability to obtain by chance such a non-uniform distribution is: $P < 3 \times 10^{-4}$.

2 Distribution of semi-major axes

We consider a particle of mass m moving in a gravitational field of a central star of mass M, the potentiel of which is $\Phi = -GM/r$. The fundamental equation of dynamics becomes [2, 4, 6, 7, 8]:

$$\frac{\acute{d}\mathcal{V}}{dt} = -\nabla\Phi. \tag{6}$$

In the stationary case, this equation can be integrated in terms of the Schrödinger equation:

$$2\mathcal{D}^2 \triangle \psi + \left(\frac{E}{m} + \frac{GM}{r}\right) \psi = 0. \tag{7}$$

the solutions of which are generalized Laguerre polynomials. Since the physics must be independent of the mass m of the test particle (because of the equivalence principle), and since the star mass M gives the natural length-unit of such a Kepler gravitational system, the parameter $\mathcal{D}$ can be written as:

$$\mathcal{D} = \frac{GM}{2w_0}, \tag{8}$$

where w_0 is a fundamental constant which has the dimension of a velocity. Its typical value for systems whose gravitational potential is of the same order as in the inner solar system, $w_0 = 144.7 \pm 0.5$ km/s, has been determined in an independent way from various gravitational systems [4,8] (planetary systems, binary stars, binary galaxies). The dimensionless ratio $\alpha_g = w_0/c$ plays the role of a gravitational coupling constant [5, 6].

This description applies to the distribution of planetesimals in the proto-planetary nebula at several embedded levels of hierarchy. Each hierarchical level (j) is characterized by a length-scale defining the parameter $\mathcal{D}_j$ (and therefore a velocity w_j) of its corresponding generalized Schrödinger equation. The matching of the solutions implies that the ratios of the velocity parameters (w_j/w_{j-1}) be integer (see Figures).

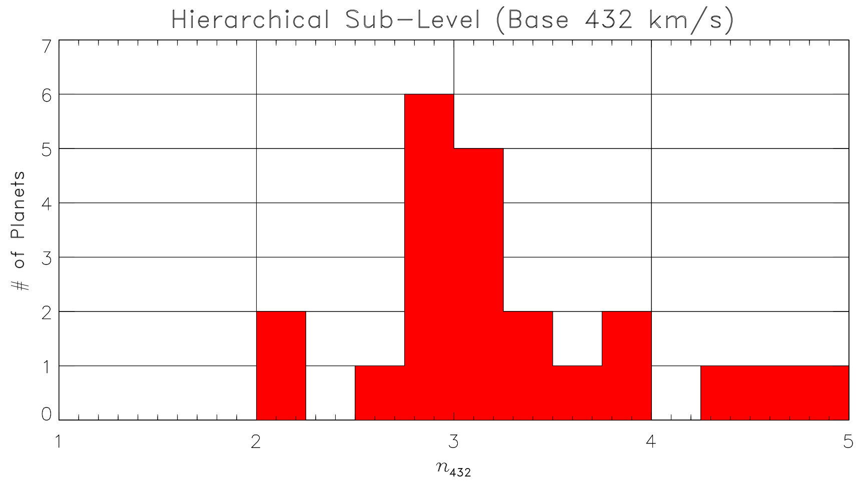

Figure 4: 23 planets and exoplanets such that $\sigma_{432} < 0.25$, $n_{432} < 5$, with $n_{432} = w_1(P/2\pi GM_*)^{1/3}$ and $w_1 = 432$ km/s.

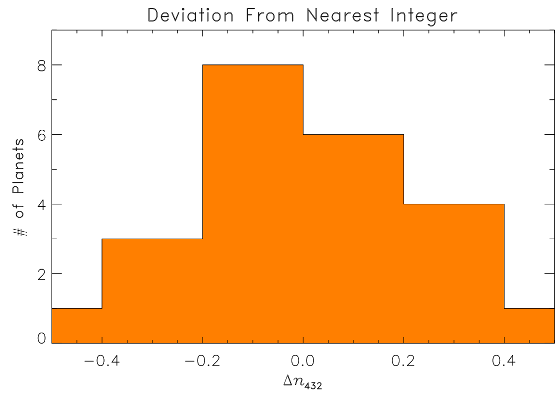

Figure 5: Histogram of the deviation from nearest integer of data in Figure 4. The probability to obtain by chance such a non-uniform distribution is $P < 0.017$.

After the end of the formation era, the distribution of the semi-major axes of planet orbits are expected to have peaks of probability for values

$$\frac{a_n}{GM} = \frac{n^2}{w^2}, \qquad (9)$$

where n is an integer and w a multiple or a submultiple of w_0, according to the level of hierarchy.

This law for the maxima of probability of semi-major axes distribution is supported by the observational data of planets of our solar system [10], their satellites [7], and also exoplanets [4,6,8,9] (see Figures 1-5 and 8-9: the data used for checking this theoretical prediction are taken from [11]) .

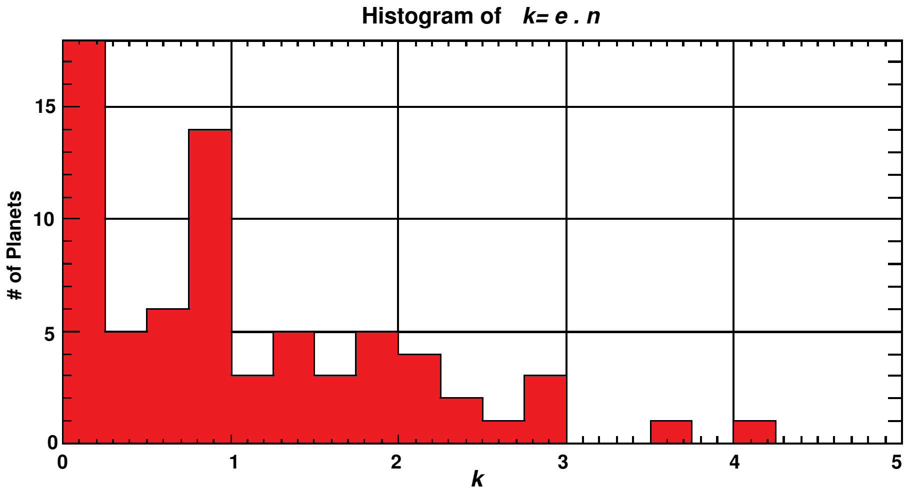

Figure 6: 83 planets and exoplanets such that $\sigma_k < 0.5$: $k = e \times [w_0(P/2\pi GM_*)^{1/3}]$ with $w_0 = 144$ km/s.

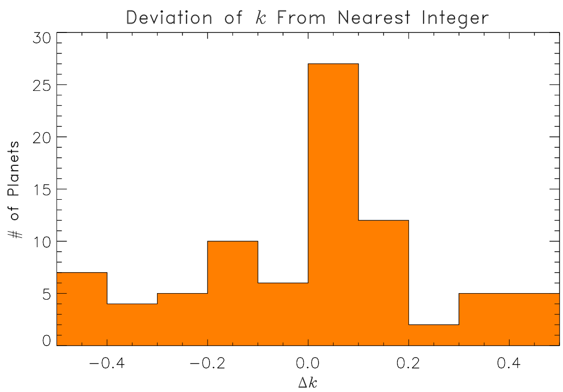

Figure 7: Histogram of the deviation from nearest integer of data in Figure 6. The probability to obtain by chance such a distribution is $P < 3 \times 10^{-5}$.

3 Distribution of Planet Eccentricities

The Schrödinger equation for the motion of a body in a Newtonian gravitational potential can be solved working in parabolic coordinates. In this case the solutions are states of well defined values for the energy E and the projections on a given axis z of the angular momentum L and of the Runge-Lenz vector A (which is a conservative quantity specific of the Kepler problem whose modulus is the eccentricity). By taking for the z-axis the semi-major axis of the orbit [8, 9], one therefore obtains a predicted quantization law for the eccentricity that reads:

$$A_z = e = \frac{k}{n}, \tag{10}$$

where k is an integer varying from 0 to $n-1$ and n is the principal 'quantum number' previously defined.

The histogram of the distribution of the product $n \times e$ for the planets of our solar system and for exo-planets (see Figures 6-8) supports the existence of peaks at integer values according to the theoretical predictions.

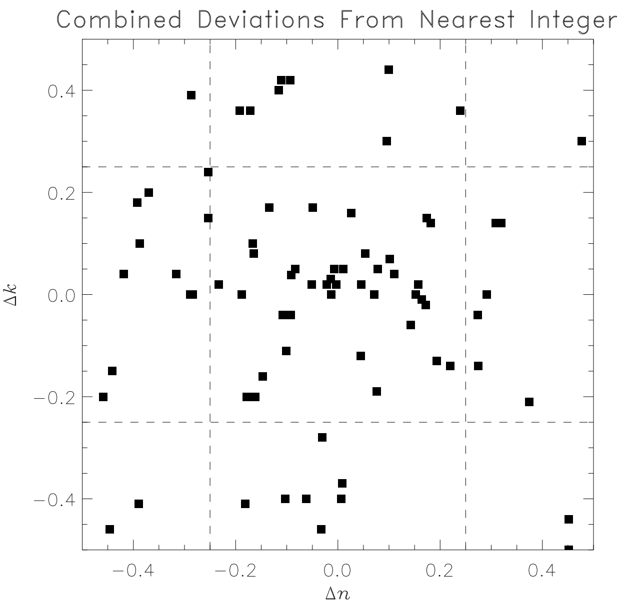

Figure 8: 78 planets and exoplanets such that $\sigma_n < 0.25$ and $\sigma_k < 0.5$. The probability to obtain by chance such a distribution is $P < 5 \times 10^{-8}$.

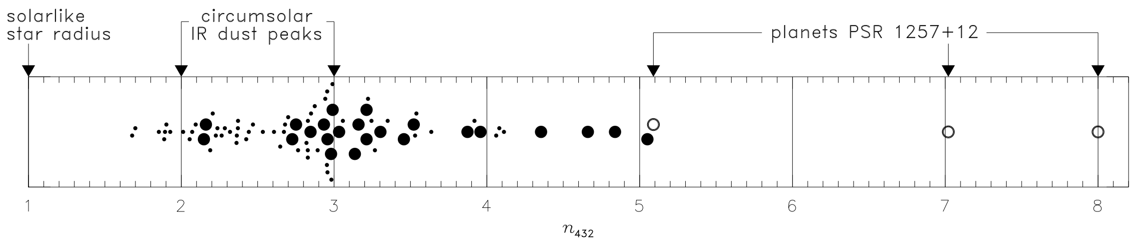

Figure 9: Comparison of the intramercurial sublevels of organization (base 432 km/s) in various planetary systems. Solar system: Sun radius ($n = 1$) and transient IR circumsolar dust peaks [8] ($n = 2$ and 3); exoplanets (large points); OGLE candidates (small points); three Wolszczan's planets around pulsar 1257+12 (with $M_{PSR} = 1.6 M_\odot$).

4 Conclusions

The results we have obtained agree in a statistically significant way with our quantitative prediction of a universality of structures in planetary systems [1,3]. Recall that this theoretical prediction has been done before the discovery of the exo-planets. One of us wrote in January 1994 [3]:"We can expect other [planetary] systems to be discovered in the forthcoming years, and new informations to be obtained about the very distant solar system (Kuiper's belt, Oort cometary cloud...). In this regard our theory is a falsifiable one, since it makes definite predictions about such observations of the near future: observables such as the distribution of eccentricities, mass, angular momentum, the preferred positions of planets and asteroids, or possibly the ratio of distance of the largest gazeous planet and the largest telluric one, are expected in our framework to be universal structures shared by any planetary system."

In conclusion, we recall that several features of the newly discovered exoplanets are a challenge to standard theories of formation, while they were quantitatively predicted in the new framework, in particular:

*The accumulation of exoplanets around $a/M = 0.043$ AU$/M_\odot$ (corresponding to a Keplerian velocity of 144 km/s), which is the fundamental 'orbital' of inner solar system-like planetary systems.

*The accumulation of exoplanets around the same a/M values as the planets of our own solar system.

*The existence of large eccentricities, possible even for isolated planets (since they are formed from accretion of groups of planetesimals defined by the same states of conservative quantities, instead of simple sweeping), and the existence of probability density peaks for their possible values.

*The existence of imbricated levels of organization for planetary systems, which begins to be unveiled for exoplanets, in correspondence with the intramercurial, inner and outer systems in our own solar system.

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