The Einstein equations on a computer: formulations and numerical solutions

Jérôme Novak (Jerome.Novak@obspm.fr)

Laboratoire Univers et Théories (LUTH)
CNRS / Observatoire de Paris / Université Paris-Diderot

based on collaboration with
Silvano Bonazzola, Philippe Grandclément,
Éric Gourgoulhon & Nicolas Vasset

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Relativistic gravity

In general relativity (1915), space-time is a four-dimensional Lorentzian manifold, where gravitational interaction is described by the metric $g_{\mu\nu}$.

**Einstein equations**

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \]

They form a set of 10 second-order non-linear PDEs, with very few (astro-)physically relevant exact solutions (Schwarzschild, Oppenheimer-Snyder, Kerr, ...). ⇒ approximate solutions: 

* e.g. linearizing around the flat (Minkowski) solution in vacuum $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

\[ \Box \left( h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \right) = -16\pi T_{\mu\nu}. \]
Gravitational waves

Astrophysical sources

Using the linearized Einstein equations:

- at first order \( h \sim \ddot{Q} \) (mass quadrupole momentum of the source), or further from the source \( h \sim \frac{G E^{(\ell \geq 2)}}{c^4 r} \).
- the total gravitational power of a source is

\[
L \sim \frac{G}{c^5} s^2 \omega^6 M^2 R^4.
\]

...introducing the Schwarzschild radius \( R_S = \frac{2GM}{c^2} \) and \( \omega = v/r \):

\[
L \sim \frac{c^5}{G} s^2 \left( \frac{R_S}{R} \right)^2 \left( \frac{v}{c} \right)^6
\]

\( \Rightarrow \) non-spherical, relativistic compact objects:

- binary neutron stars or black holes,
- supernovae and neutron star oscillations.
Gravitational waves

The effect of a wave on two tests-masses is the variation of their distance $\Delta l/l \sim h$, measured by a LASER beam.

Arms of these Michelson-type interferometers are 3 km (VIRGO) and 4 km (LIGO) long . . . almost perfect vacuum. They are acquiring data since 2005, with a very complex data analysis $\Rightarrow$ need for accurate wave patterns: perturbative and numerical approaches.
A brief history of numerical relativity

1966 : May & White, Calculations of General-Relativistic Collapse

1975 : Butterworth & Ipser, Rapidly rotating fluid bodies in general relativity

1976 : Smarr, Čadež, DeWitt & Eppley, Collision of two black holes

1985 : Stark & Piran, Gravitational-Wave Emission from Rotating Gravitational Collapse

1993 : Abrahams & Evans, Vacuum axisymmetric gravitational collapse

1999 : Shibata, Fully general relativistic simulation of coalescing binary neutron stars

2005 : Pretorius, Evolution of Binary Black-Hole Spacetimes
Formulations of Einstein equations
FOUR-DIMENSIONAL APPROACH

Classic approach in analytic studies: harmonic coordinate condition, the coordinates $\{x^\mu\}_{\mu=0...3}$ verify

$$\Box x^\mu = 0.$$  

⇒ nice form of Einstein equations, with $\Box g_{\alpha\beta} = S_{\alpha\beta}$,  
⇒ existence and uniqueness proofs in some cases.  
However, the gauge can be pathological (e.g. in presence of matter): necessity of some generalization for numerical implementation.

$$\Box x^\mu = H^\mu,$$

with an arbitrary source. Generalized Harmonic gauge

Choice of $H^\mu \iff$ choice of gauge

• arbitrary function,
• evolution toward harmonic gauge $\partial_t H_\mu = -\kappa(t) H_\mu$,
• prescription from 3+1 formulations (see later).

first successful simulation of binary black hole evolution
3+1 FORMALISM

Decomposition of spacetime and of Einstein equations

**Evolution equations:**

\[
\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} = -D_i D_j N + NR_{ij} - 2N K_{ik} K^k_j + N [KK_{ij} + 4\pi((S - E) \gamma_{ij} - 2S_{ij})]
\]

\[
K_{ij} = \frac{1}{2N} \left( \frac{\partial \gamma_{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).
\]

**Constraint equations:**

\[
R + K^2 - K_{ij} K^{ij} = 16\pi E,
\]

\[
D_j K^{ij} - D^K K = 8\pi J^i.
\]

\[
g_{\mu\nu} \, dx^\mu \, dx^\nu = -N^2 \, dt^2 + \gamma_{ij} (dx^i + \beta^i \, dt) (dx^j + \beta^j \, dt).
\]
Constrained / free formulations

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

FREE EVOLUTION

- start with initial data verifying the constraints,
- solve only the 6 evolution equations,
- recover a solution of all Einstein equations.

⇒ apparition of constraint violating modes from round-off errors. Considered cures:
   - Using of constraint damping terms and adapted gauges (many groups).
   - Solving the constraints at every time-step (efficient elliptic solver?).
Fully-constrained formulation in Dirac gauge

Proposed by Bonazzola, Gourgoulhon, Grandclément & JN (2004): Define the conformal metric (carrying the dynamical degrees of freedom)

\[ \tilde{\gamma}^{ij} = \Psi^4 \gamma^{ij} \quad \text{with} \quad \Psi = \left( \frac{\det \gamma_{ij}}{\det f_{ij}} \right)^{1/12}, \]

choose the generalized Dirac gauge

\[ \nabla^{(f)}_j \tilde{\gamma}^{ij} = 0, \]

Then, one solves 4 constraint equations + 4 gauge equations (elliptic) at each time-step. Only 2 evolution equations.
Fully-constrained formulation

Properties of the hyperbolic part

The hyperbolic part is obtained combining the evolution equations:

\[
\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} = S_{ij} \quad \text{and} \quad K^{ij} = \frac{1}{2N} \left( \frac{\partial \gamma^{ij}}{\partial t} + \ldots \right),
\]

to obtain a wave-type equation for \( \tilde{\gamma}^{ij} \).

This system of evolution equations has been studied by Cordero-Carrión et al. (2008):

- the choice of Dirac gauge implies that the system is strongly hyperbolic
- can write it as conservation laws
- no incoming characteristic in the case of black hole excision technique
Elliptic part

Uniqueness issue

From the 4 constraints and the choice of time-slicing (gauge), an elliptic system of 5 non-linear equations can be formed

- Elliptic part of Einstein equations, to be solved at every time-step
- When setting $\tilde{\gamma}^{ij} = f^{ij}$, the system reduces to the Conformal-Flatness Condition (CFC).

Because of non-linear terms, the elliptic system may not converge ⇒ the case appears for dynamical, very compact matter and GW configurations (before appearance of the black hole).
A solution to the uniqueness issue

Considering local uniqueness theorems for non-linear elliptic PDEs, it is possible to address the problem: ⇒ introducing auxiliary variables, to solve directly for the momentum constraints (Cordero-Carrión et al. (2009)

2nd fundamental form is rescaled by the conformal factor $A^{ij} = \Psi^{10} K^{ij}$, and decomposed into transverse and longitudinal parts ⇒ solving for each part:

- **longitudinal** $\iff$ momentum constraint,
- **transverse** $\iff$ zero (CFC) or evolution.
Summary of Einstein Equations

Constrained Scheme

**Evolution**

\[
\begin{align*}
\frac{\partial A^{ij}}{\partial t} &= \nabla^k \nabla_k \tilde{\gamma}^{ij} + \ldots \\
\frac{\partial \tilde{\gamma}^{ij}}{\partial t} &= 2N\Psi^{-6}A^{ij} + \ldots \\
\text{with} \quad &\det \tilde{\gamma}^{ij} = 1, \\
&\nabla_j^{(f)} \tilde{\gamma}^{ij} = 0.
\end{align*}
\]

**Constraints**

\[
\begin{align*}
\nabla_j A^{ij} &= 8\pi \Psi^{10} S^i, \\
\Delta \Psi &= -2\pi \Psi^{-1} E \\
&\quad - \Psi^{-7} \frac{A^{ij} A_{ij}}{8}, \\
\Delta N \Psi &= 2\pi N \Psi^{-1} + \ldots
\end{align*}
\]

with

\[
\lim_{r \to \infty} \tilde{\gamma}^{ij} = f^{ij}, \quad \lim_{r \to \infty} \Psi = \lim_{r \to \infty} N = 1.
\]
Spectral methods for numerical relativity
How to deal with functions on a computer?

⇒ a computer can manage only integers

In order to represent a function $\phi(x)$ (e.g. interpolate), one can use:

- a finite set of its values $\{\phi_i\}_{i=0...N}$ on a grid $\{x_i\}_{i=0...N}$,
- a finite set of its coefficients in a functional basis

$$\phi(x) \simeq \sum_{i=0}^{N} c_i \Psi_i(x).$$

In order to manipulate a function (e.g. derive), each approach leads to:

- finite differences schemes

$$\phi'(x_i) \simeq \frac{\phi(x_{i+1}) - \phi(x_i)}{x_{i+1} - x_i}$$

- spectral methods

$$\phi'(x) \simeq \sum_{i=0}^{N} c_i \Psi'_i(x)$$
Convergence of Fourier Series

\[ \phi(x) = \sqrt{1.5 + \cos(x)} + \sin^7 x \]

\[ \phi(x) \approx \sum_{i=0}^{N} a_i \Psi_i(x) \text{ with } \Psi_{2k} = \cos(kx), \Psi_{2k+1} = \sin(kx) \]

\[ N = 18 \]
Use of orthogonal polynomials

The solutions \((\lambda_i, u_i)_{i \in \mathbb{N}}\) of a singular Sturm-Liouville problem on the interval \(x \in [-1, 1]\):

\[-(pu')' + qu = \lambda wu,\]

with \(p > 0, C^1, p(\pm 1) = 0\)

- are orthogonal with respect to the measure \(w\):

\[(u_i, u_j) = \int_{-1}^{1} u_i(x)u_j(x)w(x)dx = 0 \text{ for } m \neq n,\]

- form a spectral basis such that, if \(f(x)\) is smooth \((C^\infty)\)

\[f(x) \simeq \sum_{i=0}^{N} c_i u_i(x)\]

converges faster than any power of \(N\) (usually as \(e^{-N}\)).

Gauss quadrature to compute the integrals giving the \(c_i\)'s.

Chebyshev, Legendre and, more generally any type of Jacobi polynomial enters this category.
Method of weighted residuals

General form of an ODE of unknown $u(x)$:

$$\forall x \in [a, b], \ Lu(x) = s(x), \text{ and } Bu(x)|_{x=a,b} = 0,$$

The approximate solution is sought in the form

$$\bar{u}(x) = \sum_{i=0}^{N} c_i \Psi_i(x).$$

The $\{\Psi_i\}_{i=0}^N$ are called trial functions: they belong to a finite-dimension sub-space of some Hilbert space $\mathcal{H}_{[a,b]}$. $\bar{u}$ is said to be a numerical solution if:

- $B\bar{u} = 0$ for $x = a, b$,
- $R\bar{u} = L\bar{u} - s$ is “small”.

Defining a set of test functions $\{\xi_i\}_{i=0}^N$ and a scalar product on $\mathcal{H}_{[a,b]}$, $R$ is small iff:

$$\forall i = 0 \ldots N, \quad (\xi_i, R) = 0.$$ 

It is expected that $\lim_{N \to \infty} \bar{u} = u$, “true” solution of the ODE.
# Various Numerical Methods

<table>
<thead>
<tr>
<th>Type of Trial Functions $\Psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>finite-differences methods</strong> for local, overlapping polynomials of low order,</td>
</tr>
<tr>
<td><strong>finite-elements methods</strong> for local, smooth functions, which are non-zero only on a sub-domain of $[a, b]$,</td>
</tr>
<tr>
<td><strong>spectral methods</strong> for global smooth functions on $[a, b]$.</td>
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<th>Type of Test Functions $\xi$ for Spectral Methods</th>
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<td><strong>tau method</strong>: $\xi_i(x) = \Psi_i(x)$, but some of the test conditions are replaced by the boundary conditions.</td>
</tr>
<tr>
<td><strong>collocation method</strong> (pseudospectral): $\xi_i(x) = \delta(x - x_i)$, at collocation points. Some of the test conditions are replaced by the boundary conditions.</td>
</tr>
<tr>
<td><strong>Galerkin method</strong>: the test and trial functions are chosen to fulfill the boundary conditions.</td>
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</tbody>
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Inversion of linear ODEs

Thanks to the well-known recurrence relations of Legendre and Chebyshev polynomials, it is possible to express the coefficients \( \{b_i\}_{i=0}^{N} \) of

\[
Lu(x) = \sum_{i=0}^{N} b_i \left| \frac{P_i(x)}{T_i(x)} \right|, \quad \text{with } u(x) = \sum_{i=0}^{N} a_i \left| \frac{P_i(x)}{T_i(x)} \right|.
\]

If \( L = \frac{d}{dx}, \times, \ldots \), and \( u(x) \) is represented by the vector \( \{a_i\}_{i=0}^{N} \), \( L \) can be approximated by a matrix.

Resolution of a linear ODE

\[\uparrow\]

inversion of an \((N + 1) \times (N + 1)\) matrix

With non-trivial ODE kernels, one must add the boundary conditions to the matrix to make it invertible!
Some singular operators

\[ u(x) \mapsto \frac{u(x)}{x} \] is a linear operator, inverse of \( u(x) \mapsto xu(x) \).

Its action on the coefficients \( \{a_i\}_{i=0...N} \) representing the \( N \)-order approximation to a function \( u(x) \) can be computed as the product by a regular matrix. \( \Rightarrow \) The computation in the coefficient space of \( u(x)/x \), on the interval \([-1, 1]\) always gives a finite result (both with Chebyshev and Legendre polynomials).

\( \Rightarrow \) The actual operator which is thus computed is

\[ u(x) \mapsto \frac{u(x) - u(0)}{x}. \]

\( \Rightarrow \) Compute operators in spherical coordinates, with coordinate singularities

\[ \text{e.g. } \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\theta\phi} \]
**Time discretization**

Formally, the representation (and manipulation) of $f(t)$ is the same as that of $f(x)$.

⇒ in principle, one should be able to represent a function $u(x, t)$ and solve time-dependent PDEs only using spectral methods...but this is not the way it is done! Two works:

- Hennig and Ansorg (2008): study of non-linear (1+1) wave equation, with conformal compactification in Minkowski space-time. ⇒ nice spectral convergence.

**WHY?**

- poor *a priori* knowledge of the exact time interval,
- too big matrices for full 3+1 operators ($\sim 30^4 \times 30^4$),
- finite-differences time-stepping errors can be quite small.
**Explicit / Implicit Schemes**

Let us look for the numerical solution of \((L \text{ acts only on } x)\):

\[
\forall t \geq 0, \quad \forall x \in [-1, 1], \quad \frac{\partial u(x, t)}{\partial t} = Lu(x, t),
\]

with good boundary conditions. Then, with \(\delta t\) the time-step: \(\forall J \in \mathbb{N}, \quad u^J(x) = u(x, J \times \delta t)\), it is possible to discretize the PDE as

- \(u^{J+1}(x) = u^J(x) + \delta t \, Lu^J(x)\): **explicit time scheme** (forward Euler); easy to implement, fast but limited by the CFL condition.

- \(u^{J+1}(x) - \delta t \, Lu^{J+1}(x) = u^J(x)\): **implicit time scheme** (backward Euler); one must solve an equation (ODE) to get \(u^{J+1}\), the matrix approximating it here is \(I - \delta t \, L\). Allows longer time-steps but slower and limited to second-order schemes.
Multi-domain approach

Multi-domain technique: several touching, or overlapping, domains (intervals), each one mapped on \([-1, 1]\).

- boundary between two domains can be the place of a discontinuity ⇒ recover spectral convergence,
- one can set a domain with more coefficients (collocation points) in a region where much resolution is needed ⇒ fixed mesh refinement,
- 2D or 3D, allows to build a complex domain from several simpler ones,

\[
\begin{array}{c|c|c|c|}
\text{Domain 1} & \text{Domain 2} \\
\hline
x_1 = -1 & x_1 = 1 & x_2 = -1 & x_2 = 1 \\
\hline
y = a & y = y_0 & y = y_0 & y = b \\
\end{array}
\]

Depending on the PDE, matching conditions are imposed at \( y = y_0 \) \( \iff \) boundary conditions in each domain.
MAPPINGS AND multi-D

In two spatial dimensions, the usual technique is to write a function as:

\[ f : \hat{\Omega} = [-1, 1] \times [-1, 1] \to \mathbb{R} \]

\[ f(x, y) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} c_{ij} P_i(x) P_j(y) \]

The domain \( \hat{\Omega} \) is then mapped to the real physical domain, through some mapping \( \Pi : (x, y) \mapsto (X, Y) \in \Omega \).

⇒ When computing derivatives, the Jacobian of \( \Pi \) is used.

COMPACTIFICATION

A very convenient mapping in spherical coordinates is

\[ x \in [-1, 1] \mapsto r = \frac{1}{\alpha(x - 1)}, \]

to impose boundary condition for \( r \to \infty \) at \( x = 1 \).
**Example:**

**3D Poisson Equation, with non-compact support**

To solve $\Delta \phi(r, \theta, \varphi) = s(r, \theta, \varphi)$, with $s$ extending to infinity.

- setup two domains in the radial direction: one to deal with the singularity at $r = 0$, the other with a compactified mapping.
- In each domain decompose the angular part of both fields onto spherical harmonics:

$$
\phi(\xi, \theta, \varphi) \simeq \sum_{\ell=0}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} \phi_{\ell m}(\xi) Y_{\ell m}^m(\theta, \varphi),
$$

- match between domains, with regularity conditions at $r = 0$, and boundary conditions at $r \to \infty$. 
Numerical simulation of black holes
Puncture methods

... it is not yet clear how and why they work. Hannam et al. (2007)

- black holes are described in the initial data in coordinates that do not reach the physical singularity,

⇒ the coordinates follow a wormhole through another copy of the asymptotically flat exterior spacetime,

- this is compactified so that infinity is represented by a single point, called “puncture”.

\[ \gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \text{ with } \Psi \sim \frac{1}{r}, \text{ use of } \phi = \log \Psi \text{ or } \chi = \Psi^{-4}. \]

BUT

During the evolution the time-slice loses contact with the second asymptotically flat end, and finishes on a cylinder of finite radius.

\[ \Psi(t = 0) = \mathcal{O}\left(\frac{1}{r}\right) \text{ evolves into } \Psi(t > 0) = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right). \]

Use of the shift vector \( \beta^i \) to generate motion.
Excision techniques
Apparent horizons as a boundary

- Remove a neighborhood of the central singularity from computational domain;
- Replace it with boundary conditions on this newly obtained boundary (usually, a sphere),
- Until now, imposition of apparent horizon / isolated horizon properties: zero expansion of outgoing light rays.

⇒ New views on the concept of black hole, following works by Hayward, Ashtekar and Krishnan:
  - Quasi-local approach, making the black hole a causal object;
  - For hydrodynamic, electromagnetic and gravitational waves (Dirac gauge): no incoming characteristics.
Excision technique
Kerr solution from boundary conditions

Can one recover a Kerr black hole only from boundary conditions and Einstein equations?
⇒ Many computations with CFC, but there is no time slicing in which (the spatial part of) Kerr solution can be conformally flat (Garat & Price 2000).

Vasset, JN & Jaramillo (2009) recover full Kerr solution
  - constant value \((N)\), zero expansion on the horizon \((\psi)\);
  - rotation state for \(\beta^\theta, \beta^\phi\) and isolated horizon for \(\beta^r\);
  - NO condition for \(\tilde{\gamma}^{ij}\);

+ asymptotic flatness and Einstein equations!

In particular, no symmetry requirement has been imposed in the “bulk” (only on the horizon) ⇒ illustration of the rigidity theorem by Hawking & Ellis (1973).
Many new results in numerical relativity,
The **Fully-constrained Formulation** is needed for long-term evolutions, particularly in the cases of gravitational collapse,
This formulation is now well-studied and stable.

Many of the numerical features presented here are available in the LORENE library: [http://lorene.obspm.fr](http://lorene.obspm.fr), publicly available under GPL.

Future directions:
- Implementation of FCF and excision methods in the collapse code to simulate the formation of a black hole;
- Use of excision techniques in the dynamical case ⇒most of groups are now heading toward more complex physics: electromagnetic field, realistic equation of state for matter, ...