A constrained scheme for Einstein equations in numerical relativity

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based on collaboration with
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É. Gourgoulhon & J.L. Jaramillo.

Plan

1. **Introduction**
   - Constraints issues in 3+1 formalism
   - Motivation for a fully-constrained scheme

2. **Description of the formulation and strategy**
   - Covariant 3+1 conformal decomposition
   - Einstein equations in Dirac gauge and maximal slicing
   - Integration strategy

3. **Non-uniqueness problem**
   - CFC and FCF
   - A cure in CFC
   - New constrained formulation
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3+1 FORMALISM

Decomposition of spacetime and of Einstein equations

**Evolution Equations:**

\[
\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} = -D_i D_j N + N R_{ij} - 2N K_{ik} K^k_j + N [K K_{ij} + 4\pi ((S - E) \gamma_{ij} - 2S_{ij})]
\]

\[
K^{ij} = \frac{1}{2N} \left( \frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).
\]

**Constraint Equations:**

\[
R + K^2 - K_{ij} K^{ij} = 16\pi E,
\]

\[
D_j K^{ij} - D^i K = 8\pi J^i.
\]

\[
g_{\mu\nu} \, dx^\mu \, dx^\nu = -N^2 \, dt^2 + \gamma_{ij} \, (dx^i + \beta^i \, dt) \, (dx^j + \beta^j \, dt)
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3+1 FORMALISM

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**Constraint violation**

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

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⇓

Appearance of constraint violating modes

Some cures have been investigated (and work):

- constraint-preserving boundary conditions (Lindblom *et al.* 2004)
- constraint projection (Holst *et al.* 2004)
- Using of constraint damping terms and adapted gauges
  ⇒ BSSN or Generalized Harmonic approaches.
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- start with initial data verifying the constraints,
- solve *only* the 6 evolution equations,
- recover a solution of *all* Einstein equations.

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Some reasons not to solve constraints

- Computational cost of usual elliptic solvers...
- Few results of well-posedness for mixed systems versus solid mathematical theory for pure-hyperbolic systems
- Definition of boundary conditions at finite distance and at black hole excision boundary
Motivations for a fully-constrained scheme

“Alternate” approach (although most straightforward)


⇒ Rather popular for 2D applications, but disregarded in 3D
Still, many advantages:
- constraints are verified!
- elliptic systems have good stability properties
- easy to make link with initial data
- evolution of only two scalar-like fields...
Usual conformal decomposition

Standard definition of conformal 3-metric (e.g. Baumgarte-Shapiro-Shibata-Nakamura formalism)

Dynamical degrees of freedom of the gravitational field:

York (1972): they are carried by the conformal “metric”

\[ \hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \quad \text{with} \quad \gamma := \det \gamma_{ij} \]

Problem

\[ \hat{\gamma}_{ij} = \text{tensor density of weight } -2/3 \]

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**INTRODUCTION OF A FLAT METRIC**

We introduce $f_{ij}$ (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of $\gamma_{ij}$, and $\mathcal{D}_i$ the associated covariant derivative.

**DEFINE:**

\[ \tilde{\gamma}_{ij} := \Psi^{-4} \gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^4 \tilde{\gamma}_{ij} \]

with

\[ \Psi := \left( \frac{\gamma}{f} \right)^{1/12} \]

\[ f := \det f_{ij} \]

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of $\gamma_{ij}$ and verifies $\det \tilde{\gamma}_{ij} = f$

$\Rightarrow$ no more tensor densities: only tensors.

Finally,

\[ \tilde{\gamma}^{ij} = f^{ij} + h^{ij} \]

is the deviation of the 3-metric from conformal flatness.
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Generalized Dirac gauge

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

\[ D_j \tilde{\gamma}^{ij} = D_j h^{ij} = 0 \]

where \( D_j \) denotes the covariant derivative with respect to the flat metric \( f_{ij} \).

Compare

- minimal distortion (Smarr & York 1978): \( D_j \left( \partial \tilde{\gamma}^{ij} / \partial t \right) = 0 \)
- pseudo-minimal distortion (Nakamura 1994): \( D^j \left( \partial \tilde{\gamma}_{ij} / \partial t \right) = 0 \)

Notice: Dirac gauge \( \leftrightarrow \) BSSN connection functions vanish:
\[ \tilde{\Gamma}^i = 0 \]
Generalized Dirac gauge properties

- $h^{ij}$ is transverse
- From the requirement $\det \tilde{\gamma}_{ij} = 1$, $h^{ij}$ is asymptotically traceless
- $3R_{ij}$ is a simple Laplacian in terms of $h^{ij}$
- $3R$ does not contain any second-order derivative of $h^{ij}$
- With constant mean curvature ($K = t$) and spatial harmonic coordinates ($D_j [(\gamma/f)^{1/2} \gamma^{ij}] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is locally strongly well posed
- The Conformal Flat Condition (CFC) verifies the Dirac gauge $\Rightarrow$ possibility to easily use initial data for binaries now available
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**Einstein Equations**

**Dirac gauge and maximal slicing ($K = 0$)**

### Hamiltonian Constraint

\[
\Delta(\Psi^2 N) = \Psi^6 N \left( 4\pi S + \frac{3}{4} \bar{A}_{kl} A^{kl} \right) - h^{kl} D_k D_l (\Psi^2 N) + \Psi^2 \left[ N \left( \frac{1}{16} \bar{\gamma}^{kl} D_k h^{ij} D_l \bar{\gamma}_{ij} 
      - \frac{1}{8} \bar{\gamma}^{kl} D_k h^{ij} D_j \bar{\gamma}_{il} + 2\bar{D}_k \ln \Psi \bar{D}^k \ln \Psi \right) + 2\bar{D}_k \ln \Psi \bar{D}^k N \right]
\]

### Momentum Constraint

\[
\Delta \beta^i + \frac{1}{3} D^i \left( D_j \beta^j \right) = 2A^{ij} D_j N + 16\pi N \Psi^4 J^i - 12N A^{ij} D_j \ln \Psi - 2\Delta^i_{kl} N A^{kl} 
- h^{kl} D_k D_l \beta^i - \frac{1}{3} h^{ik} D_k D_l \beta^l
\]

### Trace of Dynamical Equations

\[
\Delta N = \Psi^4 N \left[ 4\pi (E + S) + \bar{A}_{kl} A^{kl} \right] - h^{kl} D_k D_l N - 2\bar{D}_k \ln \Psi \bar{D}^k N
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\]

6 components - 3 Dirac gauge conditions - \((\det \tilde{\gamma}^{ij} = 1)\)

**2 degrees of freedom**

\[-\frac{\partial^2 A}{\partial t^2} + \Delta A = S_A\]

\[-\frac{\partial^2 \tilde{B}}{\partial t^2} + \Delta X = S_{\tilde{B}}\]

with \(A\) and \(\tilde{B}\) two scalar potentials representing the degrees of freedom.
Einstein equations

**Dirac gauge and maximal slicing \((K = 0)\)**

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**Integration Procedure**

Everything is known on slice $\Sigma_t$

\[ \downarrow \]

Evolution of $A$ and $\tilde{B}$ to next time-slice $\Sigma_{t+dt}$ (+ hydro)

\[ \downarrow \]

Deduce $h^{ij}(t+dt)$ from Dirac and trace-free conditions

\[ \downarrow \]

Deduce the trace from $\det \tilde{\gamma}^{ij} = 1$; thus $h^{ij}(t+dt)$ and $\tilde{\gamma}^{ij}(t+dt)$.

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Iterate on the system of elliptic equations for $N$, $\Psi^2 N$ and $\beta^i$ on $\Sigma_{t+dt}$.
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Everything is known on slice $\Sigma_t$

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Non-uniqueness problem
Conformal flatness condition

Within 3+1 formalism, one imposes that:

\[ \gamma_{ij} = \psi^4 f_{ij} \]

with \( f_{ij} \) the flat metric and \( \psi(t, x^1, x^2, x^3) \) the conformal factor. First devised by Isenberg in 1978 as a waveless approximation to GR, it has been widely used for generating initial data,

- discards all dynamical degrees of freedom of the gravitational field (\( A \) and \( \tilde{B} \) are zero by construction)
- exact in spherical symmetry: e.g. the Schwarzschild metric can be described within CFC

\( \Rightarrow \) captures many non-linear effects.

- The Kerr solution cannot be exactly described in CFC, but rotation can be included in BH solution.
**Einstein equations in CFC**

SET OF 5 NON-LINEAR ELLIPTIC PDEs $(K = 0)$

\[
\Delta \psi = -2\pi \psi^{-1} \left( E^* + \frac{\psi^6 K_{ij} K^{ij}}{16\pi} \right),
\]
\[
\Delta (N \psi) = 2\pi N \psi^{-1} \left( E^* + 2S^* + \frac{7\psi^6 K_{ij} K^{ij}}{16\pi} \right),
\]
\[
\Delta \beta^i + \frac{1}{3} \nabla^i \nabla_j \beta^j = 16\pi N \psi^{-2} (S^*)^i + 2\psi^{10} K^{ij} \nabla_j \frac{N}{\psi^6}.
\]

\[
E^* = \psi^6 E,
\]
\[
(S^*)^i = \psi^6 S^i, \ldots
\]

are conformally-rescaled projections of the stress-energy tensor.
Spherical collapse of matter

We consider the case of the collapse of an unstable relativistic star, governed by the equations for the hydrodynamics

\[ \frac{1}{\sqrt{-g}} \left[ \frac{\partial \sqrt{\gamma} U}{\partial t} + \frac{\partial \sqrt{-g} F_i^i}{\partial x^i} \right] = Q, \]

with \( U = (\rho W, \rho h W^2 v_i, \rho h W^2 - P - D). \)

At every time-step, we solve the equations of the CFC system (elliptic)
\[ \Rightarrow \text{exact in spherical symmetry! (isotropic gauge)} \]

- During the collapse, when the star becomes very compact, the elliptic system would no longer converge, or give a wrong solution (wrong ADM mass).
- Even for equilibrium configurations, if the iteration is done only on the metric system, it may converge to a wrong solution.
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**Collapse of gravitational waves**

Using FCF (full 3D Einstein equations), the same phenomenon is observed for the collapse of a gravitational wave packet.

- Initial data: vacuum spacetime with Gaussian gravitational wave packet,
- if the initial amplitude is sufficiently large, the waves collapse to a black hole.
- As in the fluid-CFC case, the elliptic system of the FCF suddenly starts to converge to a **wrong** solution.

⇒ effect on the ADM mass computed from $\psi$ at $r = \infty$. 
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$\Rightarrow$ effect on the ADM mass computed from $\psi$ at $r = \infty$. 

\[ N_c \]

\[ M_{\text{ADM}} \text{ [arbitrary units]} \]

\[ t \]
In the extended conformal thin sandwich approach for initial data, the system of PDEs is the same as in CFC.

Pfeiffer & York (2005) have numerically observed a parabolic branching in the solutions of this system for perturbation of Minkowski spacetime.

Some analytical studies have been performed by Baumgarte et al. (2007), which have shown the genericity of the non-uniqueness behavior.

from Pfeiffer & York (2005)
A cure in the CFC case
ORIGIN OF THE PROBLEM

In the simplified non-linear scalar-field case, of unknown function $u$

$$\Delta u = \alpha u^p + s.$$  

Local uniqueness of solutions can be proven using a maximum principle:

if $\alpha$ and $p$ have the same sign, the solution is locally unique.

In the CFC system (or elliptic part of FCF), the case appears for the Hamiltonian constraint:

$$\Delta \psi = -2\pi \psi^5 E - \frac{1}{8} \psi^5 K_{ij} K^{ij};$$

Both terms (matter and gravitational field) on the r.h.s. have wrong signs.
In the simplified non-linear scalar-field case, of unknown function $u$

$$\Delta u = \alpha u^p + s.$$ 

Local uniqueness of solutions can be proven using a maximum principle:

if $\alpha$ and $p$ have the same sign, the solution is locally unique.

In the CFC system (or elliptic part of FCF), the case appears for the Hamiltonian constraint:

$$\Delta \psi = -2\pi \psi^5 E - \frac{1}{8} \psi^5 K_{ij} K^{ij};$$

Both terms (matter and gravitational field) on the r.h.s. have wrong signs.
Let \( L, V^i \mapsto (LV)^{ij} = \nabla^i V^j + \nabla^j V^i - \frac{2}{3} f^{ij} \nabla_k V^k \).

In CFC, \( K^{ij} = \psi^{-4} \tilde{A}^{ij} \), with \( \tilde{A}^{ij} = \frac{1}{2N} (L\beta)^{ij} \),

here \( K^{ij} = \psi^{-10} \hat{A}^{ij} \), with \( \hat{A}^{ij} = (LX)^{ij} + \hat{A}^{ij}_{TT} \).

Neglecting \( \hat{A}^{ij}_{TT} \), we can solve in a hierarchical way:

1. Momentum constraints \( \Rightarrow \) linear equation for \( X^i \) from the actually computed hydrodynamic quantity \( S^*_j = \psi^6 S_j \),
2. Hamiltonian constraint \( \Rightarrow \Delta \psi = -2\pi \psi^{-1} E^* - \psi^{-7} \hat{A}^{ij} \hat{A}_{ij}/8 \),
3. linear equation for \( N\psi \),
4. linear equation for \( \beta \), from the definitions of \( \hat{A}^{ij} \).

It can be shown that the error made neglecting \( \hat{A}^{ij}_{TT} \) falls within the error of CFC approximation.
Approximate CFC

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New equations in CFC

The conformally-rescaled projections of the stress-energy tensor $E^* = \psi^6 E$, $(S^*)^i = \psi^6 S^i$, … are supposed to be known from hydrodynamics evolution.

\[ \Delta X + \frac{1}{3} \nabla^i \nabla_j X_j = 8\pi (S^*)^i, \]
\[ \hat{A}^{ij} \simeq \nabla^i X^j + \nabla^j X^i, \]
\[ \Delta \psi = - 2\pi \psi^{-1} E^* - \frac{\psi^{-7}}{8} \hat{A}^{ij} \hat{A}_{ij}, \]
\[ \Delta (N\psi) = 2\pi N\psi^{-1} (E^* + 2S^*) + N\psi^{-7} \frac{7\hat{A}_{ij} \hat{A}^{ij}}{8}, \]
\[ \Delta \beta^i + \frac{1}{3} \nabla^i \nabla_j \beta^j = \nabla_j \left( 2N\psi^{-6} \hat{A}^{ij} \right). \]
Using the code CoCoNuT combining Godunov-type methods for the solution of hydrodynamic equations and spectral methods for the gravitational fields.

- Unstable rotating neutron star initial data, with polytropic equation of state,
- approximate CFC equations are solved every time-step.
- Collapse proceeds beyond the formation of an apparent horizon;
- Results compare well with those of Baoittti et al. (2005) in GR, although in approximate CFC.

Other test: migration of unstable neutron star toward the stable branch.
New constrained formulation
NEW CONSTRAINED FORMULATION

Evolution equations

In the general case, one cannot neglect the TT-part of $\hat{A}^{ij}$ and one must therefore evolve it numerically.

\[
\begin{array}{ccc}
\text{sym. tensor} & \text{longitudinal part} & \text{transverse part} \\
\hat{A}^{ij} = & (LX)^{ij} & + \hat{A}^{ij}_{TT} \\
h^{ij} = & 0 \text{ (gauge)} & + h^{ij}
\end{array}
\]

The evolution equations are written only for the transverse parts:

\[
\frac{\partial \hat{A}^{ij}_{TT}}{\partial t} = \left[ \mathcal{L}_{\beta} \hat{A}^{ij} + N \psi^2 \Delta h^{ij} + S^{ij} \right]^{TT},
\]

\[
\frac{\partial h^{ij}}{\partial t} = \left[ \mathcal{L}_{\beta} h^{ij} + 2 N \psi^{-6} \hat{A}^{ij} - (L\beta)^{ij} \right]^{TT}.
\]
New constrained formulation

If all metric and matter quantities are supposed known at a given time-step.

1. Advance hydrodynamic quantities to new time-step,
2. advance the TT-parts of $\hat{A}^{ij}$ and $h^{ij}$,
3. obtain the longitudinal part of $\hat{A}^{ij}$ from the momentum constraint, solving a vector Poisson-like equation for $X^i$ (the $\Delta^i_{jk}$'s are obtained from $h^{ij}$):

$$\Delta X^i + \frac{1}{3} \nabla^i \nabla_j X^j = 8\pi (S^*)^i - \Delta^i_{jk} \hat{A}^{jk},$$

4. recover $\hat{A}^{ij}$ and solve the Hamiltonian constraint to obtain $\psi$ at new time-step,
5. solve for $N\psi$ and recover $\beta^i$. 

\[\text{LUTH}\]
A fully-constrained formalism of Einstein equations, aimed at obtaining stable solutions in astrophysical scenarios (with matter) has been presented, implemented and tested;

A way to cure the uniqueness problem in the elliptic part of Einstein equations has been devised;

⇒ the accuracy has been checked: the additional approximation in CFC does not introduce any new errors.

The numerical codes are present in the LORENE library: http://lorene.obspm.fr, publicly available under GPL.

Future directions:

- Implementation of the new FCF and tests in the case of gravitational wave collapse;
- Use of the CFC approach together with excision methods in the collapse code to simulate the formation of a black hole (work by N. Vasset);
**Summary - Perspectives**

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